ON SUMS OF POWERS OF TERMS
IN A LINEAR RECURRENCE

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1 – Introduction

Define the sequences \( \{U_n\}_{n=0}^{\infty} \) and \( \{V_n\}_{n=0}^{\infty} \) by

\[
\begin{align*}
U_n &= p U_{n-1} - U_{n-2}, \quad U_0 = 0, \quad U_1 = 1, \\
V_n &= p V_{n-1} - V_{n-2}, \quad V_0 = 2, \quad V_1 = p,
\end{align*}
\]

where \( p \geq 2 \) is an integer. For \( p = 2 \) \( \{U_n\} \) becomes the sequence of non-negative integers, and for this reason we may look upon \( \{U_n\} \) as a generalization of the non-negative integers. The sequence \( \{V_n\} \) bears the same relation to \( \{U_n\} \) as does the Lucas sequence to the Fibonacci sequence. For \( p > 2 \) the Binet forms are

\[
\begin{align*}
U_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,
\end{align*}
\]

where

\[
\alpha = \frac{p + \sqrt{p^2 - 4}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2 - 4}}{2}
\]

are the roots of \( x^2 - px + 1 = 0 \). We put \( \Delta = (\alpha - \beta)^2 = p^2 - 4. \)

Clary and Hemenway [2] proved

**Theorem 1.**

\[
(p + 1) \sum_{k=1}^{n} U_k^3 = (U_{n+1} - U_n + 2) \left( \sum_{k=1}^{n} U_k \right)^2.
\]

For \( p = 2 \) this reduces to the well known identity

\[
1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.
\]

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Similar results on sums of powers of integers have a long history. If \( k \) is a positive integer write
\[
T_k(n) = (1 + 2 + \cdots + n)^k \quad \text{and} \quad S_k(n) = 1^k + 2^k + \cdots + n^k .
\]
Then a result which extends (1.3) is

**Theorem 2.**

(1.4) \[
T_k(n) = \frac{1}{2k-1} \sum_{2i-1}^k \binom{k}{2i-1} S_{2k+1-2i}(n) ,
\]
the sum being taken over those \( i \) for which \( 2 \leq 2i \leq k + 1 \).

The first few instances of (1.4) are

(1.5) \[ T_2(n) = S_3(n) , \]

(1.6) \[ T_3(n) = \frac{1}{4} S_3(n) + \frac{3}{4} S_5(n) , \]

(1.7) \[ T_4(n) = \frac{1}{2} S_5(n) + \frac{1}{2} S_7(n) . \]

Theorem 1 has been rediscovered many times. It occurs in a 1952 paper of Piza [5], and according to MacDougall [4] it was known as far back as 1877 (Lampe) and 1878 (Stern). In 1997 G.L. Cohen, a colleague of the present writer, also rediscovered Theorem 1, and thus provided the motivation for this paper. Chapter xiv of Lucas [3] contains an excellent historical survey on sums of powers of integers. In a recent paper, which also contains a wealth of historical material, Beardon [1] generalized (1.3) by describing all polynomial relations that exist between any two of the \( S_i \).

Our object in this paper is to produce further identities like (1.2) which involve higher powers. Our main results are stated as Theorems 3, 4 and 5 in Section 3.

## 2 – Some preliminary results

We require the following:

(2.1) \[ \Delta U_n^3 = U_{3n} - 3 U_n , \]

(2.2) \[ \Delta^2 U_n^5 = U_{5n} - 5 U_{3n} + 10 U_n , \]

(2.3) \[ \Delta^3 U_n^7 = U_{7n} - 7 U_{5n} + 21 U_{3n} - 35 U_n , \]

(2.4) \[ U_{5n} = \Delta^2 U_n^5 + 5 \Delta U_n^3 + 5 U_n , \]
\[ U_7 = \Delta^3 U_7^7 + 7 \Delta^2 U_7^5 + 14 \Delta U_7^3 + 7 U_7, \]
\[ V_m U_n = U_{m+n} - U_{m-n}, \]
\[ \Delta U_m U_n = V_{m+n} - V_{m-n}, \]
\[ U_{2n} = U_n V_n, \]
\[ V_{2m} - 2 = \Delta U_m^2, \]
\[ U_n^2 + U_{n+1}^2 = p U_n U_{n+1} + 1, \]
\[ U_n^4 + U_{n+1}^4 = (p^2 - 2) U_n^2 U_{n+1}^2 + 2p U_n U_{n+1} + 1, \]
\[ U_n^6 + U_{n+1}^6 = (p^3 - 3p) U_n^3 U_{n+1}^3 + (3p^2 - 3) U_n^2 U_{n+1}^2 + 3p U_n U_{n+1} + 1. \]

Identities (2.1)–(2.3) are obtained from the Binet form for \( U_n \) by taking the appropriate power. Identities (2.4) and (2.5) are obtained from (2.1)–(2.3). Identities (2.6) and (2.7) are special cases of (8) and (10) respectively in [2], while (2.8)–(2.10) follow immediately from the Binet forms. Identities (2.11) and (2.12) follow from (2.10) after taking appropriate powers.

If, for the sequences \( U_n \) and \( V_n \), we highlight the dependence on the parameter \( p \) by writing \( U_n(p) \) and \( V_n(p) \), then we have the following composition formulas which appear as (17) and (18) in [2]
\[ V_r n(p) = V_n (V_r(p)), \]
\[ U_r n(p) = U_r (p) U_n (V_r(p)). \]

In the work which follows we need the following lemmas.

**Lemma 1.** If \( m \) is a positive integer, then
\[ \sum_{k=1}^{n} U_{2mk} = \frac{U_{mn} U_{m(n+1)}}{U_m}. \]

**Proof:** By using the Binet form for \( U_{2mk} \) and the formula for the sum of a geometric progression, we obtain
\[ \sum_{k=1}^{n} U_{2mk} = \frac{U_{2mn+2m} - U_{2mn} - U_{2m}}{V_{2m} - 2} \]
\[ = \frac{V_{2mn+m} U_m - U_m V_m}{\Delta U_m^2} \quad \text{(by (2.6), (2.8) and (2.9))}, \]
and the result follows from (2.7).
Lemma 2.

\[ U_{3n} U_{3(n+1)} = \Delta^2(U_n U_{n+1})^3 + 3p \Delta(U_n U_{n+1})^2 + (3\Delta + 9) U_n U_{n+1}. \]

**Proof:** From (2.1) we have

\[ U_{3n} U_{3(n+1)} = U_n U_{n+1} (\Delta U_n^2 + 3) (\Delta U_{n+1}^2 + 3) \]
\[ = U_n U_{n+1} \left( \Delta^2 U_n^2 U_{n+1}^2 + 3 \Delta (U_n^2 + U_{n+1}^2) + 9 \right), \]

and the result follows from (2.10). \[ \square \]

Lemma 3.

\[ U_{5n} U_{5(n+1)} = \Delta^4(U_n U_{n+1})^5 + 5p \Delta^3(U_n U_{n+1})^4 + 5 (2p^2 - 1) \Delta^2(U_n U_{n+1})^3 \]
\[ + 5p (2p^2 - 3) \Delta(U_n U_{n+1})^2 + 5 (\Delta^2 + 5\Delta + 5) U_n U_{n+1}. \]

**Proof:** From (2.4) we have

\[ U_{5n} U_{5(n+1)} = U_n U_{n+1} (\Delta^2 U_n^4 U_{n+1}^2 + 5 \Delta U_n^4 + 5) (\Delta^2 U_{n+1}^4 + 5 \Delta U_{n+1}^2 + 5). \]

We complete the proof by multiplying the terms in the brackets and using (2.10) and (2.11). \[ \square \]

In precisely the same manner, using (2.5) and (2.10)–(2.12), we can prove

Lemma 4.

\[ U_{7n} U_{7(n+1)} = \Delta^6(U_n U_{n+1})^7 + 7p \Delta^5(U_n U_{n+1})^6 + 7 (3p^2 - 1) \Delta^4(U_n U_{n+1})^5 \]
\[ + 35p (p^2 - 1) \Delta^3(U_n U_{n+1})^4 + 7 (5p^4 - 10p^2 + 2) \Delta^2(U_n U_{n+1})^3 \]
\[ + 7p \Delta (3\Delta^2 + 14\Delta + 14) (U_n U_{n+1})^2 \]
\[ + 7 (\Delta^3 + 7\Delta^2 + 14\Delta + 7) U_n U_{n+1}. \]

3 – The main results

From (2.1) we have \( \Delta U_{2k}^3 = U_{6k} - 3U_{2k} \), and using Lemma 1 we obtain

\[ \Delta \sum_{k=1}^{n} U_{2k}^3 = \frac{U_{3n} U_{3(n+1)}}{U_3} - 3U_n U_{n+1}. \]
By Lemma 2 this becomes
\[ U_3 \sum_{k=1}^{n} U_{2k}^3 = \Delta(U_n U_{n+1})^3 + 3p(U_n U_{n+1})^2, \]
and Lemma 1 with \( m = 1 \) yields
\[(3.1) \quad U_3 \sum_{k=1}^{n} U_{2k}^3 = \Delta \left( \sum_{k=1}^{n} U_{2k} \right)^3 + 3p \left( \sum_{k=1}^{n} U_{2k} \right)^2. \]

To convert (3.1) to a form involving consecutive subscripts we use (2.14) with \( r = 2 \). That is, in (3.1) we make the substitution \( U_{2k}(p) = pU_k(p^2 - 2) \). Finally, if we put \( U_3 = p^2 - 1 \) and \( \Delta = p^2 - 4 \), and replace \( p \) by \( \sqrt{p + 2} \) in order to restore the original parameter \( p \), we obtain

**Theorem 3.**

\[(3.2) \quad (p + 1) \sum_{k=1}^{n} U_k^3 = (p - 2) \left( \sum_{k=1}^{n} U_k \right)^3 + 3 \left( \sum_{k=1}^{n} U_k \right)^2. \]

Now (3.2) reduces to (1.3) when \( p = 2 \). We also note that (3.2) is equivalent to (1.2). Indeed we can obtain (1.2) if we first factorise the right side of (3.1) and then convert to a form involving consecutive subscripts.

Next we obtain an analogue of (3.2) involving fifth powers. From (2.2) we have \( \Delta^2 U_{2k}^5 = U_{10k} - 5U_{6k} + 10U_{2k} \), and using Lemma 1 we obtain
\[ U_3 U_5 \sum_{k=1}^{n} U_{2k}^5 = U_3 U_5 U_{5(n+1)} - 5 U_3 U_{3n} U_{3(n+1)} + 10 U_3 U_5 U_n U_{n+1}. \]

After we make the necessary substitutions using Lemma 2 and Lemma 3 this identity becomes
\[ U_3 U_5 \sum_{k=1}^{n} U_{2k}^5 = U_3 \Delta^2(U_n U_{n+1})^5 + 5p U_3 \Delta(U_n U_{n+1})^4 \]
\[ + 5p^4(U_n U_{n+1})^3 - 5p^3(U_n U_{n+1})^2. \]

Next we use Lemma 1 to replace each occurrence of \( U_n U_{n+1} \) by \( \sum_{k=1}^{n} U_{2k} \). Finally if we note that \( U_5 = p^4 - 3p^2 + 1 \), and convert to consecutive subscripts as before, we obtain
Theorem 4.

\[ (p + 1) (p^2 + p - 1) \sum_{k=1}^{n} U_k^5 = (p + 1) (p - 2)^2 \left( \sum_{k=1}^{n} U_k \right)^5 \]

\[ + 5 (p + 1) (p - 2) \left( \sum_{k=1}^{n} U_k \right)^4 \]

\[ + 5 (p + 2) \left( \sum_{k=1}^{n} U_k \right)^3 - 5 \left( \sum_{k=1}^{n} U_k \right)^2. \]

When \( p = 2 \) (3.3) becomes

\[ S_5(n) = \frac{4}{3} T_3(n) - \frac{1}{3} T_2(n), \]

which can be obtained from (1.5) and (1.6).

Next we obtain an identity involving seventh powers. Since the algebra is lengthy (but straightforward) we omit the details. Using (2.3) together with Lemma 1 we have

\[ U_3 U_5 U_7 \Delta^3 \sum_{k=1}^{n} U_{2k}^7 = U_3 U_5 U_{7n} U_{7(n+1)} - 7 U_3 U_7 U_{5n} U_{5(n+1)} \]

\[ + 21 U_5 U_7 U_{3n} U_{3(n+1)} - 35 U_3 U_5 U_7 U_n U_{n+1}. \]

Then we use Lemmas 2-4, together with Lemma 1, to express the right side as a polynomial in \( \sum_{k=1}^{n} U_{2k} \). Finally, noting that \( U_7 = p^6 - 5p^4 + 6p^2 - 1 \), we change to consecutive subscripts to obtain

Theorem 5.

\[ (p + 1) (p^2 + p - 1) (p^3 + p^2 - 2p - 1) \sum_{k=1}^{n} U_k^7 = \]

\[ = (p+1) (p-2)^3 (p^2+p-1) \left( \sum_{k=1}^{n} U_k \right)^7 + 7 (p+1) (p-2)^2 (p^2+p-1) \left( \sum_{k=1}^{n} U_k \right)^6 \]

\[ + 7 (p+1) (2p-1) (p^2-4) \left( \sum_{k=1}^{n} U_k \right)^5 + 35 p (p+1) \left( \sum_{k=1}^{n} U_k \right)^4 \]

\[ - 7 (p+2) (2p+1) \left( \sum_{k=1}^{n} U_k \right)^3 + 7 (2p+1) \left( \sum_{k=1}^{n} U_k \right)^2. \]
When $p = 2$ (3.4) becomes

$$S_7(n) = 2T_4(n) - \frac{4}{3}T_3(n) + \frac{1}{3}T_2(n),$$

which can be obtained from (1.5)–(1.7).

4 – Concluding remarks

Interestingly, in each of (3.2)–(3.4) the sum of the polynomial coefficients on the right side is equal to the polynomial coefficient on the left side. We have not been able to detect any other pattern in these coefficients. Our method of deriving these identities suggests that there are higher power analogues. Is there a more direct way to derive them? Is there a general formula which encompasses all such identities?

We conclude by making an unusual observation. If we denote the $k$-th derivative (with respect to $p$) of $V_n$ by $V_n^{(k)}$, where $V_n^{(0)} = V_n$, then (2.10)–(2.12) can be written respectively as

$$U_n^2 + U_{n+1}^2 = V_1^{(0)}U_n U_{n+1} + V_1^{(1)},$$
$$U_n^4 + U_{n+1}^4 = V_2^{(0)}U_n^2 U_{n+1}^2 + V_2^{(1)}U_n U_{n+1} + \frac{V_2^{(2)}}{2!},$$
$$U_n^6 + U_{n+1}^6 = V_3^{(0)}U_n^3 U_{n+1}^3 + V_3^{(1)}U_n^2 U_{n+1}^2 + \frac{V_3^{(2)}}{2!}U_n U_{n+1} + \frac{V_3^{(3)}}{3!}.$$

After checking that this pattern continues for several more cases, we make the following conjecture.

**Conjecture:** If $k$ is a positive integer then

$$U_n^{2k} + U_{n+1}^{2k} = \sum_{r=0}^{k} \frac{V_k^{(r)}}{r!} U_n^{k-r} U_{n+1}^{k-1}.$$ 

REFERENCES


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