SOLITON TYPE ASYMPTOTIC SOLUTIONS
OF THE CONSERVED PHASE FIELD SYSTEM

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Abstract: We consider the conserved phase field system with a small parameter in the $n$-dimensional case ($n \leq 3$). The soliton type solution describes the first stage of separation in an alloy, when a set of "superheated liquid" appears inside the "solid" part. The corresponding Hugoniot type condition is obtained and the asymptotic solution is justified.

1 – Introduction

The aim of this paper is to consider the metastable processes of solidification. We shall consider the conserved phase field system (the non-isothermal Cahn–Hilliard system), proposed by G. Caginalp [5]

$$
\frac{\partial}{\partial t} \left( \theta + \frac{l}{2} \varphi \right) = k \Delta \theta + f(x,t), \quad (x,t) \in Q,
$$

$$
-\tau_0 \frac{\partial \varphi}{\partial t} = \xi^2 \Delta \left( \xi^2 \Delta \varphi + \frac{1}{2a} (\varphi - \varphi^3) + \kappa_1 \theta \right).
$$

Here $Q = \Omega \times (0,T), \Omega \subset R^n$ is a bounded domain with smooth ($C^\infty$) boundary $\partial \Omega$, $n \leq 3$, $T < \infty$; $\Delta$ is the Laplace operator; $\theta$ is the normalized temperature; $\varphi$ is the order function (that is, the values $\varphi = 1$ and $\varphi = -1$ outside the free boundary correspond to the pure phases); $l > 0$, $k > 0$, $\kappa_1$ are constants; $f(x,t)$ is a certain smooth functions (in [5] $f = 0$); $\tau_0 > 0$, $\xi > 0$, and $a > 0$ are parameters.

The physical meaning of $\tau_0$, $\xi$, $a$, and of the whole model (1) for the phase transitions was discussed in the papers by G. Caginalp [5, 6] and by A. Novick–Cohen [17]. H. Alt and I. Pawlow [1] proposed general models for non-isothermal

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phase transitions with a conserved order function. Nevertheless, the simplest model by G. Caginalp is of independent interest since it qualitatively describes the actual physical processes in a binary alloy; the solutions of system (1) can be analysed by mathematical methods.

We shall study the structure of the solution of system (1) and analyze how to pass to the limit from the microscopic description (1) to the macroscopic description. It is well-known that we can pass to the limit as $a \to 0$ and $\xi \to 0$, and that the form of the limiting problem depends on the relations between the parameters $a$, $\xi$ and $\tau_0$ (here we have the situations in which either $\tau_0 = \text{const}$ or $\tau_0 \to 0$). We restrict our consideration to the case

$$a \ll 1, \quad \xi \ll 1, \quad \xi a^{-1/2} = \text{const}, \quad \tau_0 = \text{const}.$$ 

Let us introduce a small parameter $\varepsilon \to 0$ and set

$$(2) \quad a = \varepsilon/2, \quad \xi = \sqrt{\varepsilon}, \quad \tau_0 = \kappa, \quad \kappa = \text{const} > 0.$$ 

For simplicity, we also assume that $k = 1$ and $l = 2$. Completing (1) with natural initial and boundary conditions, we obtain our basic mathematical model

$$\frac{\partial}{\partial t} (\theta + \varphi) = \Delta \theta + f(x, t),$$

$$-\kappa \frac{\partial \varphi}{\partial t} = \Delta (\varepsilon^2 \Delta \varphi + \varphi - \varphi^3 + \kappa_1 \varepsilon \theta),$$

$$\theta|_{t=0} = \theta^0(x, \varepsilon), \quad \varphi|_{t=0} = \varphi^0(x, \varepsilon),$$

$$\frac{\partial \theta}{\partial N} \big|_{\Sigma} = 0, \quad \frac{\partial \varphi}{\partial N} \big|_{\Sigma} = 0, \quad \frac{\partial \Delta \varphi}{\partial N} \big|_{\Sigma} = 0.$$

Here $N$ is the external normal to the boundary $\partial \Omega$ and $\Sigma = [0, T] \times \partial \Omega$.

Precisely as the solution of the Cahn–Hilliard equation (the second equation in (3) with $\theta = \text{const}$), the solution of the conserved phase field system (3) is very complicated, since its behavior varies depending on different stages of the phase separation process in binary alloys. (For example, the solution is of the oscillating type [15, 16] or of the Van der Waals tanh-type [4–6, 20–22, 25]). These stages are called stable, unstable, and metastable [15, 16]. They correspond to the cases: $\varphi \geq 1$ or $\varphi \leq -1$, and $-1/\sqrt{3} \leq \varphi \leq 1/\sqrt{3}$, and $-1 < \varphi < -1/\sqrt{3}$ or $1/\sqrt{3} < \varphi < 1$ respectively. Here and below $f(x) = w - \lim_{\varepsilon \to 0} f(x, \varepsilon)$ denotes the weak limit in the $\mathcal{D}'$ sense. The numbers $\pm 1$ correspond to the zero points of the equilibrium chemical potential $W'(\varphi) = \varphi - \varphi^3$. The numbers $\pm 1/\sqrt{3}$ correspond to local maxima/minima of $W'$. At present, the solution to problem (3) with arbitrary
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initial data can be analyzed in detail only by numerical methods. However, by setting some special initial data, one can select and thoroughly study a certain type of solutions.

Naturally, solutions corresponding to experimentally observed processes must be stable at each stage, i.e., theses processes must not vary much during some time. We shall study possible stable structures of solutions by using asymptotic methods. Functions of the form

\[ \mu_M(x, t, \varepsilon) = \sum_{i=0}^{K_1} \varepsilon^i V_i(S/\varepsilon, x, t), \quad \varphi_M(x, t, \varepsilon) = \sum_{i=0}^{K_2} \varepsilon^i \Phi_i(S/\varepsilon, x, t) \]

will be called a self-similar (formal) asymptotic up to \( O(\varepsilon^{M+1}) \) solution of system (3). Here \( S = S(x, t) \in C^\infty(\bar{Q}), \Phi_i(\eta, x, t), \) and \( V_i(\eta, x, t) \) are smooth functions uniformly bounded in \( R_\eta \times \bar{Q} \) together with their derivatives; the nonnegative numbers \( K_j = K_j(M), j = 1, 2, \) are such that for any number \( M \geq 0 \) the substitution of the functions \( \theta_M, \varphi_M \) into (3) gives a discrepancy \( O(\varepsilon^{M+1}) \) on the right-hand side of (3), here \( O(\cdot) \) is a bound in the sense of \( C(R_\eta \times \bar{Q}). \)

The possible types of stable asymptotic solutions differ by the form of dependence on the fast variable \( \eta = S/\varepsilon. \) As is easy to see, there is a finite number of types for (3). To this end, we substitute \( \theta_M, \varphi_M \) into the Cahn–Hilliard equation and set the coefficient at \( \varepsilon^{-2} \) equal to zero. We obtain the equation

\[ \frac{\partial^2}{\partial \eta^2} \left( |\nabla S|^2 \frac{\partial^2 \Phi_0}{\partial \eta^2} + \Phi_0 (1 - \Phi_0^2) \right) = 0. \]

Integrating with respect to \( \eta, \) we obtain

\[ |\nabla S|^2 \frac{\partial^2 \Phi_0}{\partial \eta^2} + \Phi_0 (1 - \Phi_0^2) = C + C_1 \eta, \]

where the “constants” of integration \( C = C(x, t) \) and \( C_1 = C_1(x, t) \) are smooth functions. The assumption that the function \( \Phi_0 \) is uniformly bounded implies the equality \( C_1 \equiv 0. \) Thus, we have obtained an autonomic second-order equation, namely, the Newton equation. For this equation we can list all possible types of solutions. Multiplying this equation by \( \frac{\partial \Phi_0}{\partial \eta} \) and integrating with respect to \( \eta, \) we obtain

\[ |\nabla S| \frac{\partial \Phi_0}{\partial \eta} = \pm 2 \sqrt{E - U(\Phi_0, C)}, \]

where \( E = E(x, t) \) is the second “constant” of integration, and

\[ U(\Phi_0, C) = W(\Phi_0) - C \Phi_0, \quad W(\Phi_0) = \frac{1}{2} \Phi_0^2 - \frac{1}{4} \Phi_0^4. \]
Obviously, this equation has real solution if and only if $E \geq U$. Figure 1 shows the potential $U(\Phi_0, C)$ is plotted in the case $C < 0$ (for $C > 0$ the hump on the left is higher than the crest on the right and for $C = 0$ the potential $U(\Phi_0, 0)$ is an even function of $\Phi_0$).

![Figure 1](https://example.com/figure1.png)

We denote the maximal values of the potential by $U_0$ and $U_1$, $U_0 < U_1$ ($U_0 = U(\varphi_0, C)$ and $U_1 = U(\Phi_0^+, C)$, where $\varphi_0$ and $\Phi_0^+$ are the minimal and maximal roots of the equation $\Phi_0^3 - \Phi_0 + C = 0$, respectively). We assume that $U_2 = U(\Phi_0^*, C)$, where $\Phi_0^*$ is the point at which $U$ takes its local minimum.

Obviously, if $E > U_0$, the equation for $\Phi_0$ has no solutions with compact range, which means that the corresponding mechanical system has no finite motions.

If $E \in (U_2, U_0)$ (see Fig. 1), $\Phi_0$ has a compact range such that $U(\Phi_0, C) \leq E$. Hence, to each such value of $E$ there corresponds a periodic solution $\Phi_0(\eta, x, t)$ (see also [15–17]). Finally, for $E = U_0$ the point $\varphi_0$ at which $U = U_0$ is an isolated solution of the equation for $\Phi_0$ and, under the condition $\frac{\partial U}{\partial \Phi_0}|_{\Phi_0 = \Phi_0^*} \neq 0$ (i.e., $C \neq 0$), the point $\Phi_0^*$ such that $U(\Phi_0^*, C) = U_0$ and $\Phi_0^* \neq \varphi_0$ is a turning point and the corresponding solution is shown in Fig. 2.

![Figure 2](https://example.com/figure2.png)
It is easy to see that as $|\eta| \to \infty$ this solution exponentially tends to $\varphi_0$. This solution will be called a soliton-type solution. Finally, if $C = 0$, then $U_0 = U_1$ and $\Phi_0 = \Phi_0^+$ is also an isolated solution. In this case, the solution $\Phi_0$ is a separatrix and the explicit expression for this solution is well-known: $\Phi_0 = \tanh(\eta/\sqrt{2}|\nabla S|)$, i.e., $\Phi_0$ is the Van der Waals tanh solution.

The physical setting of our problem is the following: to simplify the problem we assume that the initial concentration $\varphi(x) \in (1/\sqrt{3}, 1)$ for all $x \in \Omega$. The set of such points with concentration will be called “solid”. At first sight, since the interval $(1/\sqrt{3}, 1)$ belongs to the domain of attraction to the point $\varphi_{eq}^+ = 1$, the concentration would seem to increase and tend to $\varphi_{eq}^+$. But it is impossible to obtain the situation in which $\varphi(x,t) = \varphi_{eq}^+$ at each point of $\Omega$, since the global mass $m(\varphi)$,

$$m(\varphi) = \int_\Omega \varphi \, dx,$$

conserves in time and $m(\varphi^0) < |\Omega|$. Thus, one must assume the appearance of subdomains $\Omega_\tau^\pm$ such that $\varphi \in (1/\sqrt{3}, 1]$ as $x \in \Omega_\tau^+$ and $\varphi \in [-1, -1/\sqrt{3})$ as $x \in \Omega_\tau^-$. After that, the next stage of the solidification starts at which the subdomains $\Omega_\tau^\pm$ transform. The soliton type asymptotic solution, constructed in the present paper, describes the first stage at which the “liquid” part $\Omega_\tau^-$ appears inside the “solid” part, but the volume of $\Omega_\tau^-$ is still small enough (note that the domains $\Omega_\tau^\pm$ are fulfilled by a substance that, in fact, is not solid or liquid).

The Van der Waals type asymptotic solution [21] describes the motion of $\Omega_\tau^\pm$ when the volumes of $\Omega_\tau^\pm$ are not small. Our construction allows us as well to show that the temperature remains a smooth function (in the leading term with respect to $\varepsilon$) during these processes. Therefore, the solidification process, described by the conserved phase field system (3), differs, in principle, from the dynamics of “solid”/“liquid”, described by both the phase field model and by (1) for small relaxation time $\tau_0$ [20].

Actually, according to the phase field model and to (1) for $\tau_0 \sim \varepsilon$, the temperature has a weak discontinuity on the free interface, whereas, according to model (3), the temperature is almost the same on the “solid” and “liquid” domains (for both $|\Omega_\tau^-| \ll 1$ and $|\Omega_\tau^-| \sim \text{const}$). So we can say that the appearing set $\Omega_\tau^-$ is a domain of “superheated liquid”. We shall consider only the case $\varphi_{eq}^0 \in (1/\sqrt{3}, 1]$. Nevertheless, it is clear that our construction also allows us to describe the similar processes in the case $\varphi_{eq}^0 \in [-1, -1/\sqrt{3})$.

The multidimensional Cahn–Hilliard equation with a small parameter was considered by B. Stoth [25] (in the spherical case) and by R.L. Pego [22]. The conserved phase field system (1) was considered by G. Caginalp [5, 6] and by G. Omel’yanov [20, 21].
Here we use the method for constructing asymptotic solutions, which is a modification of the two-scale method for obtaining solutions with localized “fast” variation. It was developed by V. Maslov, V. Tsupin, G. Omel’yanov, V. Danilov, and K. Volosov [11–14] for constructing soliton-type and traveling wave-type asymptotic solutions. A modification of this method for phase transition problems (to the phase field system and to the conserved phase field system) was proposed in the works by E. Radkevich, V. Danilov, and G. Omel’yanov [7, 8, 18–21, 24]. The main idea of this method is to construct some analogs of the internal and external expansions assuming that they are defined in the whole range of independent variables. This implies that there are no summands polynomially increasing in the “fast” variable. Obviously, no matching is needed.

Here we have the following important points. Suppose the “fast” variation of the solution $u$ is localized in a small neighborhood of the smooth surface $\Gamma_t$, for example, $u = A(x, t) \cosh^{-2} (S(x, t)/\varepsilon)$, $\Gamma_t = \{x \in \Omega, S(x, t) = 0\}$, $A \in C^\infty$. Then, to calculate the function $S$ at each instant of time $t \geq 0$, it is sufficient to define only its zero surface $\{x, \psi(x) = t\} \equiv \{x, S(x, t) = 0\}$ and the first normal derivative $S'$ on $\Gamma_t$, since in an $\varepsilon$-neighborhood of $\Gamma_t$ the smooth function $S$ cannot vary more than by $O(\varepsilon)$ and outside an $\varepsilon^{1-\delta}$-neighborhood of $\Gamma_t$ ($0 < \delta < 1$) the solution varies slowly with precision up to $O(\varepsilon^\infty)$ ($u = 0$ for the above example). Here we speak about a polynomial (with respect to the parameter $\varepsilon$) asymptotics, and hence, the discrepancy $O(\varepsilon^\infty)$ (i.e., $O(\varepsilon^M)$ for any $M > 0$) is much less than the discrepancy $O(\varepsilon^{i+1})$ arising at the $i$th step for each fixed $i$.

Furthermore, in the traditional two-scale method we first construct a rapidly varying solution $u(S/\varepsilon, x, t)$ in the “extended” space $R^1 \times \Omega \times [0, T]$, assuming that $u = u(\eta, x, t)$ and that the variables $\eta$ and $x, t$ are independent. Then we calculate the trace $u(\eta, x, t)|_{\eta = S/\varepsilon}$. This approach is absolutely justified for fast oscillating solutions. However, for solutions with localized fast variation we can construct the rapidly varying components only in the subspace $R^1 \times T_T^\delta$, where $T_T^\delta$ is an $\varepsilon^{1-\delta}$-neighborhood of the surface $T_T = \{(x, t) \in \bar{Q}, t = \psi(x)\} = \bigcup_{t \in [0, T]} \Gamma_t$. That is, we know beforehand that our final goal is only the trace $u(\eta, x, t)|_{\eta = S/\varepsilon}$.

In its turn, since $u(\eta, x, t)$ smoothly depends on “slowly” varying variables $x$ and $t$, we can calculate the solution in two stages: in the first stage we define the trace $\tilde{u} = u(\eta, x, t)|_{\ell = \psi(x)}$ of $u$ on the section $R^1 \times T_T$, in the second stage we construct a sufficiently smooth continuation of $\tilde{u}$ outside $R^1 \times T_T$. Needless to say, this continuation must be constructed sufficiently accurate, nevertheless, there is some freedom in choosing this continuation. Namely, this freedom provides the boundedness (uniformly in $\eta \in R^1, x, t \in Q$) of all terms of the asymptotic expansion. In § 2 we formalize these considerations, which are obvious enough,
and in detail describe the method for constructing the soliton-type asymptotic solution.

We have discussed only a formal asymptotic solution. However, in the strict sense, this does not imply that the discrepancy between the explicit solution and the asymptotic one is actually small in a certain sense. Therefore, in particular, one cannot use formal asymptotics for obtaining the limiting (as \( \varepsilon \to 0 \)) problem. Thus, in \( \S \) 3 we rigorously justify the asymptotics constructed.

2 – Soliton type asymptotic solution

Let us formulate the main result of this section. By \( \theta_0 = \theta_0(x,t), \varphi_0 = \varphi_0(x,t) \in (1/\sqrt{3},1), \psi = \psi(x) \), we denote the solutions of the following model problems

\[
\kappa \frac{\partial \varphi_0}{\partial t} = \Delta (\varphi_0^3 - \varphi_0), \quad x \in \Omega, \ t > 0,
\]

\[
\varphi_0 \big|_{t=0} = \varphi_0^0(x), \quad \frac{\partial \varphi_0}{\partial N} \big|_{\Sigma} = 0,
\]

\[
\frac{\partial \theta_0}{\partial t} = \Delta \theta_0 + f(x,t) - \frac{\partial \varphi_0}{\partial t}, \quad x \in \Omega, \ t > 0,
\]

\[
\theta_0 \big|_{t=0} = \varphi_0^0(x), \quad \frac{\partial \theta_0}{\partial N} \big|_{\Sigma} = 0,
\]

\[
\kappa \nu \nu = \frac{1}{3} K_t \left( 1 + G(\varphi_0) \right) + G_0 \frac{\partial \varphi_0}{\partial \nu}, \quad \psi \big|_{\Gamma_0} = 0.
\]

Here \( \nu \nu = 1/|\nabla \psi| \) is the normal velocity of motion of the surface \( \Gamma_t = \{ x \in \Omega, \psi(x) = t \} \); \( K_t = \text{div}(\nu) \) is the mean curvature of \( \Gamma_t \), \( \nu = \nabla \psi/|\nabla \psi| \) is the vector normal to \( \Gamma_t \); \( \tilde{F} = F(x,\psi(x)) \) for all continuous functions \( F(x,t); \partial/\partial \nu = (\nu, \nabla) \);

\[
G(\varphi_0) = \frac{I Q^2}{2(Q - I)}, \quad G_0 = \frac{G(\varphi_0)}{\varphi_0}, \quad \frac{\partial \varphi_0}{\partial \nu} = \left( \nu, \nabla \varphi_0(x,\psi) \right),
\]

\[
Q = \sqrt{3(\varphi_0)^2 - 1}, \quad I = \sqrt{2} \varphi_0 \ln J, \quad J = (\sqrt{2} \varphi_0 + Q)/\sqrt{b}, \quad b = 1 - (\varphi_0)^2.
\]

**Theorem 1.** Let \( \Gamma_0 = \{ x \in \Omega, \psi(x) = 0 \} \) be a sufficiently smooth closed surface of codimension 1. Let sufficiently smooth solutions of problem (8)–(10) exist, and \( \varphi_0 \in (1/\sqrt{3},1) \), and let \( \text{dist}(\Gamma_t, \partial \Omega) \geq \text{const} \) for all \( t \in [0,T] \). Then,
for any \( M \geq 0 \), there exists a formal asymptotic (up to \( \mathcal{O}(\varepsilon^{M+1}) \)) solution of
equations (3). The leading term of this asymptotic solution has the form
\[
\theta(x, t, \varepsilon) = \theta_0(x, t), \quad \varphi(x, t, \varepsilon) = \varphi_0(x, t) + \chi(\eta, x).
\]
Here
\[
\chi = -8 Q^2 \left\{ e^\xi + 8 b e^{-\xi} + 8 \tilde{\varphi}_0 \right\}^{-1},
\]
\[
\xi = \beta(\eta + \psi_1(x)), \quad \eta = (t - \psi(x))/\varepsilon, \quad \beta = Q/|\nabla \psi|,
\]
\( \psi_1 \) is a smooth function, the method for calculating this function is given below
(see formula (31) and Lemma 2).

Remark. It is not too difficult to prove that equation (10) is a quasilinear
parabolic equation, in which \( x_\nu \) (along the vector \( \nu \)) is a time like variable, and
\( x_t \) (tangential to \( \Gamma_t \)) are space like variables. So, the additional condition in (10)
is actually the initial condition \[7, 8\]. The classical solvability and uniqueness of
solutions of quasilinear parabolic problems with smooth coefficients are the result
of the realization of some matching conditions between the initial and boundary
data \[10, 23\].

At first let us consider the statement of Theorem 1. Formulas (11), (12) and
the solutions of problems (8)–(10) describe the motion of the soliton \( \chi \) on the
smooth “background” \( \varphi_0 \) (a soliton type solution was obtained also in \[9\] by nu-
merical simulations for the one-dimensional Cahn–Hilliard equation). Obviously,
the surface \( \Gamma_t \) is the set of maximum magnitude of \(|\chi|\)
\[
A = \max_{x \in \Omega} |\chi| = -\chi|_{\Gamma_t} = Q^2 \left\{ 8 + \tilde{\varphi}_0 - \varphi_0^2 \right\}^{-1}.
\]
It is easy to prove that this solution exists if and only if \( \varphi_0 \in (1/\sqrt{3}, 1) \). The
amplitude \( A \) is a monotonically increasing function, \( A'_{\varphi_0} > 0 \), and trivial calcu-
lations show that \( A \to 0 \) and \( G \to 0 \) as \( \tilde{\varphi}_0 \to 1/\sqrt{3} \). It is also clear that there
exists a value \( \varphi^* \in (1/\sqrt{3}, 1) \) such that \( A < \tilde{\varphi}_0 \) as \( \tilde{\varphi}_0 \in (1/\sqrt{3}, \varphi^*) \), and \( A > \tilde{\varphi}_0 \)
as \( \tilde{\varphi}_0 \in (\varphi^*, 1) \). Thus, moving into the domain with \( \varphi_0 \in (\varphi^*, 1) \), the soliton
solution describes how the set \( \Omega_t \) = \{ \( x \in \Omega, \varphi_0(x) + \chi < 0 \} \) with negative concen-
tration arises \( |\Omega_t| \sim \varepsilon \). Let us consider the behavior of the solution as \( \varphi_0 \) tends
to 1. Setting \( \varphi_0 = 1 - \tilde{\varphi}(x, t) \exp(-1/\delta), \delta \ll 1 \), we get the following relations
\( A = 16/9 + \mathcal{O}(e^{-1/\delta}), G = -1 + \mathcal{O}(\delta) \). Hence, \( v_\nu \sim \delta \) and the velocity of the
soliton motion decreases as \( \delta \to 0 \). On the other hand, the volume of the set \( \Omega_t \)
increases, since \( b \sim \exp(-1/\delta) \) and \(|\Omega_{t,\varepsilon}^-| \sim |\Omega_{t,\varepsilon}^+| \sim \varepsilon/\delta \) for \( \delta \ll 1 \). Thus, this solution describes the appearance of a sufficiently large domain of “superheated liquid”, since the concentration \( \varphi \sim -7/9 \) on \( \Omega_{t,\varepsilon}^- \) and the temperature \( \theta_0 \) is almost independent of \( \varphi_0 \) at these points. Nevertheless, this asymptotic solution is correct only if \( |\Omega_{t,\varepsilon}^-| \to 0 \) as \( \varepsilon \to 0 \).

We shall also see that the first corrections of the asymptotic expansions for the temperature and concentration have the form of smoothed shock waves. So, 

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\theta - \theta_0) = A_{1,\theta} H(t - \psi), \quad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\varphi - \varphi_0 - \chi) = A_{1,\varphi} H(t - \psi),
\]

where \( H \) is the Heaviside function, \( A_{1,\theta} = 2I/(\varphi_0 v_\psi) \), \( A_{1,\varphi} = 2\kappa I/(\varphi_0 Q^2 v_\psi) \) are the amplitudes of jumps on \( \Gamma_t \). It is easy to calculate that \( A_{1,\theta}, A_{1,\varphi} \) are bounded as \( \varphi_0 \to 1 \).

Let us prove Theorem 1 and consider the general method for constructing the asymptotic solution of problem (3) (with special initial data) up to an arbitrary precision.

First, we introduce some classes of functions, which we shall need for constructing asymptotic solutions with localized fast variation. That is,

\[
\mathcal{H} = \left\{ f(\eta, x, t) \in C^\infty(R^1 \times Q), \exists f^\pm \in C^\infty(Q), \lim_{\eta \to \pm \infty} \eta^m \frac{\partial^{r+|\alpha|+\gamma}}{\partial \eta^r \partial x^\alpha \partial \tau^\gamma} \left( f(\eta, x, t) - f^\pm(x, t) \right) = 0, \ \forall m, r, \alpha, \gamma \geq 0 \right\},
\]

\[
\mathcal{S} = \left\{ f(\eta, x, t) \in \mathcal{H}, \ f^+ = f^- = 0 \right\},
\]

\[
\mathcal{P} = \left\{ f(\tau, x', t) \in C^\infty(R^1_+ \times \Sigma), \lim_{\tau \to \pm \infty} \tau^m \frac{\partial^{r+|\alpha'|+\gamma}}{\partial \tau^r \partial (x')^\alpha' \partial \tau^\gamma} f(\tau, x', t) = 0, \ \forall m, r, \alpha', \gamma \geq 0 \right\}.
\]

**Lemma 1.** Let \( S(x, t) \in C^\infty(\bar{Q}) \) be such that \( \partial S/\partial t|_{\Gamma_t} \neq 0 \), where \( \Gamma_t = \{ x \in \Omega, S(x, t) = 0 \} \), \( t \geq 0 \). Then, for any function \( f(\eta, x, t) \in \mathcal{H} \), we get

\[
f\left( \frac{S(x, t)}{\varepsilon}, x, t \right) = f\left( \beta(x) \frac{t - \psi(x)}{\varepsilon}, x, t \right) + \mathcal{O}(\varepsilon),
\]

where \( t = \psi(x) \) is the equation of the surface \( S(x, t) = 0 \) and \( \beta(x) = \partial S/\partial t|_{t=\psi(x)} \).
2. Let \( \mu(\eta, x, t) \), \( \zeta(\eta, x, t) \in H \) be such that \( \mu^\pm = \pm 1 \), \( \zeta^+ = 1 \), \( \zeta^- = 0 \). Then, for any function \( f \in H \), we have the representations
\[
\begin{align*}
f &= \frac{1}{2} (f^+ + f^-) + \frac{1}{2} (f^+ - f^-) \mu(\eta, x, t) + \omega_1(\eta, x, t) , \\
f &= f^- + (f^+ - f^-) \zeta(\eta, x, t) + \omega_2(\eta, x, t) ,
\end{align*}
\]
where \( \omega_i \) are functions from \( S \).

3. The relations
\[
(t - \psi)^k f \left( \frac{t - \psi(x)}{\varepsilon}, x, t \right) = O(\varepsilon^k), \quad k \geq 0 ,
\]
\[
g(x, t) f \left( \frac{t - \psi(x)}{\varepsilon}, x, t \right) = g(x, \psi(x)) f \left( \frac{t - \psi(x)}{\varepsilon}, x, \psi(x) \right) + O(\varepsilon) ,
\]
hold for any functions \( f(\eta, x, t) \in S \), \( g(x, t) \in C^\infty(\bar{Q}) \).

The proof obviously follows from the definition (see also [12]).

Let us note that the representation \( S = t - \psi(x) \) does not mean that the solution must move with velocity of order \( O(1) \). Actually, since the function \( \psi(x) \) can increase rapidly along the direction normal to the surface \( \Gamma_t = \{ x \in \Omega, \psi(x) = t \} \), the motion of the solution can be arbitrary slowly.

Let us begin to construct the self-similar asymptotic solution of problem (3). One can show that the leading term of the asymptotic expansion for \( \theta \) must be a smooth function, since the leading term of \( \varphi \) is a soliton. This implies that the asymptotic solution has the following form
\[
\theta(x, t, \varepsilon) = \theta^M(x, t, \varepsilon) + \varepsilon \psi^M \left( \frac{S(x, t)}{\varepsilon}, \frac{x_N}{\varepsilon}, x, t, \varepsilon \right) ,
\]
\[
\varphi(x, t, \varepsilon) = \Phi^M(x, t, \varepsilon) + \mathcal{W}^M \left( \frac{S(x, t)}{\varepsilon}, \frac{x_N}{\varepsilon}, x, t, \varepsilon \right) ,
\]
(13)

where the functions \( \theta^M, \Phi^M \) (uniformly smooth with respect to \( \varepsilon \in [0, 1] \), \( x, t \in Q \), the so-called “regular part” or the “background”) give an analog of the external expansion,
\[
\theta^M(x, t, \varepsilon) = \sum_{j=0}^{M} \varepsilon^j \theta_j(x, t) , \quad \Phi^M(x, t, \varepsilon) = \sum_{j=0}^{M} \varepsilon^j \varphi_j(x, t) ,
\]
the functions $V^M, W^M$ (rapidly varying near $\Gamma_t$ and the external boundary) give an analog of the internal expansion,

$$V^M(\eta, \tau, x, t, \varepsilon) = \sum_{j=1}^{M} \varepsilon^{j-1} \left\{ U_j(\eta, x, t) + Y_j(\tau, x', t) \right\},$$

$$W^M(\eta, \tau, x, t, \varepsilon) = \chi(\eta, x, t) + \sum_{j=1}^{M} \varepsilon^j \left\{ W_j(\eta, x, t) + Z_j(\tau, x', t) \right\}.$$

Here $x_N$ is the distance from a point $x \in \Omega$ to a point $x' \in \partial\Omega$ along the internal normal,

$$S, \theta_j, \varphi_j \in C^\infty(\bar{Q}), \quad Y_j(\tau, x', t), Z_j(\tau, x', t) \in \mathcal{P},$$

$$\chi(\eta, x, t) \in \mathcal{S}, \quad U_j(\eta, x, t), W_j(\eta, x, t) \in \mathcal{H},$$

and

$$U_j^- = 0, \quad W_j^- = 0, \quad \frac{\partial S}{\partial t} \bigg|_{\Gamma_t} \neq 0, \quad \Gamma_t = \left\{ x \in \Omega, \ S(x, t) = 0 \right\}.$$

By Lemma 1, without loss of generality, we can assume that $S = t - \psi(x)$, $\chi = \chi(\eta, x)$.

Furthermore, outside small neighborhoods of the soliton support and the external boundary, i.e., as $\eta \to \pm \infty$, and $\tau \to \infty$, we have $\theta_j + U_j + Y_j \approx \theta_j + U_j^\pm$. Since all the functions in the latter relation are arbitrary for the moment, we can redefine $\theta_j$ (for example, by setting $\theta_j := \theta_j + U_j^+$ or $\theta_j := \theta_j + U_j^-$) and thus set one of the limiting values $U_j^+$ or $U_j^-$ equal to zero. For definiteness, we set $U_j^-$ and, similarly, $W_j^- = 0$.

Substituting (13) into equations (3), we get the relations

$$(15) \quad \frac{\partial}{\partial t} (\vartheta^M + \Phi^M) - \Delta \vartheta^M - f = \left[ \frac{1}{\varepsilon} \left\{ \mathcal{L}_2 V^M - \frac{\partial}{\partial \eta} W^M \right\} - ight.$$

$$- \left\{ \left( \frac{\partial}{\partial \eta} + \mathcal{L}_1 \right) V^M + \frac{\partial}{\partial t} W^M \right\} + \varepsilon \left( \Delta_x - \frac{\partial}{\partial t} \right) V^M \right]_{\eta = S/\varepsilon, \tau = x_N/\varepsilon},$$
(16) \[
\left[ \frac{1}{\varepsilon^2} \hat{\ell}_2 \left( \hat{\ell}_2 \mathcal{W}^M + \mathcal{W}^M - (\Phi^M + \mathcal{W}^M)^3 \right) + \right.
\left. + \frac{1}{\varepsilon} \left\{ \left( \kappa \frac{\partial}{\partial \eta} - \hat{\ell}_2 \hat{\ell}_1 \right) \mathcal{W}^M - \hat{\ell}_1 \left( \hat{\ell}_2 \mathcal{W}^M + \mathcal{W}^M - (\Phi^M + \mathcal{W}^M)^3 \right) \right\} \right]
+ \left\{ (\hat{\ell}_2 \Delta_x + \hat{\ell}_1^2) \mathcal{W}^M + \kappa_1 \hat{\ell}_2 \mathcal{V}^M + \kappa \frac{\partial}{\partial t} (\Phi^M + \mathcal{W}^M) \right.
+ \Delta_x \left( \hat{\ell}_2 \mathcal{W}^M + \Phi^M + \mathcal{W}^M - (\Phi^M + \mathcal{W}^M)^3 \right) \right\}
+ \kappa \frac{\partial}{\partial t}(\Phi^M + \mathcal{W}^M) + \Delta_x \left( \hat{\ell}_1 \mathcal{W}^M - \kappa_1 \vartheta^M \right) + \varepsilon \left\{ \hat{\ell}_1 (\Delta_x \mathcal{W}^M + \kappa_1 \mathcal{W}^M) + \Delta_x (\hat{\ell}_1 \mathcal{W}^M - \kappa_1 \vartheta^M) \right\} + \varepsilon^2 \Delta_x \left\{ \Delta_x (\Phi^M + \mathcal{W}^M) + \kappa_1 \mathcal{V}^M \right\} \bigg|_{\eta = S/\varepsilon, \tau = x_N/\varepsilon} = 0 .
\]

Here, as usually in the two-scale method, the expressions in square brackets are considered as functions of independent variables \( \eta, x, \) and \( t, \)
\[
\hat{\ell}_2 = |\nabla \psi|^2 \frac{\partial^2}{\partial \eta^2} + |\nabla x_N|^2 \frac{\partial^2}{\partial \tau^2} , \quad \hat{\ell}_1 = \frac{\partial}{\partial \eta} \hat{\Pi} - \frac{\partial}{\partial \tau} \hat{\Pi}_b ,
\hat{\Pi} = 2(\nabla \psi, \nabla_x) + \Delta \psi , \quad \hat{\Pi}_b = 2(\nabla x_N, \nabla_x) + \Delta x_N .
\]

First, let us obtain the regular terms of expansion (13). Passing to the limit as \( \eta \to \pm \infty, \tau \to \infty, \) from (15), (16) we get
\[
\frac{\partial}{\partial t}(\vartheta^M + \Phi^M) - \Delta \vartheta^M - f = - \frac{\partial}{\partial t} \mathcal{W}^M - \varepsilon \left( \frac{\partial}{\partial \tau} - \Delta \right) \mathcal{W}^M ,
\]
\[
(17) \kappa \frac{\partial}{\partial t}(\Phi^M + \mathcal{W}^M) + \Delta (\Phi^M + \mathcal{W}^M - (\Phi^M + \mathcal{W}^M)^3) = \varepsilon \kappa_1 \Delta \vartheta^M + \varepsilon^2 \Delta (\Delta (\Phi^M + \mathcal{W}^M) + \kappa_1 \mathcal{V}^M) .
\]

Introducing the notation
\[
(18) \quad \theta_j^\pm = \theta_j + U_j^\pm , \quad \Phi_j^\pm = \varphi_j + W_j^\pm , \quad j = 1, 2, \ldots, M ,
\]
and setting the terms of the order \( \mathcal{O}(\varepsilon^j) \) equal to zero, we get equations (8), (9) for the leading terms, as well the following equations for the lower terms of the asymptotic expansion
\[
(19) \quad \left( \frac{\partial}{\partial t} - \Delta \right) \theta_k^\pm = f_{k, \theta}(x, t) , \quad x \in \mathcal{D}_t^\pm , \quad t > 0 ,
\]
\[
\kappa \frac{\partial}{\partial t} \Phi_k^\pm - \Delta \left( (3 \varphi_0^2 - 1) \Phi_k^\pm \right) = f_{k, \varphi}(x, t) .
\]
Here \( k = 1, \ldots, M, D^+_i \) are subdomains of \( \Omega \) such that

\[
D^+_i = \left\{ x \in \Omega, \psi(x) < t \right\}, \quad D^-_i = \left\{ x \in \Omega, \psi(x) > t \right\}, \quad \Omega = D^+_i \cup D^-_i \cup \Gamma_t,
\]

\( f^\pm_k(x, t), f^\pm_k(x, t) \) are functions of the previous terms of the expansion and their derivatives. In particular, \( f^\pm_{1,0} = -\partial \Phi^\pm_1 / \partial t, f^\pm_{1,\varphi} = \kappa_1 \Delta \theta_0 \).

Now we note that the supports of fast variations of boundary-layer functions and of functions rapidly varying on a neighborhood of \( \Gamma_t \) do not intersect (up to terms \( \mathcal{O}(\varepsilon^\infty) \)), since \( \text{dist}(\partial \Omega, \Gamma_t) \geq \text{const} \). Hence the solution in a neighborhood of \( \Gamma_t \) and in a neighborhood of \( \partial \Omega \) is constructed differently.

Let us consider a neighborhood of \( \Gamma_t \). Passing to the limit as \( \varepsilon \to 1 \), we obtain that the terms in (15), (16) belong to the space \( \mathcal{S} \), since relations (17) hold. So we can use the method developed for phase transition problems by V. Danilov, G. Omel’yanov and E. Radkevich [7, 8, 20, 21, 24]: step by step we decompose the coefficients at \( \varepsilon^j \) in (15), (16), \( j = -2, -1, 0, \ldots \), into the Taylor expansion at the point \( t = \psi(x) \) and use the relation \( \eta = (t - \psi)/\varepsilon \). Further, passing to the functions of independent variables \( \eta, x \), we obtain the asymptotic solution on the surface \( T_t \). Finally, we define a sufficiently smooth extension of these functions on \( R^1_\eta \times Q \), so that the lower terms of the asymptotic expansion exist and belong to the space \( \mathcal{H} \).

Let us denote \( \hat{F} = F(\eta, x, t)|_{t=\psi(x)} \) and consider the terms \( \mathcal{O}(\varepsilon^{-2}) \) in (16):

\[
\frac{\partial^2}{\partial \eta^2} \left\{ |\nabla \psi|^2 \frac{\partial^2 \hat{\chi}}{\partial \eta^2} + \hat{\chi} - (\hat{\varphi}_0 + \hat{\chi})^3 \right\} = 0.
\]

After the integration, for \( \hat{\chi} \in \mathcal{S} \) we get

\[
|\nabla \psi|^2 \frac{\partial^2 \hat{\chi}}{\partial \eta^2} + \hat{\chi} - (\hat{\varphi}_0 + \hat{\chi})^3 = c, \quad \hat{\chi} \to 0 \quad \text{as} \ \eta \to \pm \infty.
\]

Choosing \( c = -\hat{\varphi}_0^3 \), we get that the solution on \( T_t \) has the form (12), where the “constant” of integration \( \psi_1 = \psi_1(x) \) is an arbitrary function from \( C^\infty(\Omega) \).

Let us extend \( \hat{\chi} \) (defined on the section \( (x, t) \in T_T = \bigcup_{t \in [0,T]} \Gamma_t, \eta \in R^1 \)) by the identity to \( \chi \equiv \hat{\chi}(\eta, x) \) for all \((x, t) \in Q, \eta \in R^1 \). Now, setting the terms \( \mathcal{O}(\varepsilon^{-2+k}) \) equal to zero, from (15), (16) we get the following equations

\[
\frac{\partial^2 \hat{U}_k}{\partial \eta^2} = \hat{F}_k^\theta, \quad \hat{U}_k \to 0 \quad \text{as} \ \eta \to -\infty,
\]

\[
\frac{\partial^2 \hat{W}_k}{\partial \eta^2} = \hat{F}_k^\varphi, \quad \hat{W}_k \to 0 \quad \text{as} \ \eta \to -\infty.
\]
Here
\[ \hat{L} = |\nabla \psi|^2 \frac{\partial^2}{\partial \eta^2} + 1 - 3(\hat{\varphi}_0 + \chi)^2 , \]
\( \tilde{F}_k^\varphi, \tilde{F}_k^\theta \) are functions of \( \theta_0, \varphi_0, ..., U_{k-1}, W_{k-1} \) and of their derivatives at the point \( t = \psi(x) \), \( k = 1, 2, ..., M \). In particular,
\[
\tilde{F}_1^\varphi = \frac{\partial^2}{\partial \eta^2} \left\{ \Pi \frac{\partial \chi}{\partial \eta} + 3(\hat{\varphi}_0 + \chi)^2 \varphi_1 \right\} + 3 \frac{\partial \varphi_0}{\partial t} \left( 2 + \eta \frac{\partial}{\partial \eta} \right) \frac{\partial}{\partial \eta} (\varphi_0 + \chi)^2 - \kappa |\nabla \psi|^{-2} \frac{\partial \chi}{\partial \eta} |_{t=\psi} ,
\]
\[
\tilde{F}_1^\theta = |\nabla \psi|^{-2} \frac{\partial \chi}{\partial \eta} .
\]

It is not too difficult to prove that the following statement holds (see also [7, 12]).

**Lemma 2.** The solutions \( \hat{U}_k \in \mathcal{H}, \hat{W}_k \in \mathcal{H} \) of (21), (22) exist if and only if
\[
\tilde{F}_k^\varphi \in \mathcal{S} , \quad \tilde{F}_k^\theta \in \mathcal{S} ,
\]
\[
\int_{-\infty}^{\infty} \tilde{F}_k^\varphi \, d\eta = 0 , \quad \int_{-\infty}^{\infty} \tilde{F}_k^\theta \, d\eta = 0 ,
\]
\[
\int_{-\infty}^{\infty} \tilde{f}_k^\varphi \frac{\partial \chi}{\partial \eta} \, d\eta = 0 ,
\]
where
\[
\tilde{f}_k^\varphi = \int_{-\infty}^{\eta'} \int_{-\infty}^{\eta''} \tilde{F}_k^\varphi (\eta'', x) \, d\eta'' \, d\eta' .
\]

Using (23), (24), it is easy to see that the conditions (25), (26) hold automatically for \( k = 1 \).

Further, since
\[
\tilde{f}_1^\varphi = \Pi \frac{\partial \chi}{\partial \eta} + 3(\varphi_1 + \xi \varphi_0) (2 \varphi_0 \chi + \chi^2) |_{t=\psi} + \sqrt{2} \kappa \beta \ln(J^2 R) ,
\]
\[
R = \left\{ e^\xi + 4\hat{\varphi}_0 - 2\sqrt{2} Q \right\} \left\{ e^\xi + 4\hat{\varphi}_0 + 2\sqrt{2} Q \right\}^{-1} ,
\]
where notation (12) is used, simple calculations yield the statement

**Lemma 3.** For \( k = 1 \) condition (27) is equivalent to equation (10).
Now, since (25)–(27) are satisfied for $k = 1$, we can obtain the functions $\tilde{U}_1$, $\tilde{W}_1$:

$$
\tilde{U}_1 = \tilde{U}_1^+(x) \zeta(\eta, x), \quad \tilde{W}_1 = \tilde{W}_1^+(x) \zeta(\eta, x) + \omega_1(\eta, x).
$$

Here

$$
\zeta = \frac{\sqrt{2}}{a} \ln(J^2 R) \in \mathcal{H}, \quad \zeta^+ = 1, \quad \zeta^- = 0,
$$

$$
\tilde{U}_1^+ = -a v_\nu, \quad \tilde{W}_1^+ = \tilde{U}_1^+ \kappa/Q^2,
$$

$$
\omega_1(\eta, x) = \omega_{1,1}(\eta, x) + \psi_2(x) \chi_\eta(\eta, x) \in S,
$$

$\psi_2$ is the “constant” of integration, $a = 2I/\varphi_0$.

Let us fix the functions $\theta_1(x, t), \varphi_1(x, t)$ so that $\theta_1, \varphi_1$ are sufficiently smooth on $Q$ and that $\theta_1 = \theta_1^+, \varphi_1 = \Phi_1^+$ on $\mathcal{D}_t^-$ for all $t \geq 0$. Let us define the functions $\theta_{1c}^-(x, t), \Phi_{1c}^-(x, t)$ as $\theta_{1c}^- = \theta_1, \Phi_{1c}^- = \varphi_1$ for all $(x, t) \in Q$, and let $\theta_{1c}^+, \Phi_{1c}^+$ be sufficiently smooth extensions of $\theta_1^+, \Phi_1^+$ in $\mathcal{D}_t^+ \cup \Gamma_{t, \delta}$, such that the heat equations (19) are satisfied for $k = 1$. Here $0 < \delta < 1$ is an arbitrary number and $\Gamma_{t, \delta} \subset \mathcal{D}_t^-$. We now can define the extensions $U_1, W_1$ by

$$
U_1 = u_1(x, t) \zeta(\eta, x), \quad W_1 = w_1(x, t) \zeta(\eta, x) + \omega_1(\eta, x),
$$

where

$$
u_1 = \theta_{1c}^+ - \theta_{1c}^-, \quad w_1 = \Phi_{1c}^+ - \Phi_{1c}^-.
$$

Note that outside $\Gamma_t$ we have

$$
\theta_1 + U_1 = \theta_1^+ + \left[\left(\theta_{1c}^- - \theta_{1c}^+\right) \mathcal{O}(e^{\pi \beta_\eta})\right], \quad x \in \mathcal{D}_t^+, \quad \eta = (t - \psi)/\varepsilon + \psi_1,
$$

$$
\varphi_1 + W_1 = \Phi_1^+ + \left[\left(\Phi_{1c}^- - \Phi_{1c}^+\right) \mathcal{O}(e^{\pi \beta_\eta})\right] + \omega_1.
$$

This implies that the expressions in square brackets are of maximal value $\mathcal{O}(\varepsilon)$ in an $e^{1-\mu}$-neighborhood of $\Gamma_t$, $0 < \mu < 1$, and they are exponentially small outside this neighborhood. Hence, the freedom in choosing the extensions $\theta_{1c}^+, \Phi_{1c}^+$ results in corrections of order $\mathcal{O}(\varepsilon)$ which automatically are taken into account when we construct the next approximations.

Thus we have the following conditions for jumps of $\theta_1^+, \Phi_1^+$ on $\Gamma_t$

$$
[\theta_1^+]_{\Gamma_t} = a v_\nu, \quad [\Phi_1^+]_{\Gamma_t} = \frac{\kappa a}{Q^2} v_\nu.
$$

Here we use the notation $[f^+]_{\Gamma_t} = f^-|_{\Gamma_t,+0} - f^+|_{\Gamma_t,-0}$ and note that the vector $\nu$ is directed from $\mathcal{D}_t^+$ to $\mathcal{D}_t^-$. 

**SOLITON TYPE ASYMPTOTIC SOLUTIONS**
Let us consider equations (21), (22) in the case \( k = 2 \). The right-hand sides of these equations belong to \( S \), since equation (8) and the first equation (19) hold for \( k = 1 \). Further, after some calculations, we get the following statement

**Lemma 4.** For \( k = 2 \) conditions (26) are equivalent to the equalities

\[
\left[ \frac{\partial \theta^\pm_k}{\partial \nu} \right]_{\Gamma_t} = -a v_{\nu} \left\{ \left( 1 + \frac{\kappa}{Q^2} \right) v_{\nu} + K_t - \frac{4}{abQ} \frac{\partial \varphi_0}{\partial \nu} \right\},
\]

\[
\left[ \frac{\partial}{\partial \nu} \left( (3 \varphi_0^2 - 1) \Phi^\pm_1 \right) \right]_{\Gamma_t} = -12 |\nabla \psi| \frac{\partial \varphi_0}{\partial t} \left\{ QK_t - v_{\nu} \text{div} (Q \nabla \psi) + \frac{a}{2} \frac{\partial \varphi_0}{\partial \nu} \right\}
\]

\[+ v_{\nu}^2 \frac{\kappa a}{2Q^2} (\kappa + Q^2 \Delta \psi) .\]

Finally, for \( k = 2 \), after some trivial but cumbersome calculations condition (27) can be transformed to the following linear inhomogeneous equation for the phase correction \( \psi_1 \):

\[
(31) \quad \kappa \frac{\partial \psi_1}{\partial \nu} = G_1 \hat{K}' \psi_1 + f^{\psi_1}(x), \quad \frac{\psi_1}{\Gamma_0} = 0 .
\]

Here \( \hat{K}' \) is the variation of the operator \( K_t \) from (10), the right-hand side \( f^{\psi_1}(x) \) depends on the functions \( \psi, \theta_0, \varphi_0, \theta^\pm_1, \Phi^\pm_1 \).

The following constructions are performed similarly:

1. Calculating \( \hat{U}^\pm_k, \hat{W}^\pm_k \), we get conditions for jumps of the functions \( \theta^\pm_k, \Phi^\pm_k \) on \( \Gamma_t \);

2. By using formulas similar to (28), we define the extensions \( U_k, W_k \) on \( Q \). Since \( \theta^\pm_k, \Phi^\pm_k \) is the solution of (19), we see that conditions (25) hold;

3. Conditions (26) imply conditions for the normal derivatives of \( \theta^\pm_{k-1}, \Phi^\pm_{k-1} \) on \( \Gamma_t \);

4. Condition (27) yields an equation (similar to (31)) for the “constant” of integration \( \psi_k \).

In fact, since

\[ f \left( \beta (\eta + \psi_1) + \varepsilon \beta f'_{\eta} \psi_2 + \varepsilon^2 \beta f'_{\eta} \psi_3 + \cdots \right) = f \left( \beta (\eta + \psi_1 + \varepsilon \psi_2 + \varepsilon^2 \psi_3 + \cdots) \right) + O(\varepsilon^2) , \]

the functions \( \psi_k, k \geq 1 \), are the lower corrections to the principle phase \( \psi \). So, these functions describe the front of the soliton wave more precisely.

We must also pose the initial conditions for the equations (3) as well as for the heat equations (8), (9), (19). Since \( \theta_0, \varphi_0 \) are smooth functions on \( \Omega \times [0, T] \)
and $\theta_k^\pm$, $\Phi_k^\pm$ are smooth functions on $D_t^\pm \times [0, T]$, let us define the initial data as follows

$$\theta^0 = w - \lim_{\varepsilon \to 0} \theta^0(x, \varepsilon), \quad \varphi^0 = w - \lim_{\varepsilon \to 0} \varphi^0(x, \varepsilon), \quad x \in \Omega,$$

$$\theta_k^\pm = w - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \theta^0(x, \varepsilon) - \theta_0 - \sum_{j=1}^{k-1} \varepsilon^j (\theta_j + U_j + Y_j) \right), \quad x \in D_0^\pm,$$

$$\varphi_k^\pm = w - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \varphi^0(x, \varepsilon) - \varphi_0 - \chi - \sum_{j=1}^{k-1} \varepsilon^j (\varphi_j + W_j + Z_j) \right), \quad x \in D_0^\pm.$$

These formulas yield the initial data (8), (9) and the initial data for equations (19):

$$\theta_k^\pm |_{t=0} = \theta_k^\pm(x), \quad \varphi_k^\pm |_{t=0} = \varphi_k^\pm(x), \quad x \in D_0^\pm. \quad (32)$$

Furthermore, a natural form of the initial value of $\varphi$ is $\varphi|_{t=0} = \varphi^0(\rho_0(x)/\varepsilon, \varphi(x), \varepsilon)$, where $\rho_0$ is the distance function. Nevertheless, the constructed asymptotic depends on the variable $\eta = (t - \psi(x))/\varepsilon + \psi_1$ with functions $\psi$ and $\psi_1$ unknown beforehand. Note that $\psi|_{\Gamma_0} = 0$ and $\psi_1|_{\Gamma_0} = 0$, the unit vector $\nabla(\psi/|\nabla\psi|)|_{\Gamma_0}$ is normal to $\Gamma_0$ and directed opposite to $\nabla \rho_0|_{\Gamma_0}$. Therefore, for any function $f(\eta, x) \in \mathcal{H}$ we have

$$f\left( \frac{\rho_0}{\varepsilon}, x \right) = f\left( \frac{-\psi + \varepsilon \psi_1}{|\nabla\psi| \varepsilon} + \frac{1}{\varepsilon} g_1 + g_2, x \right),$$

$$= f\left( \frac{\eta}{|\nabla\psi|}, x \right) + \varepsilon f''_\eta \left( \frac{\eta}{|\nabla\psi|}, x \right) \left( \eta \frac{\partial g_1}{\partial \nu} |_{\Gamma_0} + \frac{1}{2} \eta^2 \frac{\partial^2 g_1}{\partial \nu^2} |_{\Gamma_0} \right) + \mathcal{O}(\varepsilon^2),$$

where $g_1 = \rho_0 + \psi/|\nabla\psi|$, $g_2 = -\psi_1/|\nabla\psi|$, $\partial/\partial \nu = |\nabla\psi|^{-1} (\nabla\psi, \nabla)$, and $\eta = -\psi/e + \psi_1$ for $t = 0$. We also take into account that $f''_\eta \in \mathcal{S}$, and hence, the functions $|\eta^h f''_\eta|$ are bounded in $C$ for all $k \leq M$.

Moreover, in §3 we prove that the initial perturbations $\mathcal{O}(\varepsilon^3)$ (in the sense of $L^2(\Omega)$) do not lead to the soliton-type solution out of the stability domain. Therefore, we fix the initial data only up to the terms $\mathcal{O}(\varepsilon^2)$. The above constructions imply that the behavior of smooth (for the $\varepsilon > 0$) functions $\theta^0(x, \varepsilon), \varphi^0(x, \varepsilon)$ may be arbitrary outside an $\varepsilon$-neighborhood of $\Gamma_0$, but $\theta^0$ and the lower terms of $\theta^0$ (w.r.t. $\varepsilon$) must be of a special form in this neighborhood.

Now let us consider the boundary conditions on the external boundary $\partial \Omega$ and calculate the boundary-layer functions. The soliton part of the asymptotic solution satisfies both boundary conditions in (3) up to $\mathcal{O}(\varepsilon^\infty)$. So, for $\varphi$ on $\Sigma$, ...
a discrepancy in the second boundary condition arises only from the regular part of the solution, since \( \text{dist}(\Gamma_1, \partial \Omega) \geq \text{const} \) and, in general,

\[
\frac{\partial}{\partial N} \Delta \varphi_0 \bigg|_{\Sigma} \neq 0.
\]

Let us put \( Z_j = 0 \) for \( j = 1, 2 \) since

\[
\frac{\partial}{\partial N} \Delta \varepsilon^3 Z_3 \left( \frac{x}{\varepsilon}, x' \right) \bigg|_{x_N = 0} = \mathcal{O}(1).
\]

Then, using the construction common for the boundary layer asymptotic solutions, we obtain the equation for \( Z_3 \):

\[
\frac{\partial^2}{\partial \tau^2} Z_3 - q Z_3 = 0, \quad Z_3 \to 0, \quad \tau \to \infty,
\]

where \( q = 3 \varphi_0^2 |_{\Sigma} - 1 > 0 \). Obviously,

\[
Z_3 = c_3(x', t) \exp(-\sqrt{q} \tau).
\]

Now we can see that the boundary condition

\[
\frac{\partial^3}{\partial \tau^3} Z_3 \bigg|_{\tau = 0} = \frac{\partial}{\partial N} \Delta \varphi_0 \bigg|_{\Sigma}
\]

leads to the formula

\[
c_3 = -q^{-3/2} \left. \frac{\partial}{\partial N} \Delta \varphi_0 \right|_{\Sigma}.
\]

Further, we note that the appearance of the boundary-layer function \( \varepsilon^3 Z_3 \) necessarily implies a correction of the Neumann condition in the term \( \mathcal{O}(\varepsilon^2) \). Let, for definiteness, \( \partial \Omega \cap \partial D^* = \partial \Omega \). Then the Neumann condition for \( \Phi_\Sigma \) has the form

\[
\frac{\partial \Phi_\Sigma}{\partial N} \bigg|_{\Sigma} = \frac{\partial Z_3}{\partial \tau} \bigg|_{\tau = 0} = \frac{1}{q} \left. \frac{\partial}{\partial N} \Delta \varphi_0 \right|_{\Sigma}.
\]

The appearance of the boundary-layer functions \( Z_k \) implies the boundary-layer terms \( Y_j \) in the \( \theta \) asymptotic expansion. Since the boundary \( \partial \Omega \) is fixed, for \( j = 1, \ldots, 4 \) we have \( Y_j = 0 \) and for \( Y_5 \) we have the equation:

\[
\frac{\partial^2 Y_5}{\partial \tau^2} = \frac{\partial}{\partial t} Z_3(\tau, x', t), \quad Y_5 \to 0, \quad \tau \to \infty.
\]

Therefore

\[
Y_5 = -q^{-5/2} \Delta \varphi_0 \left. \frac{\partial^2}{\partial N \partial \tau} \Delta \varphi_0 \right|_{\Sigma} \exp(-\sqrt{q} \tau).
\]
Conversely, the appearance of $Y_5$ leads to a correction of the Neumann condition for the temperature in the term $O(\varepsilon^4)$. Thus, the Neumann condition for $\theta^-_4$ has the form

$$\frac{\partial \theta^-_4}{\partial N} \big|_\Sigma = \frac{\partial Y_5}{\partial \tau} \big|_{\tau=0} = \frac{1}{q^2} \frac{\partial^2}{\partial N \partial t} \Delta \varphi_0 \big|_\Sigma.\tag{36}$$

Finally, the asymptotic expansion in a small neighborhood of $\partial \Omega$ has the following form

$$\theta = \theta_0(x,t) + \sum_{j=1}^4 \varepsilon^j \theta^-_j(x,t) + \sum_{j=5}^M \varepsilon^j \left( \theta^-_j(x,t) + Y_j(\tau,x',t) \right) + O(\varepsilon^{M+1}),$$

$$\varphi = \varphi_0(x,t) + \sum_{j=1}^2 \varepsilon^j \Phi^-_j(x,t) + \sum_{j=3}^M \varepsilon^j \left( \Phi^-_j(x,t) + Z_j(\tau,x',t) \right) + O(\varepsilon^{M+1}).$$

Here $Z_3, Y_5 \in \mathcal{P}$ are described above, $Z_k \in \mathcal{P}, k \geq 4$, and $Y_j \in \mathcal{P}, j \geq 6$, are calculated from linear inhomogeneous problems like (32), (33), (35). In turn, we obtain the boundary conditions for problems (19):

1. Conditions (8), (9) for $\theta_0, \varphi_0$;

2. Conditions

$$\frac{\partial \theta^-_j}{\partial N} \big|_\Sigma = 0, \quad \frac{\partial \Phi^-_j}{\partial N} \big|_\Sigma = 0$$

for $\theta^-_j, j = 1, ..., 3, \Phi^-_1$;

3. Conditions (34), (36) for $\Phi^-_2, \theta^-_4$;

4. Conditions

$$\frac{\partial \theta^-_j}{\partial N} \big|_\Sigma = \frac{\partial Y_{j+1}}{\partial \tau} \big|_{\tau=0}, \quad \frac{\partial \Phi^-_k}{\partial N} \big|_\Sigma = \frac{\partial Z_{k+1}}{\partial \tau} \big|_{\tau=0}$$

for $\theta^-_j, j \geq 5$ and $\Phi^-_k, k \geq 3$.

Theorem 1 is proved. ■

Moreover, analyzing our construction, we obtain the statement.
Theorem 2. Let the assumptions of Theorem 1 hold. Then for any integer $M \geq 0$ there exist the functions

$$
\theta_{as}^M = \theta_0 + \sum_{j=1}^{M} \varepsilon^j (\theta_j + U_j + Y_j) + \varepsilon^{M+1}(U_{M+1} + Y_{M+1}) ,
$$

$$
\varphi_{as}^M = \varphi_0 + \chi + \sum_{j=1}^{M} \varepsilon^j (\varphi_j + W_j + Z_j) + \varepsilon^{M+1}(W_{M+1} + Z_{M+1}) ,
$$

such that

$$
\frac{\partial}{\partial t}(\theta_{as}^M + \varphi_{as}^M) - \Delta \theta_{as}^M - f(x,t) = \varepsilon^M F_M^\theta ,
$$

$$
\kappa \frac{\partial \varphi_{as}^M}{\partial t} + \Delta (\varepsilon^2 \Delta \varphi_{as}^M + \varphi_{as}^M - (\varphi_{as}^M)^3 + \varepsilon \kappa_1 \theta_{as}^M) = \varepsilon^M F_M^\varphi ,
$$

$$
\frac{\partial \theta_{as}^M}{\partial N} \bigg|_\Sigma = \varepsilon^{M+1} F_M^\theta , \quad \frac{\partial \varphi_{as}^M}{\partial N} \bigg|_\Sigma = 0 , \quad \frac{\partial}{\partial N} \Delta \varphi_{as}^M \bigg|_\Sigma = \varepsilon^{M-1} F_M^\varphi ,
$$

$$
\theta_{as}^M |_{t=0} = \bar{\theta} + \varepsilon (\varphi_{as}^{1+}) + \bar{\varphi} , \quad \varphi_{as}^M |_{t=0} = \varphi_0 + \chi \bigg|_{t=0} + \varepsilon (\varphi_{as}^{1+} + \bar{\varphi}) .
$$

Here $\bar{\theta}, \bar{\varphi}, F_M^{\theta,\varphi}, F_M^{\theta,\varphi}$ are (smooth for $\varepsilon > 0$) functions such that

$$
\| \hat{\theta} ; L^2(\Omega) \| + \| \hat{\varphi} ; L^2(\Omega) \| \leq c_0 \sqrt{\varepsilon} ,
$$

$$
\| F_M^\theta ; C(\hat{Q}) \| + \| F_M^\varphi ; C(\hat{Q}) \| \leq c_1 ,
$$

$$
\| F_M^\theta ; C(\Sigma) \| + \| F_M^\varphi ; C(\Sigma) \| \leq c_2 ,
$$

$$
\| F_M^\theta ; L^2(\Omega) \| + \| F_M^\varphi ; L^2(\Omega) \| \leq c_3 \sqrt{\varepsilon} ,
$$

where the constants $c_j$ are independent of $\varepsilon$.

### 3 – Justification of the soliton type asymptotic solution

In this section we shall obtain estimates for the differences between the exact $\theta, \varphi$ and asymptotic $\theta_{as}^M, \varphi_{as}^M$ solutions of problem (3). Let us introduce the notation $\sigma = \theta - \theta_{as}^M, \omega = \varphi - \varphi_{as}^M$ and let the initial data $\theta^0, \varphi^0$ exhibit a special behavior. Then, from (3) and (38)–(40), we get the following problem for the remainders $\sigma, \omega$:

$$
\frac{\partial}{\partial t} (\sigma + \omega) - \Delta \sigma = -\varepsilon^M F_M^\theta ,
$$
(43) \[ \kappa \frac{\partial \omega}{\partial t} + \Delta (\varepsilon^2 \Delta \omega + \omega (1 - 3 \varphi_M^2 - 3 \varphi_M \omega - \omega^2) + \varepsilon \kappa_1 \sigma) = -\varepsilon M F_M^\varphi, \]

(44) \[ \frac{\partial \sigma}{\partial N} \bigg|_{\Sigma} = -\varepsilon^{M+1} F_M^\varphi, \quad \frac{\partial \omega}{\partial N} \bigg|_{\Sigma} = 0, \quad \frac{\partial}{\partial N} \Delta \omega \bigg|_{\Sigma} = -\varepsilon^{M-1} F_M^\varphi, \]

\[ \sigma|_{t=0} = -\varepsilon^{M+1/2} f_M^\varphi, \quad \omega|_{t=0} = -\varepsilon^{M+1/2} \tilde{f}_M^\varphi. \]

Here \( F_M^\varphi, f_M^\varphi \) are smooth functions satisfying (41), \( f_M^\varphi \) are functions such that

(45) \[ \| f_M^\varphi;L^2(\Omega) \| + \| f_M^\varphi;L^2(\Omega) \| \leq c \sqrt{\varepsilon} \]

with constant \( c \) independent of \( \varepsilon \). To simplify the notation, we omit the superscript denoting asymptotic solutions.

The main result of this section is

**Theorem 3.** Let there exist a sufficiently smooth solution of problem (3) on the time interval \([0, T]\), where the quantity \( T > 0 \) is independent of \( \varepsilon \). Let also the assumptions of Theorem 1 be satisfied, \( M \geq 2 \) and there exist a constant \( \gamma > 0 \) such that \( \varphi_0 - 1/\sqrt{3} \geq \gamma \) uniformly in \( x \in \Omega, t \in [0, T] \). Then the estimates

(46) \[ \| \omega;L^\infty((0, T);L^2(\Omega)) \| + \| \sigma;L^\infty((0, T);L^2(\Omega)) \| \leq c \varepsilon^{M+1}, \]

\[ \| \nabla \omega;L^2(Q) \| + \| \nabla \sigma;L^2(Q) \| \leq c \varepsilon^{M+1/2}, \quad \| \Delta \omega;L^2(Q) \| \leq c \varepsilon^{M-1/2} \]

hold with constant \( c \) independent of \( \varepsilon \).

The main obstacle to the derivation of a priori estimates (46) is a rapidly varying coefficient \( \varphi_M \) in (43). This is typical for nonlinear equations and the following summand

\[ J = \int_0^t \int_\Omega \left( \nabla \omega, \nabla (\varphi_M^2 \omega) \right) dx \, dt' \]

appears on the right-hand side of the energy inequality, while on the left-hand side we have only \( \| \omega;L^\infty(0, T;L^2(\Omega)) \| \) and \( \varepsilon^2 \| \Delta \omega;L^2(Q) \|^2 \). It is clear that trivial estimates, for example,

\[ |J| \leq c_1 \max_{x, t} |\Delta \varphi_M^2| \| \omega;L^2(Q) \|^2 + \| \nabla \omega;L^2(Q) \|^2 \]

\[ \leq c_1 \varepsilon^2 \| \omega;L^2(Q) \|^2 + c \| \nabla \omega;L^2(Q) \|^2, \]

allow us to prove that the discrepancy is bounded only for the time \( T_\varepsilon \sim \varepsilon^2 \). The first point of observation is that, to overcome this difficulty, we can rewrite the “bad” summand \( (\nabla \omega, \nabla (\varphi_M^2 \omega)) \) in the form

\[ -(3\varphi_M^2 - 1) |\nabla \omega|^2 - \varepsilon^{-2} \omega^2 \Psi_1 + |\nabla \omega|^2 \chi g_1 + \varepsilon^{-2} \omega^2 \Psi_1 (\Psi_2 + \varepsilon \Psi_3), \]
where nonnegative $\Psi_1$ belongs to $\mathcal{S}$, $\Psi_2 \in \mathcal{S}$, $\Psi_3 \in \mathcal{H}$, $g_1$ is a bounded in $C$ function. Let us recall that $\varphi_0 \geq 1/\sqrt{3} + \gamma$, where $\gamma > 0$ is a fixed number. Thus, we obtain the summand $\|\sqrt{3}\varphi^2_0 - 1|\nabla \omega|; L^2(Q)\|^2 + \varepsilon^{-2}\|\sqrt{\Psi_1} \omega; L^2(Q)\|^2$ on the left-hand side of the energy inequality and the expression

$$J_1 = \int_0^t \int_\Omega \chi g_1 |\nabla \omega|^2 \, dx \, dt' + \frac{1}{\varepsilon^2} \int_0^t \int_\Omega \omega^2 \Psi_1 (\Psi_2 + \varepsilon \Psi_3) \, dx \, dt'$$

on the right-hand side. Obviously, if we again estimate the functions $\chi g_1$ and $\Psi_1 (\Psi_2 + \varepsilon \Psi_3)$ by the maximum of the modulus, there is no result, but we now can use the fact that the functions $\chi$ and $\Psi_1$ are bounded by a constant (in $C$) and localized (with precision up to $O(\varepsilon)$) in an $\varepsilon$-neighborhood of the free interface $\Gamma_t$. Here the main point is Lemma 6 about estimating integrals of the form

$$I = \int_{-\infty}^{\infty} v \left( \frac{x}{\varepsilon} \right) f(x) \, dx,$$

where $v(\eta) \in \mathcal{S}$ is a known function exponentially vanishing outside the point $\eta = 0$. To estimate $f$ we can use the norms in $L^k(R^1)$ only for $k = 1$ and $k = 2$. Lemma 6 implies that for sufficiently small $\varepsilon$ the first summand in $J_1$ is bounded from above by $o(R_\omega)$, where $R_\omega = \gamma \|\nabla \omega; L^2(Q)\|^2 + \varepsilon^{-2} \|\Delta \omega; L^2(Q)\|^2$.

Since on the left-hand side of the energy inequality we have $k R_\omega$ with a constant $k > 0$ independent of $\varepsilon$, we see that the first summand in $J_1$ is no obstacle to the derivation of a priori estimate for all finite $T$.

Similarly, by Lemma 6, we now can prove that for sufficiently small $\varepsilon$ the second summand in $J_1$ has the upper bound $k (R_\omega + 3 \varepsilon^{-2} \|\omega \sqrt{\Psi_1}; L^2(0, t; L^2(\Omega))\|^2)/4$. Therefore, the second summand is no more an obstacle to the derivation of a priori estimate for all finite $T$.

It should be noted that, justifying a boundary-layer asymptotics for the semilinear Dirichlet problem, M. Berger and L. Fraenkel [3] proved a statement similar to Lemma 6. However, in the boundary-layer situation the fast varying asymptotics $v$ is localized in a small neighborhood of the external boundary. Moreover, the discrepancy vanishes on the boundary. The condition $f|_{\partial \Omega} = 0$ used in [3] considerably simplifies the estimation of the integral $I$. In the phase transition problems the remainder does not necessarily vanish on the free boundary, so, in these problems one cannot use the estimate derived by M. Berger and L. Fraenkel.

To prove Theorem 3 we first need to obtain an auxiliary result.
**Theorem 4.** Under the assumptions of Theorem 3, the following estimates hold with constant $c$ independent of $\varepsilon$.

$$\| \omega; L^\infty((0,T); L^2(\Omega)) \| + \| \sigma; L^\infty((0,T); L^2(\Omega)) \| \leq c \varepsilon^{M+1/2},$$

$$\| \nabla \sigma; L^2(Q) \| + \| \nabla \omega; L^2(Q) \| \leq c \varepsilon^{M+1/2}, \quad \| \Delta \omega; L^2(Q) \| \leq c \varepsilon^{M-1/2},$$

Proof: Multiplying equations (42), (43) by $\sigma$, $\omega$ respectively and integrating on $\Omega$, we get the relations

$$\frac{1}{2} \frac{d}{dt} \| \sigma \|^2 + \int_\Omega \omega \sigma dx + \| \nabla \sigma \|^2 = -\varepsilon^M \int_\Omega \sigma F_M^0 dx - \varepsilon^{M+1} \int_{\partial \Omega} \sigma F_M^0 dx',$$

$$\frac{\kappa}{2} \frac{d}{dt} \| \omega \|^2 + \varepsilon^2 \| \Delta \omega \|^2 + 3 \| \omega \nabla \omega \|^2 = \int_\Omega \left( \nabla \omega, \nabla (\omega(1-3\varphi_M^2)) \right) dx - 3 \int_\Omega \left( \nabla \omega, \nabla (\varphi_M^2) \right) dx + \varepsilon \kappa_1 \int_\Omega \left( \nabla \omega, \nabla \sigma \right) dx$$

$$- \varepsilon^M \int_\Omega \omega F_M^\varphi dx + \varepsilon^{M+1} \int_{\partial \Omega} \omega F_M^\varphi dx' + \varepsilon^{M+2} \kappa_1 \int_{\partial \Omega} \omega F_M^\varphi dx'.$$

Here and below $\| f \|$ denotes the $L^2(\Omega)$ norm of $f$.

Further, multiplying (42) by $\omega$, integrating on $\Omega$ and summing with (48), we obtain

$$\frac{1}{2} \frac{d}{dt} \left\{ \| \omega \|^2 + 2 \int_\Omega \omega \sigma dx \right\} + \| \nabla \sigma \|^2 + \int_\Omega \left( \nabla \omega, \nabla \sigma \right) dx = -\varepsilon^M \int_\Omega (\omega + \sigma) F_M^0 dx - \varepsilon^{M+1} \int_{\partial \Omega} (\omega + \sigma) F_M^0 dx'.$$

Let us fix a constant $K > 1/\gamma$. Multiplying (49) by $K$ and summing with (50), we get the equality

$$\frac{1}{2} \frac{d}{dt} \left\{ (1 + \kappa K) \| \omega \|^2 + 2 \int_\Omega \omega \sigma dx \right\} + \| \nabla \sigma \|^2 + \varepsilon^2 K \| \Delta \omega \|^2 + 3K \| \omega \nabla \omega \|^2 =$$

$$= K \int_\Omega \left( \nabla \omega, \nabla (\omega(1-3\varphi_M^2)) \right) dx - 3K \int_\Omega \left( \nabla \omega, \nabla (\varphi_M(\omega)^2) \right) dx$$

$$+ (\varepsilon \kappa_1 K - 1) \int_\Omega \left( \nabla \omega, \nabla \sigma \right) dx - \varepsilon^M \int_\Omega \left\{ (\omega + \sigma) F_M^0 + \omega K F_M^\varphi \right\} dx$$

$$- \varepsilon^{M+1} \int_{\partial \Omega} \left\{ (\omega + \sigma) F_M^0 - \omega K (F_M^\varphi + \varepsilon \kappa_1 F_M^\varphi) \right\} dx'.$$

We shall analyze the terms in the right-hand side of (51).
Lemma 5. Let $\varphi_M$ be the asymptotical expansion (37). Then

(52) \[ \int_\Omega \left( \nabla \omega, \nabla (\omega (1 - 3 \varphi_M^2)) \right) dx = - \int_\Omega (3 \varphi_0^2 - 1) |\nabla \omega|^2 dx - \frac{1}{\varepsilon^2} \int_\Omega \omega^2 \Psi_1 dx + I, \]

where

\[
I = \int_\Omega |\nabla \omega|^2 (g_1 \chi + \varepsilon g_2) dx + \frac{1}{\varepsilon^2} \int_\Omega \omega^2 \left( \Psi_1 (\Psi_2 + \varepsilon \Psi_3) + \varepsilon^2 \Psi_4 \right) dx,
\]

and

(53) \[ \Psi_1 = A_1 \frac{\cosh^3 \rho + A \cosh \rho + (1 + 2A)/\alpha}{(\cosh \rho + \alpha)^4} \in \mathcal{S}, \]

\[ \rho = \xi - \frac{1}{2} \ln 8b, \quad \alpha = \varphi_0 \sqrt{2/b}, \quad A = Q^2 \sqrt{2/b}, \quad A_1 = \varphi_0 A^2 / 6b, \]

$g_i, \Psi_k$ are smooth functions such that

\[ |g_i| \leq \text{const}, \quad \Psi_2 \in \mathcal{S}, \quad \Psi_3 \in \mathcal{H}, \quad |\Psi_4| \leq \text{const}. \]

Proof of Lemma 5: Using the expansion (37) and rewriting $\varphi_M$ in the form $\varphi_M = \varphi_0 + \chi + \varepsilon \varphi_M^*$, we get

(54) \[ \int_\Omega \left( \nabla \omega, \nabla (\omega (1 - 3 \varphi_M^2)) \right) dx = - \int_\Omega (3 \varphi_0^2 - 1) |\nabla \omega|^2 dx + \]

\[ + \int_\Omega |\nabla \omega|^2 (g_1 \chi + \varepsilon g_2) dx + \frac{3}{2} \int_\Omega \omega^2 \Delta (\varphi_M^2) dx, \]

where $g_1 = -3(\chi + 2 \varphi_0), \ g_2 = (2 \varphi_0 + 2 \chi + \varepsilon \varphi_M^*) \varphi_M^*$. Obviously, $g_i$ are bounded functions. Further, simple calculations yield the relation

(55) \[ \frac{1}{2} \Delta (\varphi_M^2) = \left( \frac{1}{\varepsilon^2} |\nabla \psi|^2 \left\{ \chi_n^2 + (\varphi_0 + \chi) \chi_m \right\} + \frac{1}{\varepsilon} \tilde{\Psi} + \Psi_4 \right) \bigg|_{\eta = (t - \psi)/\varepsilon}. \]

Here

\[
\tilde{\Psi} = -2 \chi_0 \left( |\nabla \psi|^2 W_{1\eta} + (\nabla \psi, \nabla \varphi_0) \right) + (\varphi_0 + \chi) \left( |\nabla \psi|^2 W_{1\eta} - \tilde{\Pi} \chi_0 \right) + (\varphi_1 + W_1) |\nabla \psi|^2 \chi_{\eta \eta},
\]

$\Psi_4$ is bounded (in the C-sense) and $\tilde{\Pi}$ is the operator described in (16). Let us rewrite the function $\chi$ in the form $\chi = -A \{ \cosh(\rho) + \alpha \}^{-1}$, where $\rho, A > 0, \ \alpha > 0$ are described in (53). Now it is easy to calculate that

\[ 3 |\nabla \psi|^2 \left\{ \chi_n^2 + (\varphi_0 + \chi) \chi_m \right\} = \Psi_1 (1 - \Psi_2), \]
where $\Psi_1$ has the form (53) and

$$\Psi_2 = \left\{ \frac{A+1}{\alpha} \cosh^2 \rho + (\alpha^2 + 2) \cosh \rho + 2\alpha \right\} \left\{ \cosh^3 \rho + A \cosh \rho + (2A + 1) / \alpha \right\}^{-1}.$$

Obviously, $\Psi_2 \in \mathcal{S}$. Finally, let us note that the term $\tilde{\Psi}$ can be written in the form $\tilde{\Psi} = \Psi_1 \Psi_3$, where $\Psi_3$ is a function from $\mathcal{H}$, since $W_1$ satisfies (22) and $\chi_{\eta_0}$ vanish like $1 / \cosh \rho$ as $\eta \to \pm \infty$. This equality, (54), and (55) complete the proof of Lemma 5.

Now, using (44), (52) and integrating (51) with respect to $t$, we get

$$\frac{1}{2} \left\{ (1 + K \kappa) \| \omega \|^2 + \| \sigma \|^2 \right\} (t) + \int_{0}^{t} \left\{ \| \nabla \sigma \|^2 + \varepsilon^2 K \| \Delta \omega \|^2 + 3K \| \omega \nabla \omega \|^2 + K \int_{\Omega} (3 \varphi_{\omega}^2 - 1) |\nabla \omega|^2 \, dx + \frac{K}{\varepsilon^2} \int_{\Omega} \omega^2 \Psi_1 \, dx \right\} \, dt' =$$

$$= \frac{\varepsilon^{2M+1}}{2} \left\{ (1 + K \kappa) \| \varphi_{\omega} \|^2 + \| \varphi_{\sigma} \|^2 + 2 \int_{\Omega} \varphi_{\omega} \varphi_{\sigma} \, dx \right\} - \int_{\Omega} \omega \sigma \, dx$$

$$+ \int_{0}^{t} \left\{ K I - 3K \int_{\Omega} \left( \nabla \omega, \nabla (\varphi_{\omega} \varphi^2) \right) \, dx + (\varepsilon K - 1) \int_{\Omega} (\nabla \omega, \nabla \sigma) \, dx \right.$$

$$- \varepsilon^M \int_{\Omega} \left( f - \kappa \omega + \omega K \varphi_{\omega} \right) \, dx$$

$$- \varepsilon^M \int_{\Omega} \left[ (\omega + \sigma) F_{\omega}^\theta - \omega K (F_{\omega}^\theta + \varepsilon K F_{\sigma}^\theta) \right] \, dx' \right\} \, dt'.$$

It is easy to see that

$$2 \left| \int_{\Omega} \omega \sigma \, dx \right| \leq \alpha_1 \| \omega \|^2 + \frac{1}{\alpha_1} \| \sigma \|^2,$$

$$2 \left| \int_{\Omega} (\nabla \omega, \nabla \sigma) \, dx \right| \leq \alpha_2 \| \nabla \omega \|^2 + \frac{1}{\alpha_2} \| \nabla \sigma \|^2,$$

$$\alpha_1 = \frac{1}{2} \left( K \kappa + \sqrt{(K \kappa)^2 + 4} \right), \quad \alpha_2 = \frac{1}{2} \left( \gamma K - 1 + \sqrt{(\gamma K - 1)^2 + 4} \right).$$

Let us choose the constant $K$ large enough, so that $\gamma K - \alpha_2 \geq 1/2$. It is possible, since $\gamma K - \alpha_2$ varies from 0 to 1. Further, by the embedding theorem (see, for example, [10]) and (41), we get

$$\varepsilon^{M+1} \left| \int_{\partial \Omega} (\omega + \sigma) F_{\omega}^\theta - \omega K (F_{\omega}^\theta + \varepsilon K F_{\sigma}^\theta) \right| \, dx' \leq$$

$$\leq c \varepsilon^{M+1} \left( \| \omega ; L^2 (\partial \Omega) \| + \| \sigma ; L^2 (\partial \Omega) \| \right) \leq c \varepsilon^{2M+2} + \frac{1}{4} \| \omega \|^2 + \frac{1}{4} \| \sigma \|^2.$$
Here and below \( c \) denotes a universal constant, \( \| f \|_k \) is the \( H^k(\Omega) \) norm of \( f \), and \( H^k \) denotes the Sobolev space.

Therefore, choosing \( \varepsilon \) small enough, from (56) we obtain the following inequality

\[
\frac{\alpha_1 - 1}{2 \alpha_1} \left\{ \| \omega \|^2 + \| \sigma \|^2 \right\} (t) + \int_0^t \left\{ \frac{1}{4} \| \nabla \omega \|^2 + \frac{1}{4} \| \nabla \sigma \|^2 + \varepsilon^2 K \| \Delta \omega \|^2 + 3K \| \omega \| \omega \| \|^2 + \frac{K}{\varepsilon^2} \int_\Omega \omega^2 \Psi_1 \, dx \right\} \, dt' \leq \]

\[
\leq c \varepsilon^{2M+1} + \int_0^t \left\{ K |I| + c \left( \| \omega \|^2 + \| \sigma \|^2 \right) + 3K \left| \int_\Omega \left( \nabla \omega, \nabla (\varphi_M \omega^2) \right) \, dx \right| \right\} \, dt'.
\]

To estimate the integral \( I \) we shall need the following

**Lemma 6** ([13], [18]). For any nonnegative functions \( f, v \),

\[
f(x) \in L^2(R^1) \cap L^1(R^1), \quad v(x) \in S(R^1),
\]

where \( S \) is the Schwartz space, there exists a constant \( \varepsilon_0 > 0 \) such that, for all \( \varepsilon \in (0, \varepsilon_0] \),

\[
\int_{-\infty}^{\infty} f(x) v\left( \frac{x}{\varepsilon} \right) \, dx \leq \delta \| f; L^1(R^1) \| + c_v(\delta) \varepsilon^{3/2} \rho(\varepsilon) \| f; L^2(R^1) \|,
\]

where \( \delta \) is a constant such that \( \delta \leq k \varepsilon^{1/2-\mu}, \mu \in (0, 1/2), \) and \( k > 0 \) is a constant. Here \( c_v(\delta) \) is a constant depending on \( \delta \) and on \( \| v(x); L^2(R^1) \cap L^1(R^1) \| \) such that \( 0 < c_v(\delta) \leq \text{const} / \delta^2 \), and \( \rho(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \).

Let us estimate the principal terms of \( I \).

**Lemma 7.** Let \( \varepsilon \) be small enough. Then

\[
\int_{\Omega} |\nabla \omega|^2 |g_1| \chi \, dx \leq \delta_1 \| \nabla \omega \|^2 + c_\chi(\delta_1) \varepsilon^{9/2} \| \Delta \omega \|^2,
\]

\[
\frac{1}{\varepsilon^2} \int_{\Omega} \omega^2 \Psi_1 \| \Psi_2 \| \, dx \leq \frac{\delta_2}{\varepsilon^2} \int_{\Omega} \omega^2 \Psi_1 \, dx + c_\chi(\delta_2) \varepsilon^2 \| \omega \|^2,
\]

where \( \delta_1 > 0 \) are arbitrary constants.

**Proof:** Denote by \( N_\mu \) a \( \mu \) neighborhood of the interface \( \Gamma_t \), where \( \mu \geq 0 \) is a constant independent of \( \varepsilon \). Since \( \chi = O(\varepsilon^\infty) \) outside \( N_\mu \), we have

\[
\int_{\Omega} |\nabla \omega|^2 |g_1| \chi \, dx \leq c_0 \int_{N_\mu} |\nabla \omega|^2 \chi \, dx + \varepsilon^2 c_0 \int_{\Omega} |\nabla \omega|^2 \, dx,
\]

where \( c_0 = \max_{x \in \Omega} |g_1| \).
Choosing $\mu$ sufficiently small, we pass to the variables $y = (y_1, \ldots, y_n)$ in $\mathcal{N}_\mu$, where $y_1$ is the coordinate normal to $\Gamma_t = \{ x \in \Omega, t = \psi(x) \}$. Then in $\mathcal{N}_\mu = \{ y, |y_1| \leq \mu, Y_i^- \leq y_i \leq Y_i^+, i = 2, \ldots, n \}$ we have
\[
\int_{\mathcal{N}_\mu} |\nabla \omega|^2 \chi \, dx \leq \prod_{i=2}^{n} \int_{Y_i^-}^{Y_i^+} \int_{-\mu}^{\mu} |\nabla \omega|^2 \, v \left( \frac{y_1}{\varepsilon}, y, t \right) J \, dy_1 \, dy_i,
\]
where $J$ is the Jacobian of this change of variables,
\[
|\nabla \omega| = |\nabla_x \omega| \bigg|_{x=x(y,t)}^{} = \varepsilon \chi(\beta(\eta + \psi_1)) \bigg|_{x=x(y,t)}^{}.
\]
By Lemma 6 and the embedding theorem for $n = 1$, we get
\[
\int_{\mathcal{N}_\mu} |\nabla \omega|^2 \chi \, dx \leq \prod_{i=2}^{n} \int_{Y_i^-}^{Y_i^+} \left\{ \delta_1 \left( \int_{-\mu}^{\mu} |\nabla \omega|^2 \, J \, dy_1 + c \varepsilon^{3/2} \rho(\varepsilon) \left( \int_{-\mu}^{\mu} |\nabla \omega|^4 J^2 \, dy_1 \right)^{1/2} \right) \right\} dy_i
\]
\[
\leq \delta_1 \| \nabla \omega \|^2 + c \varepsilon^{3/2-k_1-k_2} \rho(\varepsilon) \left\{ \varepsilon^{4k_1/3} \| \nabla \omega \|^2 + \varepsilon^{4k_2} \left( \| \nabla \omega \|^2 + \| \Delta \omega \|^2 \right) \right\}
\]
\[
\leq \delta_1 \| \nabla \omega \|^2 + c \varepsilon^{1/2} \rho(\varepsilon) \left\{ \| \nabla \omega \|^2 + \varepsilon^4 \| \Delta \omega \|^2 \right\},
\]
where we choose $k_2 = 1 + k_1/3$ and use that $J > 0$ is a bounded smooth function. This implies estimate (58). Here and below we omit the dependence of $c_\chi(\delta)$ on the function $\chi$ and on the constant $\delta$. It is clear that, choosing $\delta$, we take into account that $c_\chi(\delta) \to \infty$ as $\delta \to 0$. Similarly,
\[
\frac{1}{\varepsilon^2} \int_{\mathcal{N}_\mu} \omega^2 \Psi_1 |\Psi_2| \, dx \leq \frac{\delta_2}{\varepsilon^2} \int_{\Omega} \omega^2 \Psi_1 \, dx + c \rho(\varepsilon) \varepsilon^{-1/2} \times
\]
\[
\times \prod_{i=2}^{n} \int_{Y_i^-}^{Y_i^+} \left\| \omega \sqrt{J \Psi_1} \right\|_{L^2(-\mu, \mu)} \left\| \omega \sqrt{J \Psi_1} \right\|^{1/2} \left\| \omega \sqrt{J \Psi_1} \right\|^{1/2} \, dy_i \leq
\]
\[
\leq \frac{\delta_2}{\varepsilon^2} \left\| \omega \sqrt{\Psi_1} \right\|^2 + c \rho(\varepsilon) \left\{ \frac{1}{\varepsilon^2} \left\| \omega \sqrt{\Psi_1} \right\|^2 + \varepsilon^2 \| \omega \|^2 + \varepsilon^4 \| \nabla \omega \|^2 \right\}.
\]
Lemma 7 is proved. \( \blacksquare \)

Further, choosing $\varepsilon$ small enough, we have the trivial estimate
\[
(60) \quad \varepsilon \int_{\Omega} |\nabla \omega|^2 |g_2| \, dx + \frac{1}{\varepsilon} \int_{\Omega} \omega^2 \Psi_1 \, dy_1 + \varepsilon \Psi_3 + \varepsilon \Psi_4 \, dx \leq
\]
\[
\leq c \| \omega \|^2 + \delta_3 \left\{ \| \nabla \omega \|^2 + \frac{1}{\varepsilon^2} \int_{\Omega} \omega^2 \Psi_1 \, dx \right\},
\]
with an arbitrary constant $\delta_3 > 0$. 

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**SOLITON TYPE ASYMPTOTIC SOLUTIONS**
Let us estimate the last term in the right-hand side of (57). Using the Galliardo-Nierenberg inequality, we get that

$$\int_\Omega \left( \nabla \omega, \nabla (\varphi_\delta \omega^2) \right) dx \leq c \int_\Omega |\nabla \omega|^2 \, dx + \frac{c}{\varepsilon} \int_\Omega |\nabla \omega| \omega^2 \, dx \leq c \|\omega\|^{(8-n)/4} \|\omega\|_2^{(4+n)/4} + \frac{c}{\varepsilon} \|\nabla \omega\| \|\omega\|^{(8-n)/4} \|\omega\|_2^{n/4} \leq \delta \left\{ \|\nabla \omega\|^2 + \varepsilon^2 \|\omega\|^3 \right\} + c \varepsilon^{-2(4+n)/(4-n)} \|\omega\|^{2(8-n)/(4-n)}.$$  

Choosing $\delta = \min \{1/12, 1/24K\}$, $i = 1, \ldots, 4$, and using (58)–(61), we can transform (57) as follows

$$U(t) + c \int_0^t \left\{ \|\nabla \omega\|^2 + \|\nabla \sigma\|^2 + \varepsilon^2 \|\Delta \omega\|^2 + \varepsilon \|\nabla \sigma\|^2 \right\} \, dt' \leq c \varepsilon^{M+1} + c \int_0^t \left\{ U(t') + \varepsilon^{-r} (U(t'))^{1+\lambda} \right\} \, dt'.$$

Here

$$U(t) = \left\{ \|\omega\|^2 + \|\sigma\|^2 \right\} (t), \quad \lambda = 4/(4-n), \quad r = 2(4+n)/(4-n).$$

Let us fix a number $T_1 \in (0, T]$, $T < \infty$, and let $t \in [0, T_1]$. Then, according to the Gronwall lemma, (62) yields

$$U(t) \leq e \left( \varepsilon^{M+1} + \varepsilon^{-r} \int_0^{T_1} U^{1+\lambda} \, dt' \right).$$

Let $z = \max_{t \in [0, T_1]} U(t)$. Then

$$z \leq e \left( \varepsilon^{M+1} + \varepsilon^{-r} T_1 z^{1+\lambda} \right).$$

To analyze the last relation, we need the following lemma proved by V.P. Maslov and P.P. Mosolov.

**Lemma 8.** Let positive numbers $p$, $q$, $\lambda$ satisfy the estimate

$$q < \frac{\lambda}{1 + \lambda} \left( p(1 + \lambda) \right)^{-1/\lambda}. $$

Then the solutions of the inequality $0 \leq z \leq q + p z^{1+\lambda}$ belong to the set $[0, Z_-] \cup [Z_+, \infty)$, where the numbers $Z_+$, $Z_-$ are such that $0 \leq Z_- < q(1 + \lambda)/\lambda < Z_+$. 

In our case $p = c T_1 \varepsilon^{-r}$. Therefore, since $\varepsilon$ is small enough, the inequality (64) holds for any $M \geq 2$. Since $z = z(T_1)$ continuously depends on $T_1$ and $z(0) \leq c \varepsilon^{2M+1} < c \varepsilon^{2M+1} (1 + \lambda)/\lambda$, we obtain the estimate

$$
\max_{t \in [0, T_1]} U(t) \leq c \varepsilon^{2M+1}
$$

with a constant $c$ independent of $\varepsilon$.

It is easy to see that (65) and (62) yield the estimates (47). This completes the proof of Theorem 4.

**Proof of Theorem 3:** Let us choose the number $M' = M + 1$, where $M \geq 2$. Then, by Theorem 4, we get

$$
\theta = \theta_{as}^{M+1} + \varepsilon^{M+3/2} \sigma_{M'} , \quad \varphi = \varphi_{as}^{M+1} + \varepsilon^{M+3/2} \omega_{M'} ,
$$

where $\sigma_{M'}$, $\omega_{M'}$ are functions from $L^\infty(0, T; L^2(\Omega))$ uniformly bounded in $\varepsilon$. Nevertheless,

$$
\theta_{as}^{M+1} = \theta_{as}^{M+1} + \varepsilon^{M+1} \gamma_{M+1}^{\theta} , \quad \varphi_{as}^{M+1} = \varphi_{as}^{M+1} + \varepsilon^{M+1} \gamma_{M+1}^{\varphi} ,
$$

$$
\gamma_{M+1}^{\theta} = \theta_{M+1}(x, t) + U_{M+1}(\eta, x, t) + Y_{M+1}(\tau, x', t) ,
$$

$$
\gamma_{M+1}^{\varphi} = \varphi_{M+1}(x, t) + W_{M+1}(\eta, x, t) + Z_{M+1}(\tau, x', t) ,
$$

where $\eta = (t - \psi(x))/\varepsilon$, $\tau = x_N/\varepsilon$. It is easy to calculate that

$$
\left\| \gamma_{M+1}^{\theta}; L^\infty(0, T; L^2(\Omega)) \right\| + \left\| \gamma_{M+1}^{\varphi}; L^\infty(0, T; L^2(\Omega)) \right\| \leq c .
$$

Therefore,

$$
\theta = \theta_{as}^{M+1} + \varepsilon^{M+1} \sigma_{M'} , \quad \varphi = \varphi_{as}^{M+1} + \varepsilon^{M+1} \omega_{M'} ,
$$

where

$$
\sigma_{M'} = \gamma_{M+1}^{\theta} + \sqrt{\varepsilon} \sigma_{M'} , \quad \omega_{M} = \gamma_{M+1}^{\varphi} + \sqrt{\varepsilon} \omega_{M'}
$$

are functions such that the estimate

$$
\left\| \gamma_{M+1}^{\theta}; L^\infty(0, T; L^2(\Omega)) \right\| + \left\| \gamma_{M+1}^{\varphi}; L^\infty(0, T; L^2(\Omega)) \right\| +
$$

$$
+ \left\| \sigma_{M'}; L^\infty(0, T; L^2(\Omega)) \right\| + \left\| \omega_{M'}; L^\infty(0, T; L^2(\Omega)) \right\| \leq \text{const}
$$

holds uniformly in $\varepsilon$. This estimate and Theorem 4 complete the proof of Theorem 3.
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