INTEGRAL REPRESENTATIONS OF GRAPHS *

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Abstract: Following the definition of graph representation modulo an integer given by Erdős and Evans in [1], we call degree of a representation to the number of prime factors in the prime factorization of its modulo. Here we study the smallest possible degree for a representation of a graph.

The starting point for this research is the concept of representation introduced in [1], and the proposed study of relations between properties of graphs and properties of their representations.

Let \( G = (V, E) \) be a graph with \( n \) vertices \( v_1, ..., v_n \). The graph \( G \) is said to be representable modulo a positive integer \( b \) if there exist distinct integers \( a_1, ..., a_n \) such that \( 0 \leq a_i < b \), and \( \text{g.c.d.} \{a_i - a_j, b\} = 1 \) if and only if \( v_i \) and \( v_j \) are adjacent. We say that \( \{a_1, ..., a_n\} \) is a representation of \( G \) modulo \( b \). We call degree of the representation to the number of prime factors, counting multiplicities, in the prime factorization of \( b \). The concept of degree was not mentioned in [1] explicitly. However we can see in the proof of the theorem of [1] that there always exists a representation of degree equal to the number of edges of the complement of a graph that results from \( G \) by adjoining an isolated vertex. We shall see that there exist representations of smaller degree. We call representation degree of \( G \), \( d_r(G) \), to the smallest possible degree for a representation of \( G \).

We say that a function \( \phi: E \to X \) is transitive if, for every \( (v_i, v_j), (v_j, v_k) \in E \) such that \( \phi(v_i, v_j) = \phi(v_j, v_k) = x \), we have \( (v_i, v_k) \in E \) and \( \phi(v_i, v_k) = x \). For example, if \( \phi: E \to X \) is one-to-one, then \( \phi \) is transitive. Given a set \( Y \), \#Y

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denotes its cardinal number. We call degree of a transitive function \( \phi: E \rightarrow X \) to \( \#\phi(E) \) and we call transitive degree of \( G \), \( d_t(G) \), to the smallest \( \#\phi(E) \), when \( \phi \) runs over the transitive functions defined in \( E \). It is not difficult to prove some properties of \( d_t(G) \). For example:

**Proposition 1.** \( d_t(G) \leq \#E \).

**Proposition 2.** \( d_t(G) = \max_H d_t(H) \), where \( H \) runs over the maximal connected subgraphs of \( G \).

**Proposition 3.** Suppose that \( G \) is connected. Then

a) \( d_t(G) = 0 \) if and only if \( \#V = 1 \).

b) \( d_t(G) = 1 \) if and only if \( \#V \geq 2 \) and \( G \) is complete.

c) \( d_t(G) = \#E \) if and only if there exists a vertex incident with all the edges of \( G \).

Let \( G' = (V', E') \) be the complement of \( G \). The following theorems are our main results. We shall prove them later.

**Theorem 4.** Let \( \phi \) be a transitive function defined in \( E' \) of degree \( d \geq 2 \). Then there exists a representation of \( G \) of degree \( d \).

**Corollary 5.** If \( d_t(G') \geq 2 \), then \( d_r(G) \leq d_t(G') \leq \#E' \).

Corollary 5 is not always true when \( d_t(G') \leq 1 \). The following proposition shows this and is easy to prove.

**Proposition 6.**

a) \( d_r(G) = 0 \) if and only if \( \#V = 1 \).

b) \( d_r(G) = 1 \) if and only if \( \#V \geq 2 \) and \( G \) is complete.

c) \( d_r(G) \leq 1 \) if and only if \( d_t(G') = 0 \).

d) If \( d_t(G') = 1 \), then \( d_r(G) = 2 \).

**Theorem 7.** Suppose that \( G' \) does not have any subgraph isomorphic to \( K_3 \). If \( G \) has a representation of degree \( d \), then there exists a transitive function defined in \( E' \) of degree \( \leq d \).
Counter-example. If \( G' \) has subgraphs isomorphic to \( K_3 \), then Theorem 7 is not always true, as the following example shows. Suppose that \( G \) is a graph with 5 vertices and only one edge. Then \( R = \{0, 3, 5, 15, 30\} \) is a representation of \( G \) modulo \( b = 3 \times 5 \times 7 = 105 \). It is not difficult to see that any transitive function defined in \( E' \) has degree greater than 3.

**Corollary 8.** If \( d_t(G_0) \geq 2 \) and \( G' \) does not have any subgraph isomorphic to \( K_3 \), then \( d_r(G_0) = d_t(G_0) \).

Let \( M(G_0) \) be the maximum number of edges incident with one vertex in \( G_0 \).

**Theorem 9.** Suppose that \( G_0 \) has no cycles. Then

a) \( d_t(G_0) = M(G_0) \).

b) If at least one of the maximal connected subgraphs of \( G_0 \) has at least 3 vertices, then

\[
(1) \quad d_r(G) = d_t(G_0) = M(G_0).
\]

**Corollary 10.** If \( G_0 \) is a tree and \( n \neq 2 \), then (1) holds.

Now we are going to prove the theorems above. We split the proof of Theorem 4 into several lemmas.

**Lemma 11.** Suppose that \( \phi: E' \rightarrow X \) is a transitive function with \( \#(\phi(E')) = 1 \). Let \( \delta \) be a positive integer. Then there exists a positive prime \( p > \delta \) and there exist distinct nonnegative integers \( a_1, \ldots, a_n \) such that \((v_i, v_j) \in E'\) if and only if \( p \) divides \( a_i - a_j \), \( i, j \in \{1, \ldots, n\}, i \neq j \).

**Proof:** Let \( H_1, \ldots, H_t \) be the maximal connected subgraphs of \( G' \). Without loss of generality, suppose that \( H_s = \{v_{k_1+\cdots+k_{s-1}+1}, \ldots, v_{k_1+\cdots+k_s}\}, k_s = \#H_s, s \in \{1, \ldots, t\} \). Let \( p \) be a prime > \( \max\{t, \delta\} \). If \( i = k_1 + \cdots + k_{s-1} + j, 1 \leq j \leq k_s \), let \( a_i = s + j p \). Since \( \#(\phi(E')) = 1 \), the graphs \( H_i \) are complete. It is easy to conclude that the lemma is satisfied. \[\blacksquare\]

**Lemma 12.** Let \( \alpha \) and \( \beta \) be integers with \( \gcd(\alpha, \beta) = 1 \). Let \( p \) be a prime. Then there exists at most one \( \epsilon \in \{0, \ldots, p-1\} \) such that \( \epsilon \beta + \alpha \in (p) \), where \((p)\) denotes the principal ideal, of the ring of the integers, generated by \( p \).

**Proof:** Firstly, suppose that \( p \) divides \( \beta \). Then \( p \) does not divide \( \alpha \) and, therefore, \( \epsilon \beta + \alpha \notin (p) \), for every integer \( \epsilon \). Now suppose that \( p \) does not divide \( \beta \)
and that there exist \( \epsilon_1, \epsilon_2 \in \{1, \ldots, p-1\} \) such that \( \epsilon_1 \neq \epsilon_2 \) and \( \epsilon_1 \beta + \alpha, \epsilon_2 \beta + \alpha \in (p) \). Then \( (\epsilon_1 - \epsilon_2) \beta \in (p) \). As \( p \) is prime, \( p \) divides \( \epsilon_1 - \epsilon_2 \) or \( p \) divides \( \beta \), what is impossible.

**Lemma 13.** Let \( \alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_s \) be integers such that \( \gcd(\alpha_j, \beta_j) = 1 \), \( j \in \{1, \ldots, s\} \). Let \( b = p_1 \cdots p_r \), where \( p_1, \ldots, p_r \) are positive primes. If \( \min \{p_i : 1 \leq i \leq r\} > s r \), then there exists an integer \( \gamma \) such that

\[ \gcd(\gamma \beta_j + \alpha_j, b) = 1, \quad j \in \{1, \ldots, s\}. \]

**Proof:** Let \( m = \min \{p_i\} \). From the previous lemma, it can easily be deduced that there exists \( \gamma \in \{0, \ldots, m-1\} \) such that \( \gamma \beta_j + \alpha_j \notin (p_i) \), \( j \in \{1, \ldots, s\} \), \( i \in \{1, \ldots, r\} \). That is, \( \gamma \) satisfies (2).

**Lemma 14.** Let \( \phi : E' \to X \) be a transitive function. Suppose that \( d = \# \phi(E') \geq 2 \) and \( \phi(E') = \{x_1, \ldots, x_d\} \). Let \( \delta \) be a positive integer. Then there exist distinct positive primes \( p_1, \ldots, p_d \) and there exist distinct integers \( a_1, \ldots, a_n \) such that:

i) \( 0 \leq a_i < p_1 \cdots p_d, \; i \in \{1, \ldots, n\} \).

ii) \( \gcd(a_i - a_j, p_1 \cdots p_d) = 1 \) if and only if \( (v_i, v_j) \notin E', \; i, j \in \{1, \ldots, n\}, \; i \neq j \).

iii) \( \gcd(a_i - a_j, p_1 \cdots p_d) = p_u \) if and only if \( (v_i, v_j) \in E', \phi(v_i, v_j) = x_u, \; i, j \in \{1, \ldots, n\}, \; i \neq j, \; u \in \{1, \ldots, d\} \).

iv) \( \min \{p_1, \ldots, p_d\} > \delta \).

**Proof:** By induction on \( n \). As \( d \geq 2 \), we have \( n \geq 3 \). Let \( G_0 = (V_0, E_0) \) be the subgraph that we obtain from \( G' \) deleting \( v_n \) and all the edges incident with \( v_n \). Without loss of generality, we assume that \( E_0 \neq E' \) and \( \phi(E_0) = \{x_1, \ldots, x_e\} \). We choose \( p_1, \ldots, p_e \) and \( a_1, \ldots, a_{n-1} \) as follows. Note that \( e \leq 1 \) when \( n = 3 \).

If \( e \geq 2 \), then, by the induction assumption, there exist distinct primes \( p_1, \ldots, p_e \) and there exist distinct integers \( a_1, \ldots, a_{n-1} \) such that:

i_0) \( 0 \leq a_i < p_1 \cdots p_e, \; i \in \{1, \ldots, n-1\} \).

ii_0) \( \gcd(a_i - a_j, p_1 \cdots p_e) = 1 \) if and only if \( (v_i, v_j) \notin E_0, \; i, j \in \{1, \ldots, n-1\}, \; i \neq j \).

iii_0) \( \gcd(a_i - a_j, p_1 \cdots p_e) = p_u \) if and only if \( (v_i, v_j) \in E_0, \phi(v_i, v_j) = x_u, \; i, j \in \{1, \ldots, n-1\}, \; i \neq j, \; u \in \{1, \ldots, e\} \).

iv_0) \( \min \{p_1, \ldots, p_e\} > \max \{\delta, (n-1)d\} \).
If $e = 1$, then, according to Lemma 11, there exists a prime $p_1$ and there exist distinct nonnegative integers $a_1, \ldots, a_{n-1}$ satisfying (ii$_0$), (iii$_0$) and (iv$_0$).

If $e = 0$, take $a_i = i - 1, i \in \{1, \ldots, n-1\}$.

In any case $e \geq 0$, we choose primes $p_{e+1}, \ldots, p_d$ such that:

I) $p_1, \ldots, p_d$ are distinct.

II) None of the primes $p_{e+1}, \ldots, p_d$ divide $a_i - a_j, i, j \in \{1, \ldots, n-1\}, i \neq j$.

III) $\min\{p_1, \ldots, p_d\} > \max\{\delta, (n-1)d\}$.

Without loss of generality, suppose that $v_1, \ldots, v_t$ are the vertices of $G'$ incident with $v_n$. Let $x_{k_i} = \phi(v_i, v_n), i \in \{1, \ldots, t\}$. Without loss of generality, suppose that $k_1, \ldots, k_r$ are pairwise distinct and $k_i \in \{k_1, \ldots, k_r\}$ whenever $i \in \{r+1, \ldots, t\}$.

According to the Chinese Remainder Theorem, there exists an integer $z$ such that

\begin{equation}
\tag{3}
z - a_j \in (p_{k_j}), \quad j \in \{1, \ldots, r\}.
\end{equation}

Let $i \in \{r+1, \ldots, t\}$ and suppose that $k_i = k_j$, where $j \in \{1, \ldots, r\}$. As $\phi$ is transitive, $(v_i, v_j) \in E'$ and $\phi(v_i, v_j) = x_{k_i}$. Therefore $k_i \in \{1, \ldots, e\}$. From (ii$_0$), it follows that $a_i - a_j \in (p_{k_j})$. Thus $z - a_i = (z - a_j) + (a_j - a_i) \in (p_{k_j})$.

Now suppose that $z - a_i \in (p_{k_i})$, with $i \in \{1, \ldots, n-1\}, j \in \{1, \ldots, r\}$. From (3), $a_i - a_j \in (p_{k_j})$. Bearing in mind (ii), (iii$_0$) and (iv$_0$), we conclude that $k_j \in \{1, \ldots, e\}$, $(v_i, v_j) \in E_0$ and $\phi(v_i, v_j) = x_{k_j}$. From the transitivity of $\phi$, $(v_i, v_n) \in E'$ and $\phi(v_i, v_n) = x_{k_j}$. Therefore, $i \in \{1, \ldots, t\}$ and $k_i = k_j$.

It is not difficult to prove that

\begin{equation}
\tag{4}
g.c.d\{p_{k_1} \cdots p_{k_r}, z - a_i\} = p_{k_i}, \quad i \in \{1, \ldots, t\},
\end{equation}

\begin{equation}
\tag{5}
g.c.d\{p_{k_1} \cdots p_{k_r}, z - a_i\} = 1, \quad i \in \{t+1, \ldots, n-1\}.
\end{equation}

Using Lemma 13, it follows from (4), (5) and III) that there exists an integer $\gamma$ such that

\begin{equation}
\tag{6}
g.c.d\left\{\gamma \frac{p_{k_1} \cdots p_{k_r}}{p_{k_i}} + \frac{z - a_i}{p_{k_i}}, b\right\} = 1, \quad i \in \{1, \ldots, t\},
\end{equation}

\begin{equation}
\tag{7}
g.c.d\left\{\gamma p_{k_1} \cdots p_{k_r} + z - a_i, b\right\} = 1, \quad i \in \{t+1, \ldots, n-1\},
\end{equation}

where $b = p_1 \cdots p_d$. Let $a_n = \gamma p_{k_1} \cdots p_{k_r} + z + wb$, where $w$ is an integer chosen so that $0 \leq a_n < b$. Then (6) and (7) take the forms

\begin{equation}
\tag{6'}
g.c.d\{a_n - a_i, b\} = p_{k_i}, \quad i \in \{1, \ldots, t\},
\end{equation}

\begin{equation}
\tag{7'}
g.c.d\{a_n - a_i, b\} = 1, \quad i \in \{t+1, \ldots, n-1\}.
\end{equation}
Clearly \( a_n \) is different from \( a_i, \ i \in \{1, ..., n-1\} \), and conditions i)–iv) are satisfied.

Now Theorem 4 follows immediately from Lemma 14.

**Proof of Theorem 7:** Let \( R = \{a_1, ..., a_n\} \) be a representation of \( G \) modulo \( b = p_1 \cdots p_d \), where \( p_1, ..., p_d \) are primes. Suppose that \( a_1, ..., a_n \) are ordered so that \( \gcd\{a_i - a_j, b\} = 1 \) if and only if \( (v_i, v_j) \in E \). For each \( (v_i, v_j) \in E' \), let \( \phi(v_i, v_j) \) be an element of \( \{p_1, ..., p_d\} \) such that \( \phi(v_i, v_j) \) divides \( a_i - a_j \). It is easy to see that \( \phi: E' \to \{p_1, ..., p_d\} \) is transitive.

**Proof of Theorem 9:** a) For each \( i \in \{1, ..., n\} \), we denote by \( E_i(G') \) the set of all the edges incident with \( v_i \) in \( G' \). Given a transitive function \( \phi: E' \to X \), the restriction of \( \phi \) to \( E_i(G') \) is one-to-one. Therefore, \( \#\phi(E') \geq \#E_i(G') \). Consequently, \( d_i(G') \geq \max\{\#E_i(G')\} \).

Now we prove that \( d_i(G') \leq \max\{\#E_i(G')\} \) by induction on \#\(E'\). If \( E' \) is empty, this is trivial. Suppose that \( \#E' \geq 1 \). Then there exists \( i \in \{1, ..., n\} \) such that \( \#E_i(G') = 1 \). Without loss of generality, assume that \( E_n(G') = \{(v_{n-1}, v_n)\} \). Let \( G = (V, \bar{E}) \), where \( \bar{E} = E' \setminus \{(v_{n-1}, v_n)\} \). By the induction assumption, \( d_i(\bar{G}) \leq \max\{\#E_i(\bar{G})\} \). Let \( \psi: \bar{E} \to X \) be a transitive function of degree \( d_i(\bar{G}) \). If there exists \( i \in \{1, ..., n-2\} \) such that \( \#E_{n-1}(\bar{G}) < \#E_i(\bar{G}) \) (\( = \#E_i(G') \)), let \( x \) be an element of \( \psi(E_i(\bar{G})) \setminus \psi(E_{n-1}(\bar{G})) \). If \( \#E_i(\bar{G}) \leq \#E_{n-1}(\bar{G}) \), \( i \in \{1, ..., n-2\} \), let \( x \) be an element that does not belong to \( X \). Let \( \phi: E' \to X \cup \{x\} \) be the extension function of \( \psi \) satisfying \( \phi(v_{n-1}, v_n) = x \). It is easy to see that \( \phi \) is transitive and

\[ d_i(G') \leq \#\phi(E') \leq \max\{\#E_i(G')\} \, . \]

b) Since \( G' \) is acyclic, the hypothesis of b) is equivalent to \( d_i(G') \geq 2 \). Thus b) follows from a) and Corollary 8.

**REFERENCES**


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