SUBDIRECT PRODUCTS OF A BAND AND A SEMIGROUP*

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Abstract: Subdirect products of a band and a semigroup have been studied in various special cases by a number of authors. In the present paper, using the constructions and the methods from our earlier papers, we give characterizations of all subdirect products of a band and a semigroup.

Introduction and preliminaries

Subdirect products of a band and a semigroup have been studied in various special cases by a number of authors. A characterization of all subdirect products of a rectangular band and a semigroup was given by J.L. Chrislock and T. Tamura [3]. Subdirect products connected with sturdy bands of semigroups were investigated by the authors in [4], and in the semilattice case by M. Petrich [9, 10]. Spined products of a band and a semigroup, predominantly with respect to the greatest semilattice homomorphic image of this band, were also considered many times. More information about these can be found in [6]. A characterization of all subdirect products of a band and a semilattice of semigroups contained in their spined product were given by the authors in [6]. A band composition used in this paper, which is an extension of Petrich’s construction from [9], has been also explored by the authors in [4–7].

In the present paper we consider such compositions in which all members of the related system of homomorphisms are one-to-one, and using this, by Theorem 1 we describe all subdirect products of a band and a semigroup. In Theorem 2 we give an alternative construction of such products, similar to the ones of J.L.

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Chrislock and T. Tamura [3] and H. Mitsch [8]. Theorem 3 shows that all subdirect products of a given semigroup and a band can be obtained from the subdirect product of this semigroup and of the greatest semilattice homomorphic image of this band, using spined products. In Theorem 4 we give a characterization of subdirect products of a band and a semilattice of semigroups. Finally, Section 3 is devoted to the study of subdirect products of a band and a group. The results obtained there are generalizations of some results of M. Petrich [9, 10], H. Mitsch [6] and of the authors [4].

Let \( B \) be a band. By \( \preceq \) we will denote the natural partial order on \( B \), i.e. a relation on \( B \) defined by: \( j \preceq i \iff ij = ji = j \ (i, j \in B) \), and \( \preceq \) will denote a quasi-order on \( B \) defined by: \( j \preceq i \iff jij (i, j \in B) \). Clearly, \( \preceq \) and \( \preceq \) coincide if and only if \( B \) is a semilattice. Further, for \( i \in B \), \([i] \) will denote the class of \( i \) with respect to the smallest semilattice congruence on \( B \). It is easy to verify that \( j \preceq i \iff [j] \preceq [i] \) for all \( i, j \in B \).

Let \( B \) be a band. To each \( i \in B \) we associate a semigroup \( S_i \) and an over-semigroup \( D_i \) of \( S_i \) such that \( D_i \cap D_j = \emptyset \), if \( i \neq j \). For \( i, j \in B \), \( i \geq j \), let \( \phi_{i,j} \) be a mapping of \( S_i \) into \( D_j \) and suppose that the family of \( \phi_{i,j} \) satisfies the following conditions:

1. \( \phi_{i,i} \) is the identity mapping on \( S_i \), for each \( i \in B \);
2. \((S_i, \phi_{i,j}) (S_j, \phi_{j,i}) \subseteq S_{ij} \), for all \( i, j \in B \);
3. \([(a \phi_{i,j}) (b \phi_{j,k})] \phi_{i,j,k} = (a \phi_{i,k}) (b \phi_{j,k}), \) for \( a \in S_i, \ b \in S_j, \ i \geq j, \ i, j, k \in B \).

Define a multiplication \( * \) on \( S = \bigcup_{i \in B} S_i \) by: \( a * b = (a \phi_{i,j}) (b \phi_{j,i}) \), for \( a \in S_i, \ b \in S_j \). Then \( S \) is a band \( B \) of semigroups \( S_i \), \( i \in B \), in notation \( S = (B; S_i, \phi_{i,j}, D_i) \) [6]. If we assume \( i = j \) in (3), then we obtain that \( \phi_{i,k} \) is a homomorphism, for all \( i, k \in B, \ i \geq k \). If all \( \phi_{i,j} \) are one-to-one, then we write \( S = (B; S_i, \phi_{i,j}, D_i) \).

Further, if \( D_i = S_i \), for each \( i \in B \), then we write \( S = (B; S_i, \phi_{i,j}) \). Here the condition (2) can be omitted. If \( S = (B; S_i, \phi_{i,j}) \) and if \( \{ \phi_{i,j} \mid i, j \in B, \ i \geq j \} \) is a transitive system of homomorphisms, i.e. if \( \phi_{i,j} \phi_{j,k} = \phi_{i,k}, \) for \( i \geq j \geq k \), then we will write \( S = [B; S_i, \phi_{i,j}] \), and we will say that \( S \) is a strong band \( B \) of semigroups \( S_i \). If \( S = [B; S_i, \phi_{i,j}] \) and all \( \phi_{i,j} \) are one-to-one, then we will write \( S = (B; S_i, \phi_{i,j}) \) and we will say that \( S \) is a sturdy band \( B \) of semigroups \( S_i \).

In the case when \( B \) is a semilattice, we obtain a strong (sturdy) semilattice of semigroups.

For undefined notions and notations we refer to [9] and [10].

It is easy to prove the following
Lemma 1. Let $S = (B; S_i, \phi_{i,j}, D_i)$ and let $T$ be a subsemigroup of $S$. Then $B' = \{i \in B \mid S_i \cap T \neq \emptyset\}$ is a subsemigroup of $B$ and if $T_i = T \cap S_i$, $i \in B'$, and for $i, j \in B'$, $i \geq j$, $\psi_{i,j}$ is the restriction of $\phi_{i,j}$ onto $T_i$, then $T = (B'; T_i, \psi_{i,j}, D_i)$.

2 – The main results

In this section we will give various characterizations of subdirect products of a band and a semigroup, in the general case. The following is the main theorem of this paper:

Theorem 1. Let $S = (B; S_i, \phi_{i,j}, D_i)$ and let $\xi$ be a relation on $S$ defined by:

(4) $a \xi b$ if and only if $a \in S_i$, $b \in S_j$, $i, j \in B$, and there exists $k \in B$ such that $k \leq i, j$, and $a \phi_{i,k} = b \phi_{j,k}$, for every $l \in B$, $l \leq k$.

Then $\xi$ is a congruence on $S$. Furthermore, if $S = (B; S_i, \phi_{i,j}, D_i)$, then $S$ is a subdirect product of a band and a semigroup $T$, then $S = (B; S_i, \phi_{i,j}, D_i)$, where for each $i \in B$, $S_i$ is isomorphic to some subsemigroup of $T$.

Proof: Clearly, $\xi$ is reflexive and symmetric. Assume $a, b, c \in S$ such that $a \xi b$ and $b \xi c$. Let $a \in S_i$, $b \in S_j$, $c \in S_k$, $i, j, k \in B$. Then there exists $m_1, m_2 \in B$ such that $m_1 \leq i, j$ and $m_2 \leq j, k$, and $a \phi_{i,l} = b \phi_{j,l}$, $b \phi_{j,l} = c \phi_{k,l}$, for all $l_1, l_2 \in B$, $l_1 \leq m_1$ and $l_2 \leq m_2$. Clearly, there exists $m \in B$ such that $m \leq m_1, m_2$, and for every $l \in B$, $l \leq m$, we obtain that $l \leq m_1, m_2$, whence $a \phi_{i,l} = b \phi_{j,l} = c \phi_{k,l}$. Therefore, $a \xi c$, so $\xi$ is transitive.

Let $a, b, c \in S$, $a \xi b$. Assume that $a \in S_i$, $b \in S_j$, $c \in S_k$, $i, j, k \in B$. Then there exists $m_0 \in B$, $m_0 \leq i, j$, such that $a \phi_{i,l} = b \phi_{j,l}$, for every $l \in B$, $l \leq m_0$. Assume that $m \in B$ is such that $m \leq m_0, ik, jk$, and that $l \in B$, $l \leq m$. Then $l \leq m_0$, whence

$$(a * c) \phi_{i,l} = (a \phi_{i,l})(c \phi_{k,l}) = (b \phi_{j,l})(c \phi_{k,l}) = (b * c) \phi_{j,k,l}.$$  

Thus, $a * c \xi b * c$. Similarly we prove that $c * a \xi c * b$. Hence, $\xi$ is a congruence on $S$.

Let $S = (B; S_i, \phi_{i,j}, D_i)$. Assume that $(a, b) \in \xi \cap \eta$, where $\eta$ is a band congruence on $S$ such that $S/\eta \cong B$. Then $a, b \in S_i$, for some $i \in B$, and there exists $k \in B$, $k \leq i$, such that $a \phi_{i,k} = b \phi_{i,k}$, whence $a = b$, since $\phi_{i,k}$ is one-to-
one. Therefore, \( \xi \cap \eta = \varepsilon \), where \( \varepsilon \) is the equality relation. Thus, \( S \) is a subdirect product of \( B \) and \( S/\xi \).

Conversely, let \( S \subseteq T \times B \) be a subdirect product of a semigroup \( T \) and a band \( B \). For \( i \in B \), let \( S_i = (T \times \{i\}) \cap S \). Clearly, \( S_i \neq \emptyset \) and it is isomorphic to a subsemigroup of \( T \), for each \( i \in B \), and \( S \) is a band \( B \) of semigroups \( S_i, i \in B \).

Let \( D_i = T \times \{i\}, i \in B \), and for \( i, j \in B, i \geq j \), let \( \phi_{i,j} : S_i \to D_j \) be a mapping defined by:

\[
(a, i) \phi_{i,j} = (a, j) \quad ((a, i) \in S_i).
\]

Now it is easy to verify that \( S = (B; S_i, \phi_{i,j}, D_i) \).

**Remark.** Note that if \( S = [B; S_i, \phi_{i,j}] \) and \( \xi \) is a congruence on \( S \) defined as in (4), then \( S/\xi \) is the well-known direct limit of the family \( S_i, i \in B \), carried by \( B \).

Considering the mappings of a band \( B \) into the set \( \mathcal{G}(T) \) of all subsemigroups of a semigroup \( T \), satisfying some suitable conditions, we give another characterization of subdirect products of \( B \) and \( T \), similar to the ones of J.L. Chrislock and T. Tamura [3] and H. Mitsch [8].

**Theorem 2.** Let \( B \) be a band, let \( T \) be a semigroup and let \( \mu : B \to \mathcal{G}(T) \) be a mapping satisfying the following conditions:

i) \( \bigcup_{i \in B} i \mu = T \);

ii) \( (i \mu) \cdot (j \mu) \subseteq (ij) \mu \), for all \( i, j \in B \).

Then \( S = \{(i, a) \in B \times T \mid a \in i \mu \} \) is a subdirect product of \( B \) and \( T \), in notation \( S = (B; \mu, T) \).

Conversely, any subdirect product of \( B \) and \( T \) can be obtained in this way.

**Proof:** The proof is similar to the proofs of Theorem 1 [3] and Theorem 7 [8].

Let \( B \) be a band, let \( T \) be a semigroup, let \( \mu : B \to \mathcal{G}(T) \) be a mapping satisfying i) of the previous theorem and let \( \mu \) be antitone, i.e. let for all \( i, j \in B, i \geq j \) implies \( i \mu \subseteq j \mu \). Then clearly \( \mu \) satisfies ii). A semigroup \( S \) constructed by such a mapping as in the previous theorem will be denoted by \( S = [B; \mu; T] \).

By Theorem 2 we obtain the following two corollaries. The first of them is in fact Proposition 1 [4], and the first part of the second corollary is the result of M. Petrich [10, p. 87–88], [9, p. 98].
Corollary 1. If $S$ is a sturdy band $B$ of semigroups, then $S = [B; \mu; S/\xi]$, where $\xi$ is a relation defined as in (4).

Conversely, if $S = [B; \mu; T]$, then $S$ is a sturdy band $B$ of semigroups $S_i = i\mu$, $i \in B$.

Corollary 2. If $S$ is a sturdy semilattice $Y$ of semigroups, then $S = [Y; \mu; S/\xi]$, where $\xi$ is a relation defined as in (4).

Conversely, if $S = [Y; \mu; T]$, where $Y$ is a semilattice, then $S$ is a sturdy band $B$ of semigroups $S_i = i\mu$, $i \in B$.

If $P$ and $Q$ are two semigroups with a common homomorphic image $Y$, then the spined product of $P$ and $Q$ with respect to $Y$ is $S = \{(a,b) \in P \times Q \mid a \varphi = b \psi\}$, where $\varphi : P \to Y$ and $\psi : Q \to Y$ are homomorphisms onto $Y$. If $P_\alpha = \alpha \varphi^{-1}$, $Q_\alpha = \alpha \psi^{-1}$, $\alpha \in Y$, then $S = \bigcup_{\alpha \in Y} (P_\alpha \times Q_\alpha)$. Clearly, spined products are easier for construction than other subdirect products, so it is of interest the following result that reduces the problem of construction of subdirect products of a given semigroup and a band to the problem of construction of subdirect products of this semigroup and of the greatest semilattice homomorphic image of this band.

Theorem 3. Let $B$ be a band, let $Y$ be its greatest semilattice homomorphic image and let $T$ be a semigroup. Then a semigroup $S$ is a subdirect product of $B$ and $T$ if and only if it is a spined product, with respect to $Y$, of $B$ and of a subdirect product of $Y$ and $T$.

Proof: Let $B$ be a semilattice $Y$ of rectangular bands $B_\alpha$, $\alpha \in Y$.

Let $S \subseteq B \times T$ be a subdirect product of $B$ and $T$. Define a mapping $\varphi$ of $S$ into $Y \times T$ by:

$$(i, a) \varphi = ([i], a) \quad ((i, a) \in S).$$

By a routine verification we obtain that $\varphi$ is a homomorphism. Let us prove that $P = S \varphi$ is a subdirect product of $Y$ and $T$. Indeed, for $\alpha \in Y$, $\alpha = [i]$ for some $i \in B$, and $(i, a) \in S$ for some $a \in T$; hence $(\alpha, a) = ([i], a) = (i, a) \varphi \in P$. Similarly we prove that for $a \in T$ there exists $\alpha \in Y$ such that $(\alpha, a) \in P$. Therefore, $P$ is a subdirect product of $Y$ and $T$.

For $\alpha \in Y$, let $P_\alpha = (\{\alpha\} \times T) \cap P$. Clearly, $P$ is a semilattice $Y$ of semigroups $P_\alpha$, $\alpha \in Y$. Define a mapping $\psi$ of $S$ into $B \times P$ by:

$$(i, a) \psi = (i, ([i], a)) \quad ((i, a) \in S).$$
It is not hard to verify that $\psi$ is an embedding of $S$ into $B \times T$. Assume $(i, a) \in S$. Then $i \in B_\alpha$, for some $\alpha \in Y$, whence

$$(i, a) \psi = (i, ([i], a)) = (i, (\alpha, a)) \in B_\alpha \times P_\alpha.$$ 

Thus, $S \psi \subseteq \bigcup_{\alpha \in Y} (B_\alpha \times P_\alpha)$. On the other hand, if $\alpha \in Y$ and $(i, (\alpha, a)) \in B_\alpha \times P_\alpha$, then $i \in B_\alpha$, so

$$(i, (\alpha, a)) = (i, a) \psi \in S \psi.$$ 

Therefore, $S \psi = \bigcup_{\alpha \in Y} (B_\alpha \times P_\alpha)$, so $S$ is a spined product of $B$ and $P$ with respect to $Y$.

Conversely, let $S \subseteq B \times P$ be a spined product of $B$ and $P$, with respect to $Y$, where $P$ is a subdirect product of $Y$ and $T$, i.e. let $S = \bigcup_{\alpha \in Y} (B_\alpha \times P_\alpha)$, where $P_\alpha = (\{\alpha\} \times T) \cap P$, $\alpha \in Y$. Define a mapping $\phi$ of $S$ into $B \times T$ by:

$$(i, (\alpha, a)) \phi = (i, a) \quad ((i, (\alpha, a)) \in S).$$

Then $\phi$ is an embedding of $S$ into $B \times T$. It remains to prove that $Q = S \phi$ is a subdirect product of $B$ and $T$. Indeed, for $i \in B$, $i \in B_\alpha$, for some $\alpha \in Y$, and there exists $a \in T$ such that $(\alpha, a) \in P$, since $P$ is a subdirect product of $Y$ and $T$, whence $(i, (\alpha, a)) \in S$ and $(i, a) = (i, (\alpha, a)) \phi \in Q$. Similarly we prove that for any $a \in T$ there exists $i \in B$ such that $(i, a) \in Q$. Therefore, $Q$ is a subdirect product of $B$ and $T$.

An element of a semigroup is $\pi$-regular if some of its power is regular, and a semigroup is $\pi$-regular if each of its element is $\pi$-regular.

**Corollary 3.** The following conditions on a semigroup $S$ are equivalent:

i) $S$ is $\pi$-regular and a subdirect product of a band and a semilattice of groups;

ii) $S$ is regular and a subdirect product of a band and a semilattice of groups;

iii) $S$ is a spined product of a band and a semilattice of groups.

**Proof:** The authors in [1] proved that if a semigroup is a subdirect product of semilattices of groups, then it is a semilattice of groups if and only if it is $\pi$-regular. By this and by Theorem 3 we obtain i)$\iff$iii). The equivalence ii)$\iff$iii) was proved by M. Petrich [11].

By the well-known Tamura’s result [12], any semigroup can be represented as a semilattice of semilattice indecomposable semigroups. Also, M. Petrich in
Theorem III 7.2 [9] proved that every semilattice of semigroups can be composed as \((Y; S_\alpha, \phi_{\alpha, \beta}, D_\alpha)\). Therefore, every semigroup \(S\) can be represented as \(S = (Y; S_\alpha, \phi_{\alpha, \beta}, D_\alpha)\), where \(Y\) is a semilattice, so it is of interest to consider subdirect products of a band and a semilattice of semigroups. This we will do in the next theorem.

Let \(B\) be a band and let \(Y\) be a semilattice. Assume that \(P\) is a subdirect product of \(B\) and \(Y\) an let \(\pi\) and \(\varpi\) be projection homomorphisms of \(P\) onto \(B\) and \(Y\), respectively. It is easy to verify that for \(i, j \in P\), \(i \leq j\) in \(P\) if and only if \(i\pi \leq j\pi\) in \(B\) and \(i\varpi \leq j\varpi\) in \(Y\). Define a quasi-order \(\preceq\) on \(P\) by:

\[
i \preceq j \iff i\pi \leq j\pi \quad \text{and} \quad i\varpi = j\varpi \quad (i, j \in P).
\]

If \(S = (P; S_i, \phi_{i,j}, D_i)\) and if \(\phi_{i,j}\) is one-to-one for all \(i, j \in P\) such that \(i \succ j\), then we will write \(S = (B, Y, P; S_i, \phi_{i,j}, D_i)\).

**Theorem 4.** Let \(B\) be a band and let \(Y\) be a semilattice.

Let \(P\) be a subdirect product of \(B\) and \(Y\), let \(S = (B, Y, P; S_i, \phi_{i,j}, D_i)\) and define relations \(\eta\) and \(\xi\) on \(S\) by:

1. \((a \eta b)\) if and only if \(a \in S_i\), \(b \in S_j\), \(i, j \in P\), and \(i\pi = j\pi\);
2. \((a \xi b)\) if and only if \(a \in S_i\), \(b \in S_j\), \(i, j \in P\), \(i\varpi = j\varpi\), and there exists \(k \in P\), \(k \preceq i, j\), such that \(a\phi_{i,l} = b\phi_{j,l}\), for each \(l \in P\), \(l \leq k\).

Then \(\eta\) and \(\xi\) are congruences on \(S\), \(S/\eta\) is isomorphic to \(B\), \(S/\xi\) is a semilattice \(Y\) of semigroups, and \(S\) is a subdirect product of \(S/\eta\) and \(S/\xi\).

Conversely, every subdirect product of \(B\) and a semigroup that is a semilattice \(Y\) of semigroups can be obtained in this way.

**Proof:** Clearly, \(\eta\) is a congruence on \(S\), \(S/\eta\) is isomorphic to \(B\) and \(\xi\) is reflexive and symmetric.

Assume that \(a, b, c \in S\) are such that \(a \xi b\) and \(b \xi c\). Let \(a \in S_i\), \(b \in S_j\), \(c \in S_k\), \(i, j, k \in P\), \(i\varpi = j\varpi = k\varpi\). By the hypothesis, there exist \(m_1, m_2 \in P\) such that \(m_1 \preceq i, j\) and \(m_2 \preceq j, k\), and \(a\phi_{i,l} = b\phi_{j,l}, b\phi_{j,l} = c\phi_{k,l}, \) for all \(l_1, l_2 \in P\) such that \(l_1 \leq m_1, l_2 \leq m_2\). Now for \(m = m_1 m_2\), \(m \preceq m_1, m_2\), so for any \(l \in P\), \(l \leq m\), we obtain that \(a\phi_{i,l} = c\phi_{k,l}\). Therefore, \(a\xi c\), so \(\xi\) is transitive.

Assume that \(a, b, c \in S\) are such that \(a \xi b\). Let \(a \in S_i\), \(b \in S_j\), \(c \in S_k\), \(i, j, k \in P\). By the hypothesis, \(i\varpi = j\varpi\), whence \((ik)\varpi = (jk)\varpi\), since \(\varpi\) is a homomorphism. Also, there exists \(m_0 \in P\) such that \(m_0 \preceq i, j\) and \(a\phi_{i,l} = b\phi_{j,l}\), for each \(l \in P\), \(l \leq m_0\). Let \(m = m_0 k\). Then \(m \preceq ik, jk\) and for any \(l \in P\), \(l \leq m\) we have

\[(a \ast c)\phi_{ik,l} = (a\phi_{i,l})(c\phi_{k,l}) = (b\phi_{j,l})(c\phi_{k,l}) = (b \ast c)\phi_{jk,l},\]
since \( l \leq m_0 \). Therefore, \( a \ast c \xi b \ast c \), and similarly \( c \ast a \xi c \ast b \), so \( \xi \) is a congruence on \( S \).

Assume that \((a, b) \in \eta \cap \xi \). Then \( a \in S_i, b \in S_j, i, j \in P \), and \( i \varpi = j \varpi \), whence \( i = j \). Also, there exists \( k \in P, k \ll i \), such that \( a \phi_{i,k} = b \phi_{i,k} \), whence \( a = b \), since \( \phi_{i,k} \) is one-to-one. Therefore, \( \eta \cap \xi = \varepsilon \), so \( S \) is a subdirect product of \( S/\eta \) and \( S/\xi \). Clearly, \( S/\xi \) is a semilattice \( Y \) of semigroups \( T_\alpha = S_\alpha \xi^\delta, \alpha \in Y \), where \( S_\alpha = \bigcup_{i \in P_\alpha} S_i \) and \( P_\alpha = \{ i \in P \mid i \pi = \alpha \}, \alpha \in Y \).

Conversely, let \( S \subseteq B \times T \) be a subdirect product of \( B \) and a semigroup \( T \) that is a semilattice \( Y \) of semigroups \( T_\alpha, \alpha \in Y \). Let \( P = \{ (i, \alpha) \in B \times Y \mid ((i) \times T_\alpha) \cap S \neq \emptyset \} \). It is easy to check that \( P \) is a subdirect product of \( B \) and \( Y \). Let \( \pi \) and \( \varpi \) denote the projection homomorphisms of \( P \) onto \( B \) and \( Y \), respectively, and for \( i \in P \), let \( S_i = (\{ i \pi \} \times T_{i \varpi}) \cap S \). Clearly, \( S \) is a band \( P \) of semigroups \( S_i, i \in P \). By Theorem III 7.2 [9], \( T = (Y; T_\alpha, \phi_{\alpha, \beta}, D_\alpha) \). Now, for \( i \in P \), let \( D_i = \{ i \pi \} \times D_{i \varpi} \) and for \( i, j \in P, i \succeq j \), define a mapping \( \phi_{i,j} \) of \( S_i \) into \( S_j \) by:

\[
(i \pi, a) \phi_{i,j} = (j \pi, a \phi_{i \varpi, j \varpi}) \quad (a \in T_{i \varpi}).
\]

Now it is easy to show that \( S = (B, Y, P; S_i, \phi_{i,j}, D_i) \).

3 – Subdirect products of a band and a group

Subdirect products of a band and a group were considered in various special cases by M. Petrich [9-11], H. Mitsch [8] and the authors [4]. In this section we will characterize such products in the general case.

Let \( E(S) \) denote the set of all idempotents of a semigroup \( S \). An element \( a \) of a semigroup \( S \) is \( E \)-inversive if there exists \( x \in S \) such that \( ax \in E(S) \), or equivalently, if there exists \( x \in S \) such that \( x = ax \) [2]. A semigroup \( S \) is \( E \)-inversive if each of its elements is \( E \)-inversive. For more informations about such semigroups we refer to [2] and [8].

**Lemma 2.** Let \( S \) be a subdirect product of a band \( B \) and an \( E \)-inversive semigroup \( T \). Then \( S \) is also \( E \)-inversive.

**Proof:** Let \( S \subseteq B \times T \), \((i, a) \in S \). For \( a \in T \) there exists \( x \in T \) such that \( ax \in E(T) \) and there exists \( j \in B \) such that \((j, x) \in S \). Therefore, \((i, a)(j, x) = (ij, ax) \in E(S) \), so \( S \) is \( E \)-inversive.

Note that if \( S = (B; S_i, \phi_{i,j}, D_i) \), then \( D = \bigcup_{i \in B} D_i \) need not be a semigroup. One very interesting case when the multiplication on \( S \) can be extended to a
multiplication on \( D \) will be considered in the following

**Theorem 5.** Let \( S = (B; S_i, \phi_{i,j}, D_i) \), where \( D_i, i \in B \), are cancellative semigroups and \( D_k = \{ a \phi_{i,k} \mid a \in S_i, i \geq k \} \), for each \( k \in B \). Then

i) For all \( i, j \in B, i \geq j, \phi_{i,j} \) can be extended up to a homomorphism \( \varphi_{i,j} \) of \( D_i \) into \( D_j \) such that there exists a composition \( D = \{ B; D_i, \varphi_{i,j} \} \);

ii) If \( S = (B; S_i, \phi_{i,j}, D_i) \), then \( D = (B; D_i, \varphi_{i,j}) \);

iii) If \( S \) is \( E \)-inversive, then \( D \) is also \( E \)-inversive.

**Proof:**
i) Assume that \( k, l \in B \) are such that \( k \geq l \). For \( a \in D_k \), by the hypothesis, \( a = x \phi_{i,k} \), for \( x \in S_i, i \in B, i \geq k \), and we define a mapping \( \varphi_{i,j} \) of \( D_k \) into \( D_l \) by

\[
    a \varphi_{k,l} = x \phi_{i,l} .
\]

To prove that \( \varphi \) is well-defined, it is necessary and sufficient to prove that for \( x \in S_i, y \in S_j, i, j \geq k \geq l \), \( x \phi_{i,k} = y \phi_{j,k} \) implies \( x \phi_{i,l} = y \phi_{j,l} \). Indeed, by \( x \phi_{i,k} = y \phi_{j,k} \), for arbitrary \( u, v \in S_k \),

\[
    (u \phi_{k,l})(x \phi_{i,l})(v \phi_{k,l}) = (u \times u \times v)(\phi_{k,l}) = u(x \phi_{i,k})v \phi_{k,l} = u(y \phi_{j,k})v \phi_{k,l}
\]

\[
    = (u \times y \times v)(\phi_{k,l}) = (u \phi_{k,l})(y \phi_{j,l})(v \phi_{k,l}) ,
\]

so by the cancellativity in \( D_l, x \phi_{i,l} = y \phi_{k,l} \). Hence, \( \varphi_{k,l} \) is well-defined and clearly, it is an extension of \( \phi_{k,l} \).

Assume that \( a \in D_k, b \in D_l \), \( a = x \phi_{i,k} \), \( b = y \phi_{j,l} \), \( x \in S_i, y \in S_j, i, j, k, l \in B \), \( i \geq k \), \( j \geq l \), and assume that \( m \in B, m \leq k, l \). Then by (3) and by the definition of mappings \( \varphi_{i,j} \) we obtain

\[
    \varphi_{k,m} = \left[ (a \varphi_{k,l})(b \varphi_{l,m}) \right] = \left[ (x \phi_{i,k})(y \phi_{j,l}) \right] = \left[ (x \phi_{i,k})(y \phi_{j,l}) \right] = \left[ (x \times y \phi_{i,j}) \right] \varphi_{l,m} = \left[ (x \times y \phi_{i,j}) \right] \varphi_{l,m} = (x \times y \phi_{i,j,m}) = (a \varphi_{k,m})(b \varphi_{l,m}) .
\]

Therefore, there exists a composition \( D = (B; D_i, \varphi_{i,j}) \). Since \( D_i, i \in B \), are cancellative, then \( D = [B; D_i, \varphi_{i,j}] \).

ii) Let all \( \phi_{i,j} \) be one-to-one. Assume that \( a \varphi_{k,l} = b \varphi_{k,l} \), for \( a, b \in D_k \), \( k, l \in B \), \( k \geq l \). Then \( a = x \phi_{i,k}, b = y \phi_{j,l}, x \in S_i, y \in S_j, i, j \in B, i, j \geq k \). Let \( u, v \in S_k \) be arbitrary. By \( a \varphi_{k,l} = b \varphi_{k,l} \), it follows that \( x \phi_{i,l} = y \phi_{j,l} \), whence

\[
    (u \times u \times v)(\phi_{k,l}) = (u \phi_{k,l})(x \phi_{i,l})(v \phi_{k,l}) = (u \phi_{k,l})(y \phi_{j,l})(v \phi_{k,l}) = (u \times y \times v)(\phi_{k,l}) .
\]
Since $\phi_{k,l}$ is one-to-one, then $u * x * v = u * y * v$, whence
\[ u(x \phi_{i,k}) v = u * x * v = u * y * v = u(y \phi_{j,k}) v . \]
Now, by the cancellativity in $D_k$, $x \phi_{i,k} = y \phi_{j,k}$, i.e. $a = b$. Therefore, $\varphi_{k,l}$ is one-to-one.

iii) Assume that $a \in D$. Then $a \in D_k$, $k \in B$, and $a = x \phi_{i,k}$, $x \in S_i$, $i \in B$, $i \geq k$. Now, $x \ast y \in E(S)$, for some $y \in S_j$, $j \in B$, so
\[ a \ast y = (a \varphi_{k,kj})(u \varphi_{j,kj}) = (x \phi_{i,kj})(y \phi_{j,kj}) \]
\[ = [(x \phi_{i,ij})(y \phi_{j,ij})] \phi_{ij,kj} = (x \ast y) \phi_{ij,kj} \in E(D) . \]
Thus, $D$ is also $E$-inversive.

A semigroup containing exactly one idempotent will be called a \textit{unipotent semigroup}, and a semigroup without idempotents will be called an \textit{idempotent-free semigroup}. Now we go to the main theorem of this section.

\textbf{Theorem 6.} The following conditions on a semigroup $S$ are equivalent:

i) $S$ is a subdirect product of a band and a group;

ii) $S$ is $E$-inversive, $S = \langle B; S_i, \phi_{i,j}, D_i \rangle$, and for every $i \in B$, $D_i$ is cancellative;

iii) $S$ is $E$-inversive, $S = \langle B; S_i, \phi_{i,j}, D_i \rangle$, and for every $i \in B$, $D_i$ is either a unipotent monoid or an idempotent-free semigroup;

iv) $S$ is $E$-inversive and it can be embedded into a sturdy band of cancellative semigroups;

v) $S$ is $E$-inversive and it can be embedded into a sturdy band of unipotent monoids and idempotent-free semigroups;

vi) $S$ is $E$-inversive and it can be embedded into a spined product of a band and a sturdy semilattice of cancellative semigroups;

vii) $S$ is $E$-inversive and it can be embedded into a spined product of a band and a sturdy semilattice of unipotent monoids and idempotent-free semigroups.

\textbf{Proof:} i)$\Rightarrow$ii) Let $S \subseteq B \times G$ be a subdirect product of a band $B$ and a group $G$. For $i \in B$, let $D_i = \{i\} \times G$, $S_i = S \cap D_i$. Clearly, $S_i \neq \emptyset$ and $D_i$ is a cancellative semigroup, for each $i \in B$. If for $i,j \in B$, $i \geq j$, we define
a mapping $\phi_{ij} : S_i \rightarrow D_j$ by $(i, a) \phi_{ij} = (j, a)$, then it is easy to verify that $S = (B; S_i, \phi_{ij}, D_i)$ and by Lemma 2, $S$ is $E$-inversive.

ii) $\Rightarrow$ v) Let ii) hold. Without loss of generality we can assume that $D_k = \{a \phi_{i,k} \mid i \in B, i \geq k, a \in S_i\}$, for each $k \in B$. By Theorem 5, $S$ can be embedded into $D = (B; D_i, \varphi_{ij})$ and $D$ is $E$-inversive.

Let $i \in B$ be such that $E(D_i) \neq \emptyset$. Assume that $a \in D_i, e \in E(D_i)$. Since $D$ is $E$-inversive, then $x = x * e * a * x$, for some $x \in D$. If $x \in D_j, j \in B$, then clearly $i \geq j$ and $(e * a * x) \varphi_{ij,j}, e \varphi_{ij,j} \in E(D_j)$, since $e * a * x \in E(D_{ij})$, $e \in E(D_i)$. By the cancellativity in $D_j$, $|E(D_j)| = 1$, whence $e \varphi_{ij,j} = (e * a * x) \varphi_{ij,j} = (e \varphi_{ij,j}) (a \varphi_{ij,j}) x$. Now, by the cancellativity in $D_j$, $e \varphi_{ij,j} = (a \varphi_{ij,j}) x$, whence

$$[(e * a) \varphi_{ij,j}] x = (e * a * x) \varphi_{ij,j} = e \varphi_{ij,j} = (a \varphi_{ij,j}) x,$$

and again by the cancellativity in $D_j$, $(e * a) \varphi_{ij,j} = a \varphi_{ij,j}$. Therefore, $e * a = a$, since $\varphi_{ij,j}$ is one-to-one. Similarly we prove that $a * e = a$. Hence, $D_j$ is a monoid. Since $D_j$ is cancellative, then it is unipotent.

v) $\Rightarrow$ iii) This follows immediately.

iii) $\Rightarrow$ i) Let iii) hold. By Theorem 1, $S$ is a subdirect product of $B$ and a semigroup $S/\xi$, where $\xi$ is a congruence defined as in (4). Clearly, $e \xi f$, for all $e, f \in E(S)$. Let $u = e \xi^2, e \in E(S)$. Assume $v \in S/\xi$. Then $v = a \xi^2$, for some $a \in S$. Since $S$ is $E$-inversive, then $x = x * a * x$, for some $x \in S$. If $a \in S_i, x \in S_j, i, j \in B$, then $i \geq j$, $x * a = e \in E(S_{ji})$ and $a * e \in S_{ij}$. Assume $k \in B, k \geq i, iji$. Then

$$(a * e) \phi_{iji,k} = (a \phi_{i,k}) (e \phi_{ji,k}) = (a \phi_{i,k}),$$

since $e \phi_{ji,k}$ is the identity of $D_k$. Thus, $a * e \xi a$, whence $v = a \xi^2 = (a * e) \xi^2 = (a \xi^2) (e \xi^2) = v u$, and similarly $v = u v$. On the other hand, $u = e \xi^2 = (x * a) \xi^2 = (x \xi^2) (a \xi^2) = (x \xi^2) v$, and similarly $u = v(x \xi^2)$. Hence, $S/\xi$ is a group.

ii) $\iff$ iv) This follows by Theorem 5 and Lemma 1.

iv) $\iff$ v) and v) $\iff$ vii) This follows by Theorem 3 [6].

Similarly we can prove the following

**Corollary 4.** The following conditions on a semigroup $S$ are equivalent:

i) $S = [B, \mu, G]$, where $B$ is a band and $G$ is a group;

ii) $S$ is $E$-inversive and a sturdy band of cancellative semigroups;

iii) $S$ is $E$-inversive and a sturdy band of unipotent monoids and idempotent-free semigroups;
iv) $S$ is $E$-inversive and a spined product of a band and a sturdy semilattice of cancellative semigroups;

v) $S$ is $E$-inversive and a spined product of a band and a sturdy semilattice of unipotent monoids and idempotent-free semigroups.

**Corollary 5.** [4] A semigroup $S$ is a sturdy band of groups if and only if it is regular and a subdirect product of a band and a group.

**Corollary 5.** [9, 10] A semigroup $S$ is a sturdy semilattice of groups if and only if it is regular and a subdirect product of a semilattice and a group.

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