EXISTENCE THEOREMS
FOR SOME ELLIPTIC SYSTEMS

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Abstract: We investigate the existence of solutions of systems of semilinear elliptic equations. The proof makes use of the Leray–Schauder degree theory. We also study the corresponding linear problem.

1 – Introduction

In this paper we consider the following elliptic system

\begin{equation}
\begin{cases}
-\Delta u_j = f_j(x, u_1, ..., u_m), & j = 1, ..., m \quad \text{in } \Omega, \\
u_j = \psi_j, & j = 1, ..., m \quad \text{on } \partial \Omega,
\end{cases}
\end{equation}

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ ($n \geq 1$) of class $C^{2,\alpha}$ for some $\alpha \in (0, 1)$, $m \geq 1$ is an integer and $f_j : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}$, $j = 1, ..., m$, are locally Hölder continuous functions with exponent $\alpha$. When $\psi_j \in C^{2,\alpha}(\partial \Omega)$, $j = 1, ..., m$, we seek a solution $u = (u_1, ..., u_m) \in (C^{2,\alpha}(\overline{\Omega}))^m$.

Let $0 < \mu_1 < \mu_2 \leq ... \leq \mu_k \leq ...$ be the eigenvalues of the operator $-\Delta$ on $\Omega$ with Dirichlet boundary conditions. We shall note $\varphi_1$ the positive eigenfunction corresponding to $\mu_1$.

**Theorem 1.** Suppose that there are constants $a_{jk} \geq 0$ and $c_j \geq 0$, $j, k = 1, ..., m$ such that

\begin{equation}
|f_j(x, u_1, ..., u_m)| \leq \sum_{k=1}^{m} a_{jk} |u_k| + c_j,
\end{equation}

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for \( j = 1, \ldots, m \) and \((x, u_1, \ldots, u_m) \in \Omega \times \mathbb{R}^m\), with

\[
(1.3) \quad \mu_1 > \rho(A),
\]

where \( \rho(A) \) denotes the spectral radius of \( A = (a_{jk})_{1 \leq j, k \leq m}\).

Then for any \((\psi_1, \ldots, \psi_m) \in (C^{2,\alpha}(\partial\Omega))^m\), problem (1.1) has a solution \( u = (u_1, \ldots, u_m) \in (C^{2,\alpha}(\Omega))^m\).

**Remark 1.** It will be clear from the proof that, at least in the case \( n = 1 \),

theorem 1 remains true for zero boundary conditions if (1.2) is replaced by

\[
u_j f_j(x, u_1, \ldots, u_m) \leq \sum_{k=1}^m a_{jk} |u_k| + c_j |u_j|,
\]

which in some instances may be a weaker growth condition; roughly speaking

\( f_j \) may contain a term in \( u_j \) that is linearly bounded from above or below only,

according to the sign of \( u_j \).

In Section 4 we shall give an example showing that our condition is sharp.

When \( n = 1, m = 2 \) and \( f_1(x, u_1, u_2) = -u_2 \) problem (1.1) reduces to

\[
(1.4) \quad \begin{cases}
\frac{d^4 u}{dx^4} = f(x, u, u''), & a < x < b, \\
u(a) = u_a, \quad u(b) = u_b, \quad u''(a) = u_{pi}, \quad u''(b) = u_{pi},
\end{cases}
\]

where \( b - a < +\infty \) and \( f \in C([a, b] \times \mathbb{R}^2)\).

Aftabizadeh [1] and Yang [7] proved the existence of a solution of (1.4) (with

\( a = 0, b = 1 \)) when

\[
|f(x, u, v)| \leq \alpha |u| + \beta |v| + \gamma,
\]

where \( \alpha, \beta, \gamma \geq 0 \) are such that \( \alpha / \pi^4 + \beta / \pi^2 < 1 \).

When \( n \geq 1 \) and \( f_j(x, u_1, \ldots, u_m) = -u_{j+1} \) for \( j = 1, \ldots, m - 1 \) (if \( m \geq 2 \)),

problem (1.1) reduces to

\[
(1.5) \quad \begin{cases}
\Delta^m u = f(x, u, \Delta u, \ldots, \Delta^{m-1} u) \quad \text{in} \ \Omega, \\
\Delta^j u = \psi_j, \quad j = 0, \ldots, m - 1 \quad \text{on} \ \partial\Omega,
\end{cases}
\]

where \( f: \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R} \) is a locally Hölder continuous function with exponent \( \alpha \).

Chen and Nee [4] proved the existence of a solution of (1.5) under the condition

\[
|f(x, u_1, \ldots, u_m)| \leq \sum_{k=1}^m a_k |u_k| + c,
\]
where $a_k \geq 0$, $c \geq 0$ are such that

\begin{equation}
\sum_{k=1}^{m} \frac{a_k}{\mu_1^{m-k+1}} < 1.
\end{equation}

We wish to point out that the condition of solvability in the above examples coincides with that given in theorem 1 (see remark 2).

**Remark 2.** For problem (1.5) the matrix $A$ defined in theorem 1 is such that, when $m \geq 2$, $a_{jj+1} = 1$ for $1 \leq j \leq m - 1$, $a_{jk} = 0$ for $k \neq j + 1$, $1 \leq j \leq m - 1$, $1 \leq k \leq m$ and $a_{mk} = a_k$ for $1 \leq k \leq m$. In Section 2 we shall show that condition (1.3) is equivalent to condition (1.6).

In both cases the proof makes use of the Leray–Schauder degree theory [2]. Therefore the underlying technique is the establishment of a priori estimates.

Note that we can assume that $\psi_j = 0$ for $j = 1, ..., m$. Indeed let $\chi_j \in C^{2,\alpha}(\Omega)$ be such that

\begin{align*}
\Delta \chi_j &= 0, \quad j = 1, ..., m \quad \text{in } \Omega, \\
\chi_j &= \psi_j, \quad j = 1, ..., m \quad \text{on } \partial \Omega.
\end{align*}

Define $v_j = u_j - \chi_j$, $j = 1, ..., m$. Then problem (1.1) is equivalent to the following boundary value problem

\begin{equation}
\begin{cases}
-\Delta v_j = f_j(x, v_1 + \chi_1, ..., v_m + \chi_m), & j = 1, ..., m \quad \text{in } \Omega, \\
v_j = 0, & j = 1, ..., m \quad \text{on } \partial \Omega,
\end{cases}
\end{equation}

and the functions

\begin{equation}
g_j(x, v_1, ..., v_m) = f_j(x, v_1 + \chi_1, ..., v_m + \chi_m), \quad j = 1, ..., m
\end{equation}

still satisfy (1.2) with different $c_j$.

In Section 2, in order that the paper be self-contained, we provide preliminary results from the theory of nonnegative matrices. In Section 3 we prove our a priori bounds. Theorem 1 is proved in Section 4. Finally in Section 5 we study the corresponding linear problem.

\section{Preliminaries}

In this section, in order that the paper be self-contained, we provide preliminary results from the theory of nonnegative matrices. We refer the reader to Berman and Plemmons [3] for proofs. We consider the proper cone

\[ \mathbb{R}_+^m = \left\{ x = (x_1, ..., x_m) \in \mathbb{R}^m; \ x_j \geq 0, \ j = 1, ..., m \right\}. \]
Definition 1. An $m \times m$ matrix $M$ is called $\mathbb{R}^+_m$-monotone if

$$Mx \in \mathbb{R}^+_m \Rightarrow x \in \mathbb{R}^+_m.$$  

The following theorems are parts of some results proved in [3] (theorem 1.3.2, p. 6, theorem 1.3.12, p. 10, corollary 2.1.12, p. 28 and theorems 5.2.3, 5.2.6, p. 113).

Theorem 2. Let $N$ be an $m \times m$ nonnegative matrix (i.e. $N = (n_{jk})_{1 \leq j, k \leq m}$ with $n_{jk} \geq 0$ for $j, k = 1, \ldots, m$). Then $\rho(N)$ is an eigenvalue of $N$.

Theorem 3. Let $M = \alpha I - N$, where $\alpha \in \mathbb{R}$ and $N$ is an $m \times m$ nonnegative matrix. If $Mx \in \mathbb{R}^+_m$ for some $x \in \mathbb{R}^+_m$, then $\rho(N) \leq \alpha$.

Theorem 4. Let $N$ be an $m \times m$ nonnegative matrix. If $x$ is a positive (i.e. $x = (x_j)_{1 \leq j \leq m}$ with $x_j > 0$ for $j = 1, \ldots, m$) eigenvector of $N$ then $x$ corresponds to $\rho(N)$.

Theorem 5. An $m \times m$ matrix $M$ is $\mathbb{R}^+_m$-monotone if and only if it is nonsingular and $M^{-1}$ is nonnegative.

Theorem 6. Let $M = \alpha I - N$, where $\alpha \in \mathbb{R}$ and $N$ is an $m \times m$ nonnegative matrix. Then the following are equivalent:

i) The matrix $M$ is $\mathbb{R}^+_m$-monotone.

ii) $\rho(N) < \alpha$.

We conclude this section with the proof of the assertion of remark 2. We first note that condition (1.6) can be written $\det(\mu I - A) > 0$. Then we use the following lemma.

Lemma 1. Let $N = (n_{jk})_{1 \leq j, k \leq m}$ be a nonnegative matrix such that, when $m \geq 2$, $n_{jk} = 0$ for $k \neq j + 1$, $1 \leq j \leq m - 1$, $1 \leq k \leq m$. If $\alpha > 0$ the following are equivalent:

i) $\det(\alpha I - N) > 0$ (resp. $\det(\alpha I - N) = 0$).

ii) $\alpha > \rho(N)$ (resp. $\alpha = \rho(N)$).

Proof: i)$\Rightarrow$ii): Since the lemma is obvious when $m = 1$, we assume $m \geq 2$. Let $\lambda \in \mathbb{R}$. We have

$$\det(\lambda I - N) = \lambda^m - \left\{n_{mm} \lambda^{m-1} + \sum_{k=1}^{m-1} n_{mk} n_{kk+1} \cdots n_{m-1m} \lambda^{k-1}\right\} .$$
Suppose first that $n_{m-1} = 0$. Then $\det(\lambda I - N) = \lambda^{m-1}(\lambda - n_{mm})$. Clearly $\rho(N) = n_{mm}$ and since $\alpha > 0$ the result follows.

Now if $n_{m-1} > 0$ we claim that we can assume that $n_{jj+1} > 0$ for $j = 1, \ldots, m - 1$. Indeed if $n_{jj+1} = 0$ for some $j \in \{1, \ldots, m - 2\}$ (thus necessarily $m \geq 3$), we define $h = \max\{j \in \{1, \ldots, m - 2\}; n_{jj+1} = 0\}$. Then

$$\det(\lambda I - N) = \lambda^h \det(\lambda I - Q),$$

where $Q = (q_{jk})_{1 \leq j, k \leq m - h}$ is an $(m - h) \times (m - h)$ nonnegative matrix such that $q_{jj+1} > 0$ for $1 \leq j \leq m - h - 1$ and $q_{jk} = 0$ for $k \neq j + 1$, $1 \leq j \leq m - h - 1$, $1 \leq k \leq m - h$. Clearly $\rho(N) = \rho(Q)$. Since $\alpha > 0$, $\det(\alpha I - N) > 0$ (resp. $\det(\alpha I - Q) = 0$) if and only if $\det(\alpha I - Q) > 0$ (resp. $\det(\alpha I - Q) = 0$). Thus our claim is proved. Now let $x_m > 0$ and define the column vector $x = (x_j)_{1 \leq j \leq m}$ by

$$x_j = \alpha^{j-m} n_{jj+1} \cdots n_{m-1} x_m \quad \text{for} \quad j = 1, \ldots, m - 1.$$ 

Then $(\alpha I - N)x = y = (y_j)_{1 \leq j \leq m}$ where $y_j = 0$ for $j = 1, \ldots, m - 1$ and $y_m = \alpha^{1-m} x_m \det(\alpha I - N)$. Using theorem 3 we get $\rho(N) \leq \alpha$. Then the result follows with the help of theorem 2.

$\text{ii)} \Rightarrow \text{i)}$: Since $\rho(N)$ is an eigenvalue of $N$, the result is clear. \qed

3 – A priori bounds

We first introduce the following problems

$$\begin{align*}
(3.1)_t & \quad \begin{cases}
-\Delta u_j = t f_j(x, u_1, \ldots, u_m), & j = 1, \ldots, m \quad \text{in} \ \Omega, \\
u_j = 0, & j = 1, \ldots, m \quad \text{on} \ \partial \Omega,
\end{cases}
\end{align*}$$

where $t \in [0, 1]$ is the Leray–Schauder homotopy parameter.

**Theorem 7.** Under the assumptions of theorem 1, there exists a constant $M > 0$ such that for any $t \in [0, 1]$ and any solution $u = (u_1, \ldots, u_m) \in (C^{2, \alpha}(\bar{\Omega}))^m$ of (3.1)$_t$ we have

$$\|u_j\|_{L^\infty(\Omega)} \leq M, \quad j = 1, \ldots, m.$$

**Proof:** Multiplying the differential equation in (3.1)$_t$ by $u_j$, integrating over $\Omega$ and using (1.2) we obtain

$$\int_\Omega |\nabla u_j|^2 \, dx = t \int_\Omega u_j f_j(x, u_1, \ldots, u_m) \, dx$$

$$\leq \sum_{k=1}^m a_{jk} \int_\Omega |u_j u_k| \, dx + c_j \int_\Omega |u_j| \, dx$$

$$\leq M \int_\Omega |u_j| \, dx$$

Thus

$$\int_\Omega |u_j| \, dx \leq M.$$
for \( j = 1, \ldots, m \). By first using the Schwarz inequality and then the Poincaré inequality we get

\[
\int_{\Omega} |\nabla u_j|^2 \, dx \leq \sum_{k=1}^{m} a_{jk} \left( \int_{\Omega} u_j^2 \, dx \right)^{1/2} \left( \int_{\Omega} u_k^2 \, dx \right)^{1/2} + c_j |\Omega|^{1/2} \left( \int_{\Omega} u_j^2 \, dx \right)^{1/2} \leq \\
\sum_{k=1}^{m} \frac{a_{jk}}{\mu_1} \left( \int_{\Omega} |\nabla u_j|^2 \, dx \right)^{1/2} \left( \int_{\Omega} |\nabla u_k|^2 \, dx \right)^{1/2} + \frac{c_j}{\sqrt{\mu_1}} |\Omega|^{1/2} \left( \int_{\Omega} |\nabla u_j|^2 \, dx \right)^{1/2}
\]

for \( j = 1, \ldots, m \) from which we deduce

\[
\|\nabla u_j\|_{L^2(\Omega)} \leq \sum_{k=1}^{m} \frac{a_{jk}}{\mu_1} \|\nabla u_k\|_{L^2(\Omega)} + \frac{c_j}{\sqrt{\mu_1}} |\Omega|^{1/2}, \quad j = 1, \ldots, m .
\]

Let \( x \) and \( b \) denote the column vectors

\[
x = \left( \|\nabla u_j\|_{L^2(\Omega)} \right)_{1 \leq j \leq m} \quad \text{and} \quad b = \left( \frac{c_j}{\sqrt{\mu_1}} |\Omega|^{1/2} \right)_{1 \leq j \leq m} .
\]

(3.2) can be written

\[
b - (I - \mu_1^{-1} A) x \in \mathbb{R}_+^m .
\]

(1.3) and theorem 6 imply that \( I - \mu_1^{-1} A \) is \( \mathbb{R}_+^m \)-monotone. Hence using theorem 5 we obtain

\[
(I - \mu_1^{-1} A)^{-1} b - x \in \mathbb{R}_+^m .
\]

From (3.3) and the Poincaré inequality it follows that

\[
\|u_j\|_{W^{1,2}(\Omega)} \leq C, \quad j = 1, \ldots, m .
\]

where \( C \) is a positive constant. Now for \( 1 < p < +\infty \) we have the following estimates

\[
\|u_j\|_{W^{2,p}(\Omega)} \leq C \|\Delta u_j\|_{L^p(\Omega)}, \quad j = 1, \ldots, m ,
\]

([6], lemma 9.17, p. 242) for some positive constant \( C \). Moreover from the differential equations in (3.1) and condition (1.2) we deduce

\[
\|\Delta u_j\|_{L^p(\Omega)} \leq C \sum_{k=1}^{m} \|u_k\|_{L^p(\Omega)}, \quad j = 1, \ldots, m ,
\]

for another positive constant \( C \).

Now if \( n = 1 \), (3.4) and the Sobolev imbedding theorem imply \( L^\infty \) bounds.
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If \( n = 2 \), (3.4) and the Sobolev imbedding theorem imply that, for \( 1 < p < +\infty \), there exists \( C > 0 \) such that

\[
\|u_j\|_{L^p(\Omega)} \leq C, \quad j = 1, \ldots, m.
\]

Then using (3.5)–(3.7) and the Sobolev imbedding theorem we obtain the \( L^\infty \) bounds.

Finally if \( n \geq 3 \), the conclusion follows from a classical bootstrapping procedure (see [2]) using (3.4)–(3.6) and the Sobolev imbedding theorem. The proof of the theorem is complete. ■

4 – Proof of theorem 1

We recall from Section 1 that it is sufficient to deal with zero boundary conditions.

We shall note \( G(x, y) \) the Green’s function of the operator \( -\Delta \) on \( \Omega \) with Dirichlet boundary conditions. Consider the function space \( X = (C(\Omega))^m \) endowed with the norm

\[
\|u\| = \max_{1 \leq j \leq m} \left( \|u_j\|_{L^\infty(\Omega)} \right) \quad \text{for} \quad u = (u_1, \ldots, u_m) \in X.
\]

Then \( X \) is a Banach space. Regularity theory implies that solving (3.1) is equivalent to finding a solution \( u = (u_1, \ldots, u_m) \in X \) of the following system of integral equations

\[
u_j(x) = t \int_\Omega G(x, y) f_j(y, u_1(y), \ldots, u_m(y)) \, dy, \quad j = 1, \ldots, m.
\]

Now define a map \( T_t: X \to X \) by \( T_t u = v = (v_1, \ldots, v_m) \) where

\[
v_j(x) = \int_\Omega G(x, y) f_j(y, u_1(y), \ldots, u_m(y)) \, dy, \quad j = 1, \ldots, m.
\]

It is well-known that \( T_t \) is continuous and compact for \( t \in [0, 1] \). Regularity theory implies that solving (1.1) (with \( \psi_j = 0, \ j = 1, \ldots, m \)) is equivalent to finding a fixed point of the map \( T_1 \) in \( X \). Let \( M \) be the constant appearing in theorem 7. Consider the ball \( B_M \) in \( X \):

\[
B_M = \left\{ u \in X; \|u\| < M + 1 \right\}.
\]

Theorem 7 implies that \( T_t \) has no fixed point on \( \partial B_M \). Let \( I: X \to X \) be the identity map. By the homotopy invariance of the Leray–Schauder degree we have

\[
\deg(I - T_1, B_M, 0) = \deg(I - T_1, B_M, 0) = \deg(I - T_0, B_M, 0) = \deg(I, B_M, 0) = 1.
\]
Consequently, $T_1$ has a fixed point in $B_M$. The theorem is proved. ■

**Remark 3.** If there exist constants $a_{jk} \geq 0$, $j, k = 1, \ldots, m$, such that

$$|f_j(x, u_1, \ldots, u_m) - f_j(x, v_1, \ldots, v_m)| \leq \sum_{k=1}^m a_{jk} |u_k - v_k|$$

for $j = 1, \ldots, m$ and $(x, u_1, \ldots, u_m), (x, v_1, \ldots, v_m) \in \overline{\Omega} \times \mathbb{R}^m$ with $A = (a_{jk})_{1 \leq j, k \leq m}$ satisfying (1.3), then the solution of (1.1) is unique. The argument is similar to the proof of theorem 7.

**Example 1:** Let

$$f_j(x, u_1, \ldots, u_m) = \sum_{k=1}^m a_{jk} u_k$$

for $j = 1, \ldots, m$ and $(x, u_1, \ldots, u_m) \in \overline{\Omega} \times \mathbb{R}^m$ where $a_{jk} \geq 0$ are constants, $j, k = 1, \ldots, m$. Let $b$ denote the column vector

$$b = \left(- \int_{\partial \Omega} \varphi_j \frac{\partial \varphi_1}{\partial \nu} \, ds \right)_{1 \leq j \leq m}$$

and $A = (a_{jk})_{1 \leq j, k \leq m}$. Suppose that $\mu_1 = \rho(A)$. By theorem 2 $\det(\mu_1 I - A) = 0$. The Hopf boundary lemma ([6], lemma 3.4, p. 33) implies that $\frac{\partial \varphi_1}{\partial \nu} < 0$ on $\partial \Omega$. Therefore we can choose $\varphi_j \in C^{2,\alpha}(\partial \Omega)$, $j = 1, \ldots, m$, such that $b \notin R(\mu_1 I - A)$. Then problem (1.1) has no solution. Indeed, suppose that problem (1.1) has a solution $u = (u_1, \ldots, u_m) \in (C^{2,\alpha}(\overline{\Omega}))^m$. Multiplying the differential equation in (1.1) by $\varphi_1$ and using Green’s formula we obtain

$$-\int_{\Omega} \varphi_1 \Delta u_j \, dx = -\int_{\Omega} u_j \Delta \varphi_1 \, dx + \int_{\partial \Omega} \varphi_j \frac{\partial \varphi_1}{\partial \nu} \, ds = \mu_1 \int_{\Omega} u_j \varphi_1 \, dx + \int_{\partial \Omega} \varphi_j \frac{\partial \varphi_1}{\partial \nu} \, ds = \sum_{k=1}^m a_{jk} \int_{\Omega} u_k \varphi_1 \, dx \quad j = 1, \ldots, m,$$

where $\nu$ is the unit outward normal to $\partial \Omega$. This yields

$$(\mu_1 I - A) x = b,$$

where $x$ denotes the column vector

$$x = \left(\int_{\Omega} u_j \varphi_1 \, dx \right)_{1 \leq j \leq m}$$

and we reach a contradiction.
The above example shows that our condition is sharp.

5 – The linear problem

In this section we consider the following boundary value problem:

\[ -\delta u_j = \sum_{k=1}^{m} a_{jk} u_k, \quad j = 1, \ldots, m \quad \text{in} \ \Omega, \tag{5.1} \]

\[ u_j = 0, \quad j = 1, \ldots, m \quad \text{on} \ \partial \Omega, \tag{5.2} \]

where \( m \geq 1 \) and \( a_{jk} \in \mathbb{R} \) for \( 1 \leq j, k \leq m \). We define \( A_m = (a_{jk})_{1 \leq j,k \leq m} \). Below \( u = (u_1, \ldots, u_m) \geq 0 \) (resp. \( > 0 \)) means \( u_j \geq 0 \) (resp. \( u_j > 0 \)) for \( j = 1, \ldots, m \).

**Lemma 2.** Let \( u = (u_1, \ldots, u_m) \in (C^{2,\alpha}(\Omega))^m \) be a nonnegative nontrivial solution of problem (5.1), (5.2). Then \( \det(\mu_1 I - A_m) = 0 \).

**Proof:** Arguing as in example 1 we get

\[ (\mu_1 I - A_m) x = 0 \]

where \( x \) is the column vector \( x = (\int_{\Omega} u_j \varphi_1 \, dx)_{1 \leq j \leq m} \). Since there exists \( j \in \{1, \ldots, m\} \) such that \( \int_{\Omega} u_j \varphi_1 \, dx \neq 0 \), we have necessarily

\[ \det(\mu_1 I - A_m) = 0 \]

and the lemma is proved. \( \blacksquare \)

**Lemma 3.** Assume that \( a_{jk} \geq 0 \) for \( j, k = 1, \ldots, m \). Let \( u = (u_1, \ldots, u_m) \in (C^{2,\alpha}(\Omega))^m \) be a positive solution of problem (5.1), (5.2). Then \( \mu_1 = \rho(A_m) \).

**Proof:** Indeed, using the above notations we still get \( (\mu_1 I - A_m) x = 0 \) and the result follows from theorem 4. \( \blacksquare \)

**Remark 4.** Assume that \( m = 2 \) and that problem (5.1), (5.2) has a positive solution \( u = (u_1, u_2) \in (C^{2,\alpha}(\Omega))^2 \). Then we have

\[ \mu_1 = a_{11} \quad (\text{resp.} \ a_1 = a_{22}) \quad \iff \quad a_{12} = 0 \quad (\text{resp.} \ a_{21} = 0), \tag{5.3} \]

\[ \mu_1 > a_{11} \quad (\text{resp.} \ a_1 > a_{22}) \quad \iff \quad a_{12} > 0 \quad (\text{resp.} \ a_{21} > 0). \tag{5.4} \]

Indeed, arguing as in example 1 we get

\[ (\mu_1 - a_{11}) \int_{\Omega} u_1 \varphi_1 \, dx = a_{12} \int_{\Omega} u_2 \varphi_1 \, dx \]

and
\[(\mu_1 - a_{22}) \int_{\Omega} u_2 \varphi_1 \, dx = a_{21} \int_{\Omega} u_1 \varphi_1 \, dx,\]

from which we deduce (5.3) and (5.4).

Now we give two examples.

**Example 2:** Assume \( m = 2 \) and \( \det A_2 \notin \{\mu_k; k \geq 2\} \). Then problem (5.1), (5.2) has a positive solution \( u = (u_1, u_2) \in (C^{2,\alpha}(\Omega))^2 \) if and only if

\[
\text{det}(\mu_1 I - A_2) = 0
\]

and one of the following conditions holds:

i) \( \mu_1 = a_{11}, a_{12} = 0 \) and \( (\mu_1 - a_{22}) a_{21} > 0 \).
Then the solution is given by \( u_1 = C \varphi_1 \) and \( u_2 = \frac{a_{22}}{\mu_1 - a_{22}} C \varphi_1 \) for some constant \( C > 0 \).

ii) \( \mu_1 = a_{22}, a_{21} = 0 \) and \( (\mu_1 - a_{11}) a_{12} > 0 \).
Then the solution is given by \( u_1 = C \varphi_1 \) and \( u_2 = \frac{a_{12}}{a_{11} - a_{22}} C \varphi_1 \) for some constant \( C > 0 \).

iii) \( \mu_1 = a_{11} = a_{22}, a_{12} = a_{21} = 0 \).
Then the solution is given by \( u_1 = C \varphi_1 \) and \( u_2 = C' \varphi_1 \) for some constants \( C, C' > 0 \).

iv) \( (\mu_1 - a_{11}) a_{12} > 0 \) and \( (\mu_1 - a_{22}) a_{21} > 0 \).
Then the solution is given by \( u_1 = C \varphi_1 \) and \( u_2 = \frac{a_{22}}{\mu_1 - a_{22}} C \varphi_1 = \frac{a_{11} - a_{12}}{a_{12}} C \varphi_1 \) for some constant \( C > 0 \).

**Proof:** Assume that problem (5.1), (5.2) has a positive solution \( u = (u_1, u_2) \in (C^{2,\alpha}(\Omega))^2 \). By lemma 2 condition (5.5) is satisfied.

Define \( D(\lambda) = \text{det}(\lambda I - A_2) \). \( D \) is a polynomial of degree 2. Since \( D(\mu_1) = 0 \), the roots of \( D \) are real. We denote by \( \mu \) the other root. Since \( \mu_1 = \text{det} A_2 \), our assumption implies \( \mu \neq \mu_k \) for all \( k \geq 2 \).

Now denote by \( \varphi_j \) the eigenfunction corresponding to \( \mu_j \) (with \( \varphi_1 > 0 \) in \( \Omega \)). These form a complete orthonormal set in \( W^{1,2}_0(\Omega) \), hence total in \( C^{2,\alpha} \). If \( u = (u_1, u_2) \in (C^{2,\alpha}(\Omega))^2 \) is a solution of problem (5.1), (5.2) the corresponding Fourier coefficients \( u_{1j} \) and \( u_{2j} \) satisfy the linear system

\[
\begin{align*}
(\mu_j - a_{11}) u_{1j} - a_{12} u_{2j} &= 0, \\
-a_{21} u_{1j} + (\mu_j - a_{22}) u_{2j} &= 0,
\end{align*}
\]

from which it immediately follows that \( u_{1j} = u_{2j} = 0 \) for \( j \geq 2 \). Using (5.3) and (5.4) of remark 4 we easily verify that one of the conditions i)–iv) holds. The relation between \( u_{1j} \) and \( u_{2j} \) is easily checked in each case.
The converse is obvious. ■

**Example 3:** Assume \( m = 2 \). If there exists \( k \geq 2 \) such that \( \det A_2 = \mu_1 \mu_k \), then problem (5.1), (5.2) has a positive solution \( u = (u_1, u_2) \in (C^{2,o}(\Omega))^2 \) if and only if (5.5) is satisfied and one of the following conditions holds:

i) \( \mu_1 = a_{11}, a_{12} = 0 \) and \( a_{21} < 0 \).
Then the solution is given by \( u_1 = C \varphi_1 \) and \( u_2 = \frac{a_{21}}{\mu_1 - \mu_k} C \varphi_1 + v \) where \( v \) is an eigenfunction corresponding to \( \mu_k \) and \( C > 0 \) is a constant such that \( u_2 > 0 \) in \( \Omega \).

ii) \( \mu_1 = a_{22}, a_{21} = 0 \) and \( a_{12} < 0 \).
Then the solution is given by \( u_2 = C \varphi_1 \) and \( u_1 = \frac{a_{12}}{\mu_1 - \mu_k} C \varphi_1 + v \) where \( v \) is an eigenfunction corresponding to \( \mu_k \) and \( C > 0 \) is a constant such that \( u_1 > 0 \) in \( \Omega \).

iii) \( (\mu_1 - a_{11}) a_{12} > 0 \) and \( (\mu_1 - a_{22}) a_{21} > 0 \).
Then the solution is given by \( u_1 = \frac{\mu_1 - a_{22}}{a_{21}} C \varphi_1 + \frac{\mu_k - a_{12}}{a_{21}} v \) and \( u_2 = C \varphi_1 + v \) where \( v \) is an eigenfunction corresponding to \( \mu_k \) and \( C > 0 \) is a constant such that \( u_1 > 0 \) and \( u_2 > 0 \) in \( \Omega \).

**Proof:** Assume that problem (5.1), (5.2) has a positive solution \( u = (u_1, u_2) \in (C^{2,o}(\Omega))^2 \). As in example 2 (5.5) is satisfied. We keep the notations of the proof of example 2. Our assumption implies that \( \mu = \mu_k \) for some \( k \geq 2 \). Using the same argument we obtain (5.6) from which it immediately follows that \( u_{1j} = u_{2j} = 0 \) except possibly for \( j = 1 \) and the indices such that \( \mu_j = \mu_k \).

Using (5.3) and (5.4) of remark 4 we easily show that one of the conditions i)--iii) holds. The relations between the coefficients of the expansions of \( u_1 \) and \( u_2 \) in the eigenfunctions are easily checked according to the various possibilities i)--iii).

The converse is obvious. ■

**Remark 5.** Assume \( a_{jk} \geq 0 \), \( j, k = 1, 2 \), and \( \mu_1 = \rho(A_2) \).
If problem (5.1), (5.2) has a positive solution \( u = (u_1, u_2) \in (C^{2,o}(\Omega))^2 \) then \( \det A_2 \leq \mu_1^2 \) since \( \det A_2 = \mu \mu_1 \).

If \( \det A_2 = \mu_1^2 \), let \( a_{jj} = \mu_1 \) for \( j = 1, 2 \) and \( a_{12} = a_{21} = 0 \). Then iii) of example 2 gives the existence of infinitely many positive solutions.

If \( \det A_2 < \mu_1^2 \), first let

\[
A_2 = \begin{pmatrix} \mu_1 & 0 \\ a_{21} & a_{22} \end{pmatrix} \quad \text{with} \quad \mu_1 > a_{22} \quad \text{and} \quad a_{21} > 0
\]

or

\[
A_2 = \begin{pmatrix} a_{11} & a_{12} \\ 0 & \mu_1 \end{pmatrix} \quad \text{with} \quad \mu_1 > a_{11} \quad \text{and} \quad a_{12} > 0.
\]
Then i) or ii) of example 2 gives the existence of infinitely many positive solutions. Now let
\[ A_2 = \begin{pmatrix} \mu_1 - \varepsilon_1 & a_{12} \\ a_{21} & \mu_1 - \varepsilon_2 \end{pmatrix} \]
with \( 0 < \varepsilon_j < \mu_1 \) for \( j = 1, 2, a_{12}, a_{21} > 0 \) and \( \varepsilon_1 \varepsilon_2 = a_{12} a_{21} \). Then iv) of example 2 gives the existence of infinitely many positive solutions.

**Remark 6.** If \( a_{jk} \geq 0, j, k = 1, 2, \) and \( \mu_1 > \rho(A_2) \) then the only solution of problem (5.1), (5.2) is the trivial solution (see remark 3). If \( \mu_1 = \rho(A_2) \), infinitely many positive solutions may exist by remark 5.

The next result was proved in [5].

**Theorem 8.** Assume that \( a_{jk} \) in (5.1) are such that
\[ a_{jj+1} = \lambda_{j+1} \] \[ a_{m1} = \lambda_1 \]
and
\[ a_{jk} = 0, \quad \text{otherwise}. \]

Then problem (5.1), (5.2) has a positive solution \( u = (u_1, ..., u_m) \in (C^{2,\alpha}(\Omega))^m \) if and only if
\[ \lambda_j > 0, \quad j = 1, ..., m \quad \text{and} \quad \lambda_1 \cdots \lambda_m = \mu_1^m. \]
The solution is given by \( u_j = c_j \varphi_1 \) where \( c_1 > 0 \) is an arbitrary constant and \( c_j = c_1 (\lambda_2 \cdots \lambda_j)^{-1} (\lambda_1 \cdots \lambda_m)^{(j-1)/m} \) for \( j = 2, ..., m \).

**Remark 7.** By lemma 1 condition (5.7) is equivalent to
\[ \lambda_j > 0, \quad j = 1, ..., m \quad \text{and} \quad \mu_1 = \rho(A_m). \]

Now with the notations of theorem 8, if \( \lambda_j \geq 0, j = 1, ..., m \) and \( \mu_1 > \rho(A_m) \), then the only solution of problem (5.1), (5.2) is the trivial solution (see remark 3). If \( \lambda_j > 0, j = 1, ..., m \) and \( \mu_1 = \rho(A_m) \), theorem 8 shows that there exist infinitely many positive solutions.

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