Abstract: The main aim of this paper is to establish the equalities

\[ H_b(E, F) = H_{ub}(E, F) \]
\[ H_b(E^*_\beta, F^*_\beta) = H_{ub}(E^*_\beta, F^*_\beta) \]

for the case where \( E \) and \( F \) are Fréchet spaces in the relation with the linear topological invariants \( (H_{ub}), (LB^\infty) \) and \( (DN) \).

1 – Introduction

Let \( E, F \) be locally convex spaces. By \( H(E, F) \) we denote the space of all \( F \)-valued holomorphic mappings on \( E \). Instead of \( H(E, \mathbb{C}) \) we write \( H(E) \). Each element of \( H(E, F) \) is called an entire mapping. By \( H_b(E, F) \) we denote the space of all entire mappings which are bounded on all bounded subsets of \( E \). The mappings in \( H_b(E, F) \) are called of bounded type. An entire mapping \( f \in H(E, F) \) is called of uniformly bounded type if it is bounded on multiples of some neighbourhood of 0 in \( E \). We denote by \( H_{ub}(E, F) \) the space of all entire mappings of uniformly bounded type.

A locally convex space \( E \) has the property \( (H_{ub}) \) and is written shortly \( E \in (H_{ub}) \) if \( H(E) = H_{ub}(E) \). The property \( (H_{ub}) \) has been investigated by some authors. Colombeau and Mujica have proved that \( H(E) = H_{ub}(E) \) for each \( (DFM) \)-space \( E \) (Ex. 3.11 in [2], p.163) while Nachbin has shown that \( H_{ub}(E) \not\subset H(E) \) for the nuclear Fréchet space \( E = H(\mathbb{C}) \) (Ex. 3.12 in [2], p.165).
Meise–Vogt have also proved that a nuclear locally convex space $E$ satisfies $H(E) = H_{ab}(E)$ if and only if entire mappings on $E$ are universally extendable in the following sense, whenever $E$ is a topological linear subspace of a locally convex space $F$ with the topology defined by a fundamental system of continuous semi-norms induced by semi-inner products, then each $f \in H(E)$ has a holomorphic extension to $F$ (Proposition 6.21 in [2], p. 421).

Next they have given some sufficient conditions for the equality $H(E) = H_{ab}(E)$ in terms of the linear topological invariants $(\Omega)$ and $(\bar{\Omega})$ (Theorem 3.3 and 3.9 in [8]) and in the case $E$ is a nuclear Fréchet space they have shown that $(\bar{\Omega}) \Rightarrow (H_{ab}) \Rightarrow (LB^{\infty})$ (Remark 3.11 in [8]). By Vogt (Ex. 5.5 in [15]) the class $(LB^{\infty})$ is strictly larger than the class $(\bar{\Omega})$. However we do not know whether one of the above implications can be reversed.

In this paper we will establish the relations

1. \[ H_b(E, F) = H_{ab}(E, F) \]
2. \[ H_b(E^{*}_{\beta}, F^{*}_{\beta}) = H_{ab}(E^{*}_{\beta}, F^{*}_{\beta}) \]

for Fréchet-valued (resp. $DF$-valued) entire mappings on Fréchet spaces (resp. $DF$ spaces) in the relation with linear topological invariants ($H_{ab}$), ($LB^{\infty}$) and $(DN)$. Note that under various assumptions (1) has been considered by some authors [3], [4], [5], [6]. It should be noticed that if $E$ is a Fréchet space that is not a Banach space then the scalar valued equality $H_{ab}(E) = H_b(E)$ does not imply the equality $H_{ab}(E, F) = H_b(E, F)$ for all Fréchet spaces $F$. It is enough to consider the case $F = E$.

Beside the introduction the article contains four sections. In the second one we recall some definitions and fix the notations. The section 3 is devoted to prove the equality (2). The main aim of section 4 is to prove that (1) holds in a special case where $F = H(\mathcal{C}, A)$, $A$ is a Banach space. In order to obtain the result in this case we modify some techniques of Vogt (Proposition 1.3 and 1.4 in [15]) for continuous linear maps to holomorphic mappings of bounded type. From the results obtained in the section 4 as a special case we prove, in the section 5, the equality (1) under the assumption that $E$ has the property $(H_{ab})$ and $F$ has the property $(DN)$. 
2 – Preliminaries

2.1. We shall use standard notations from the theory of locally convex spaces as presented in the books of R. Meise and D. Vogt [9] and Schaefer [13]. All locally convex spaces \( E \) are assumed to be complex vector spaces and Hausdorff.

For a locally convex space \( E \) by \( \mathcal{U}(E) \) we denote a neighbourhood basis of \( 0 \in E \). For each \( U \in \mathcal{U}(E) \) by \( E_U \) we denote the Banach space associated to the neighbourhood \( U \). Let \( V \in \mathcal{U}(E) \), \( V \subset U \), \( \omega_{UV} : E_V \to E_U \) denotes the canonical map from \( E_V \) to \( E_U \).

A locally convex space \( E \) is called to be Schwartz if for each \( U \in \mathcal{U}(E) \) there exists \( V \in \mathcal{U}(E) \), \( V \subset U \) such that \( \omega_{UV} : E_V \to E_U \) is compact.

For each locally convex space \( E \), \( E^\beta \) denotes the topological dual space \( E^* \) of \( E \) equipped with the strong topology \( \beta(E^*, E) \).

Now assume that \( E \) is a Fréchet space. We always consider that its locally convex structure is generated by an increasing system \( (\| \cdot \|_k)_{k \geq 1} \) of semi-norms. For \( k \geq 1 \) \( E_k \) will denote the Banach space associated to the semi-norm \( \| \cdot \|_k \).

Let \( E \) be a Fréchet space and \( u \in E^* \). For each \( k \geq 1 \) we define

\[
\| u \|^*_k = \sup \left\{ |u(x)| : \| x \|_k \leq 1 \right\}.
\]

Now we say that \( E \) has the property (\( LB^\infty \)) if

\[
(LB^\infty) \quad \forall \{\rho_n\} \uparrow +\infty \quad \forall p \quad \exists q \quad \forall n_0 \quad \exists N_0 \geq n_0, C > 0 \quad \forall u \in E^* \quad \exists k \quad n_0 \leq k \leq N_0 : \quad \| u \|^{1+\rho_k} \leq C \| u \|_k \| u \|^{*\rho_k}.
\]

\( E \) is said to have the property (\( DN \)) if

\[
(DN) \quad \exists p, d > 0 \quad \forall q \quad \exists k, C > 0 \quad \forall x \in E : \quad \| x \|^{1+d} \leq C \| x \|_k \| x \|_p^d.
\]

The properties (\( LB^\infty \)) and (\( DN \)) and some others are introduced and investigated by Vogt [15], [16], [17].

From now on, to be brief, whenever \( E \) has the property (\( H_{ub} \)) (resp. (\( LB^\infty \)), (\( DN \)), ...) we write \( E \in (H_{ub}) \) (resp. \( E \in (LB^\infty) \), \( E \in (DN) \), ...).

2.2. Holomorphic mappings. Let \( E, F \) be locally convex spaces and \( D \) be a non empty open subset of \( E \).

A mapping \( f : D \to F \) is called Gâteaux-holomorphic if for each \( x \in D, a \in E \) and \( u \in F^* \) the \( \mathbb{C} \)-valued function of one complex variable

\[
\lambda \to u \circ f(x + \lambda a)
\]
is holomorphic on some neighbourhood of 0 in \( \mathbb{C} \). A mapping \( f : D \to F \) is called holomorphic if \( f \) is Gâteaux-holomorphic and continuous. By \( H(D, F) \) we denote the space of all \( F \)-valued holomorphic mappings on \( D \), the compact-open topology on \( H(D, F) \) is denoted by \( \tau_0 \). For details concerning holomorphic mappings on locally convex spaces we refer to the books of Dineen [2] and Noverraz [12].

3 – \( DF \)-valued holomorphic mappings of uniformly bounded type and the linear topological invariants \( (LB^\infty) \) and \( (DN) \)

In the section we investigate the connection between \( DF \)-valued holomorphic mappings of uniformly bounded type on \( DF \)-spaces and the linear topological invariants \( (LB^\infty) \) and \( (DN) \). We prove the following

3.1. Theorem. Let \( E \) be a Fréchet space. Then

a) \( E \) has the property \( (DN) \) if and only if \( H_{ab}(E^*_\beta, F^*_\beta) = H_b(E^*_\beta, F^*_\beta) \) holds for every Fréchet space \( F \) having the property \( (LB^\infty) \).

b) \( E \) has the property \( (LB^\infty) \) if and only if \( H_{ab}(F^*_\beta, E^*_\beta) = H_b(F^*_\beta, E^*_\beta) \) holds for every Fréchet space \( F \) having the property \( (DN) \).

Proof: a) Assume that \( E \in (DN) \), obviously \( H_{ab}(E^*_\beta, F^*_\beta) \subset H_b(E^*_\beta, F^*_\beta) \). Let \( f : E^*_\beta \to F^*_\beta \) be a holomorphic mapping of bounded type. Consider the linear map \( \hat{f} : H_b(F^*_\beta) \to H_b(E^*_\beta) \) given by \( \hat{f}(g) = g \circ f \) for all \( g \in H_b(F^*_\beta) \). It is easy to see that \( F \) is a subspace of \( H_b(F^*_\beta) \). Hence \( \hat{f} : F \to H_b(E^*_\beta) \) is linear and continuous. Since \( E \in (DN) \) by (Theorem 3 in [10]) \( H_b(E^*_\beta) \) also has the property \( (DN) \). Now from \( F \in (LB^\infty) \) we infer that there exists a neighbourhood \( V \) of 0 in \( F \) for which \( \hat{f}(V) \) is bounded in \( H_b(E^*_\beta) \) (Theorem 6.2 in [15]). This yields that

\[
\sup \{ |\hat{f}(x)(u)| : x \in V, \ u \in B \} = \sup \{ |f(u)(x)| : x \in V, \ u \in B \} < +\infty
\]

for every bounded subset \( B \subset E^*_\beta \). Hence \( f : E^*_\beta \to (F_V)^* \) is holomorphic and of bounded type (Proposition 7 in [3]).

Conversely, by (Theorem 2.1 in [15]) it suffices to show that

\[
L(A_1(\alpha), E) = LB(A_1(\alpha), E)
\]

for the exponential sequence \( \alpha = (\alpha_n) \) where \( \alpha_n = n \) for \( n \geq 1 \).

Let \( f : A_1(\alpha) \to E \) be a continuous linear map. Since \( f \) maps bounded subsets of \( A_1(\alpha) \) to bounded subsets of \( E \) then \( f^* \in L\left( E^*_\beta, (A_1(\alpha))^*_\beta \right) \) where \( f^* \) is the
dual map of $f$. In view of $\Lambda_1(\alpha) \in (LB^\infty)$ and by applying the hypothesis we obtain that $f^* \in LB\left(E_\beta^*,(\Lambda_1(\alpha))_\beta\right)$. Hence $f \in LB(\Lambda_1(\alpha), E)$.

b) Necessity follows from a).

Conversely, by (Theorem 5.2 in [16]) it suffices to show that

$$L(E, \Lambda_\infty^\infty(\alpha)) = LB(E, \Lambda_\infty^\infty(\alpha))$$

where $\alpha_n = n$ for all $n \geq 1$ and

$$\Lambda_\infty^\infty(\alpha) = \left\{ \xi = (\xi_j)_{j \geq 1} : \|\xi\|_k = \sup |\xi_j|^{\alpha_j} < +\infty \text{ for all } k \geq 1 \right\}$$

and $\{\rho_k\} \uparrow +\infty$.

Let $f : E \to \Lambda_\infty^\infty(\alpha)$ be a continuous linear map. As in a) $f^* \in L((\Lambda_\infty^\infty(\alpha))_\beta^*, E_\beta^*)$. It is easy to check that $\Lambda_\infty^\infty(\alpha)$ has the property $(DN)$ and, hence, $f^* \in LB((\Lambda_\infty^\infty(\alpha))_\beta^*, E_\beta^*)$. From an argument as in a) we obtain that $f \in LB(E, \Lambda_\infty^\infty(\alpha))$ which completes the proof of 3.1 Theorem. 

### 4 – Fréchet-valued holomorphic mappings of uniformly bounded type and the linear topological invariant $(H_{ub})$

The main aim of this section is to prove the following technical result which is crucial for the proof of 5.1 Theorem.

#### 4.1. Theorem. Let $E$ be a Fréchet-Schwartz space having the property $(H_{ub})$ and $A$ be a Banach space. Then $\forall\{\rho_n\} \uparrow +\infty \ \exists k > 0 \ \forall p, s > 0 \ \forall r > 0 \ \forall n$ sufficiently large $\exists N_0 > n, \ C > 0 \ \forall f \in H_0(E, A) \ \exists n \leq N^* \leq N_0$:

$$\|f\|_{k, r}^{1+\rho_{N^*}} \leq C \|f\|_{N^*, \rho_{N^*}} \cdot \|f\|_{p, \rho_{N^*}}$$

where

$$\|f\|_{k, r} = \sup \left\{ \|f(x)\| : \|x\|_k \leq r \right\}$$

for $f \in H_0(E, A)$.

In order to derive the proof of this theorem first we establish the stability of the property $(H_{ub})$ under the finite products (see 4.2 Proposition below). 4.2 Proposition is a key ingredient in the proof of 4.1 Theorem. Moreover, next we modify some techniques of Vogt (Proposition 1.3, 1.4 in [15]) which are used for establishing (1) for continuous linear maps to holomorphic mappings of bounded type.
Now we state and prove the following

4.2. Proposition. Let $E$ and $F$ be Fréchet-Schwartz spaces having the property $(H_{ub})$. Then $E \times F$ has also the property $(H_{ub})$.

Proof: Given $f \in H(E \times F)$. Consider the holomorphic mapping $f_E : E \to (H(F), \tau_0)$ associated to $f$. Since $F \in (H_{ub})$, by (Proposition 4.1 in [8]), $(H(F), \tau_0)_{bor}$ is a regular inductive limit of $H_b(F_\alpha)$, $\alpha \in \mathbb{N}$, the Banach space of holomorphic mappings of bounded type on $F_\alpha$ where $F_\alpha$ is the Banach space associated to the continuous semi-norm $\| \cdot \|_\alpha$ of $F$. First we prove that there exist $p, \alpha \geq 1$ such that

$$f_E(U_p) \subset H_b(F_\alpha).$$

Indeed, otherwise, for each $p \geq 1$, $\alpha \geq 1$ there exists $x^p_\alpha \in U_p$ and $f_E(x^p_\alpha) \notin H_b(F_\alpha)$. Since $\{x^p_\alpha\}_{p \geq 1} \to 0$ and $(H(F), \tau_0)_{bor} = \lim \text{ind} H_b(F_\alpha)$ is regular we can find $\alpha_0$ such that

$$f_E(x^p_\alpha) \subset H_b(F_{\alpha_0}) \quad \text{for all} \quad p \geq 1.$$

This is impossible because $f_E(x^\alpha_0) \notin H_b(F_{\alpha_0})$. Thus there exists $p$ and $\alpha$ such that $f_E(U_p) \subset H_b(F_\alpha)$. Similarly there exist $q > p, \beta > \alpha$ such that $f^F(V_\beta) \subset H_b(E_q)$ where $f^F : F \to (H(E), \tau_0)$ is the holomorphic mapping induced by $f$.

Consider the mapping

$$g : (U_q \times F_\beta) \cup (E_q \times V_\beta) \subset E_q \times F_\beta \to \mathbb{C}$$

defined by $f_E$ and $f^F$. Notice that $g$ is separately holomorphic. By a result of N.T. Van–Zeriahi (Théorème 1.1 in [11]) $g$ extends to Gâteaux-holomorphic mapping $\hat{g}$ on $E_q \times F_\beta$ such that $f$ is Gâteaux-holomorphically factorized through $\hat{g}$ by $\omega_q \times \omega_\beta : E \times F \to E_q \times F_\beta$.

By shrinking $U_q$ and $V_\beta$ we may assume that $f$ is bounded on $U_q \times V_\beta$. Hence by the Zorn theorem $\hat{g}$ is holomorphic on $E_q \times F_\beta$.

On the other hand, since $E$ and $F$ are Schwartz spaces we can find $k \geq q$ and $\gamma \geq \beta$ such that the canonical maps $\omega_{qk} : E_k \to E_q$, $\omega_{\beta\gamma} : F_\gamma \to F_\beta$ are compact. Hence $\hat{g} \in H_b(E_k \times F_\gamma)$ and $f$ is factorized through $\hat{g}$ by $\omega_k \times \omega_\gamma$. Hence $f \in H_{ub}(E \times F)$. $
$

Remark. In the above proposition, if we take $F = \mathbb{C}$ then we have $H_b(E \times \mathbb{C}) = H_{ub}(E \times \mathbb{C})$. However, $H_b(E \times \mathbb{C}) = H_b(E, H(\mathbb{C}))$, $H_{ab}(E \times \mathbb{C}) = H_{ab}(E, H(\mathbb{C}))$ and, hence, (1) holds for the case $F = H(\mathbb{C})$. But it is known that $H(\mathbb{C})$ has the
property \((DN)\). Below, in 5.1 Theorem, we shall show that (1) holds under the assumptions \(E \in (H_{ub})\) and \(F \in (DN)\).

Now in order to obtain the proof of 4.1 Theorem we shall establish some equivalent conditions for which (1) holds.

First we fix some notations. Let \(E\) (resp. \(F\)) be a Fréchet space with the topology defined by an increasing system of semi-norms \((\| \cdot \|_\gamma)_{\gamma \geq 1}\) (resp. \((\| \cdot \|_k)_{k \geq 1}\)). For each \(k, \gamma, r > 0\) (or \(\rho > 0\)) and \(f \in H(E, F)\) we define

\[
\|f\|_{k, \gamma, r} = \sup \left\{ \|f(x)\|_{k} : \|x\|_\gamma \leq r \right\}.
\]

Through this section we always assume that \(E\) is a Fréchet space having the property \((H_{ub})\). Now we have the following

4.3. Proposition. The following assertions are equivalent

(i) \(H_b(E, F) = H_{ub}(E, F)\).

(ii) \(\forall \{\gamma(n)\} \uparrow \forall \{\rho_n\} \uparrow +\infty \exists k \forall r > 0 \forall n \exists N_0, C > 0 \forall f \in H_b(E, F)\)

\[
\|f\|_{n, \gamma(k), r} \leq C \max_{1 \leq N \leq N_0} \|f\|_{N, \gamma(N), \rho N}.
\]

Proof: (i) \(\Rightarrow\) (ii) Given \(\{\gamma(n)\} \uparrow\) and \(\{\rho_n\} \uparrow +\infty\). Put

\[
G = \left\{ f \in H_b(E, F) : \|f\|_{n, \gamma(n), \rho_n} < +\infty, \forall n \right\}.
\]

Since \(H_b(E, F) = H_{ub}(E, F)\) then \(G\) is a Fréchet space equipped with the topology defined by the system of semi-norms

\[
q_m(f) = \sup \left\{ \|f\|_{n, \gamma(n), \rho_n} : n = 1, 2, ..., m \right\}
\]

for \(f \in G\). For each \(k \in \mathbb{N}\), define

\[
H_k = \left\{ f \in H_b(E, F) : \|f\|_{n, \gamma(k), r} < +\infty \text{ for all } n, r > 0 \right\}.
\]

\(H_k\) is a Fréchet space under the topology defined by the systems of semi-norms

\[
p_{n, r}(f) = \|f\|_{n, \gamma(k), r}.
\]

We note that \(H_k \subset H_{k+1}\) for all \(k \geq 1\). By the hypothesis \(H_b(E, F) = H_{ub}(E, F)\) it follows that \(G \subset \bigcup_{k \geq 1} H_k\). All these spaces are continuously embedded in \(H_b(E, F)\).
By the factorization theorem of Grothendieck (Theorem 24.33 in [9], p. 290) there exists $k$ such that $G$ is continuously embedded in $H_k$. Hence $\forall r > 0 \ \forall n \exists N_0, C > 0$ such that

$$p_{n,r}(f) \leq C \max_{N \leq N_0} q_N(f)$$

for $f \in H_b(E, F)$. This shows that (4) holds.

(ii) $\Rightarrow$ (i) is trivial.

Now we need the following result which shows that (1) holds for the case $F$ is a Banach space.

4.4. Lemma. Let $E$ be a Fréchet space having the property $(H_{ub})$ and $F$ a Banach space. Then

$$H_b(E, F) = H_{ub}(E, F).$$

Proof: See the proof of (i) $\Rightarrow$ (iii) of Proposition 2.5 in [4].

Let $A$ be a Banach space and $B = (b_{j,k})_{j,k \geq 1}$ a Köthe matrix. We define

$$\Lambda^\infty(B, A) := \{ a = (a_i)_{i \geq 1}: a_i \in A, \|a\|_n = \sup \|a_i\|_{b_{j,n}} < +\infty \text{ for all } n \geq 1 \}.$$ 

$\Lambda^\infty(B, A)$ is a Fréchet space under the topology defined by the system of seminorms $(\| \cdot \|)_n \geq 1$.

When $A = \mathbb{C}$ we write $\Lambda^\infty(B)$ instead of $\Lambda^\infty(B, \mathbb{C})$.

For a comprehensive survey on the theory of Köthe sequence spaces we refer the readers to the book of Meise–Vogt (Chapters 27-31, p. 326–403 in [9]).

Let $E \in (H_{ub})$. Then we have the following

4.5. Proposition. Let $A$ be a Banach space. The following assertions are equivalent

(i) $H_b(E, \Lambda^\infty(B, A)) = H_{ub}(E, \Lambda^\infty(B, A))$.

(ii) $\forall \{\gamma(n)\} \uparrow \forall \{\rho_n\} \uparrow +\infty \exists k \forall r > 0 \ \forall n \exists N_0, C > 0$

$$b_{j,n}\|f\|_{\gamma(k),r} \leq C \max_{1 \leq N \leq N_0} b_{j,N}\|f\|_{\gamma(N),\rho_N}$$

for all $j \geq 1$ and for $f \in H_b(E, A)$. 
**Proof:** (i)⇒(ii) Let \( f \in H_b(E, A) \).

Put \( g_j = f \otimes e_j \in H_b(E, \Lambda^\infty(B, A)) \) where \( \{e_j\}_{j \geq 1} \) are vectors in \( \Lambda^\infty(B) \) of the form \( e_j = (0, 0, ..., 0, 1, 0, ...) \). By applying 4.3 Proposition to \( g_j \) and using \( k g_j k_{\gamma(n),r} = b_j,n f_{\gamma(n),r} \) we obtain (ii).

(ii)⇒(i) Let \( f \in H_b(E, \Lambda^\infty(B, A)) \) be given. Since \( \Lambda^\infty(B, A) = \{a = (a_i)_{i \geq 1}: a_i \in A, \|a\|_n = \sup_i \|a_i\|b_{i,n} < +\infty \text{ for all } n \geq 1\} \) it implies that \( f = (f_i)_{i \geq 1} \) where \( f_i \in H_b(E, A) \). From \( E \in (H_{ub}) \) and \( f \in H_b(E, \Lambda^\infty(B, A)) \) it follows that for each \( n \geq 1 \) if \( f \) can be considered as a holomorphic mapping of bounded type with values in the Banach space \( \Lambda^\infty(B, A)_n \) induced by the continuous semi-norm \( \|\cdot\|_n \) then 4.4 Lemma implies that there exists \( \gamma(n) \geq 1 \) such that \[
M(n, \gamma(n), r) = \sup \{\|f(x)\|_n: \|x\|_{\gamma(n)} \leq r\} < +\infty
\]
for all \( r > 0 \).

We may assume that the sequence \( \{\gamma(n)\} \) is increasing.

Take some sequence \( \{\rho_n\} \uparrow +\infty \) and by using (ii) for \( \{\gamma(n)\} \uparrow \) and \( \{\rho_n\} \) we derive that \( \exists k \forall r > 0 \forall n \exists N_0, C > 0: \)

\[
b_{i,n} f_i_{\gamma(n),r} \leq C \max_{1 \leq N \leq N_0} b_{i,N} f_i_{\gamma(N),\rho_N}.
\]

Hence \[
\|f\|_{n,\gamma(k),r} = \sup_i b_{i,n} f_i_{\gamma(k),r}
\leq C \max_{1 \leq N \leq N_0} \sup_i b_{i,N} f_i_{\gamma(N),\rho_N}
= C \max_{1 \leq N \leq N_0} \|f\|_{N,\gamma(N),\rho_N}.
\]

It follows that \( f \in H_{ub}(E, \Lambda^\infty(B, A)) \).

**Proof of 4.1 Theorem:** By 4.2 Proposition, we have \( E \times \mathbb{C} \in (H_{ub}) \). Using 4.4 Lemma for \( F = A \), we get \[
H_b(E \times \mathbb{C}, A) = H_{ub}(E \times \mathbb{C}, A).
\]
We have $H(C, A)$ is topologically isomorphic to $H(C_\otimes, A)$ [14] (Also see Ex. 4.91, p. 313 in [2]). Moreover, the Fréchet-nuclear space $H(C)$ is topologically isomorphic to $\Lambda_\infty^\infty(\alpha)$, where

$$\Lambda_\infty^\infty(\alpha) = \left\{ \xi = (\xi_j) \subset C^N : \|\xi\|_k = \sup_j |\xi_j| e^{\rho_{\alpha_j}} < +\infty, \text{ for all } k \right\}$$

and $\alpha = (\alpha_j)$, $\alpha_j = j$, $\rho = \{\rho_k\} \uparrow +\infty$.

Hence

$$H(C, A) = H(C_\otimes, A) = H(C_\otimes_\pi, A) = \Lambda_\infty^\infty(\alpha) \otimes_\pi A = \Lambda^\infty(B, A).$$

Now we have

$$H_b(E \times C, A) = H_b(E, H(C, A)) = H_b(E, \Lambda^\infty(B, A)).$$

Hence

$$H_b(E, \Lambda^\infty(B, A)) = H_{ub}(E, \Lambda^\infty(B, A)).$$

Now by applying 4.5 Proposition to the sequence $\{\gamma(n) = n\}$ and $\{\rho_k\} \uparrow +\infty$ as above we have

$$\exists k > 0 \ \forall r > 0 \ \forall n > k \ \exists N_0 > n, \ \ D > 0 \ \forall f \in H_b(E, A)$$

$$e^{\rho_{n_j}} \|f\|_{k, r} \leq D \max_{1 \leq N \leq N_0} e^{\rho_{N_j}} \|f\|_{N, \rho_N} \text{ for all } j \geq 1. \quad (6)$$

For each $n$ we can choose $j_0$ such that for $j \geq j_0$

$$e^{(\rho_{n-1}\rho_n)}D < 1. \quad (7)$$

For $k \leq N \leq n - 1$ and $j \geq j_0$ the following inequality holds

$$D e^{\rho_{N_j}} \|f\|_{N, \rho_N} < e^{\rho_{n_j}} \|f\|_{k, r} \quad (8)$$

for $r \geq \rho_{n-1}$.

Indeed, in the converse case, we assume that there exist $k \leq N \leq n - 1$ and $j \geq j_0$ such that

$$e^{\rho_{n_j}} \|f\|_{k, r} \leq D e^{\rho_{N_j}} \|f\|_{N, \rho_N}$$

for $r \geq \rho_{n-1}$.

It follows that

$$\frac{\|f\|_{k, r}}{\|f\|_{N, \rho_N}} \leq D \cdot e^{(\rho_N - \rho_n)j} < 1. \quad (9)$$
However, since $N \geq k$ it implies that $U_N \subset U_k$ and
\[
\left\{ \|f(x)\| : \frac{x}{\rho_N} \in U_N \right\} \subset \left\{ \|f(x)\| : \frac{x}{r} \in U_k \right\}
\]
for $r \geq \rho_{n-1}$. This shows that
\[
1 \leq \frac{\|f\|_{k,r}}{\|f\|_{N,\rho_N}}
\]
and, hence, it contradicts to (9).

Therefore, for $j \geq j_0$ and $r \geq \rho_{n-1}$
\[
(10) \quad e^{\rho_{n-j}} \frac{\|f\|_{k,r}}{\|f\|_{p,\rho_s}} \leq D \max \left\{ e^{\rho_{N-j}} \|f\|_{N,\rho_N} : N = 1, 2, \ldots, k - 1, n, \ldots, N_0 \right\}.
\]
Now let $f \in H_b(E, A)$ and $p, s$ be given. If $\|f\|_{p,\rho_s} = +\infty$ then (3) holds.
Now assume that $\|f\|_{p,\rho_s} < +\infty$. Let $j$ be the smallest natural number larger or equal to $j_0$ such that
\[
D \frac{\|f\|_{p,\rho_s}}{\|f\|_{p,\rho_s}} \leq e^{(\rho_{n-\rho_{k-1}})^j} \|f\|_{k,r}.
\]
Then
\[
(11) \quad e^{(\rho_{n-\rho_{k-1}})^{(j-1)}} \frac{\|f\|_{k,r}}{\|f\|_{p,\rho_s}} \leq \frac{\|f\|_{p,\rho_s}}{\|f\|_{p,\rho_s}} \leq e^{(\rho_{n-\rho_{k-1}})^j} \|f\|_{k,r}.
\]
For $j$ such that (11) holds there exists $n \leq N^* \leq N_0$ which satisfies
\[
e^{\rho_{N^*j}} \|f\|_{N^*,\rho_{N^*}} = \max_{1 \leq N \leq N_0} e^{\rho_{N^*j}} \|f\|_{N,\rho_N}.
\]
Indeed, otherwise there exists $1 \leq N^* \leq k - 1$ such that
\[
e^{\rho_{N^*j}} \|f\|_{N^*,\rho_{N^*}} = \max_{1 \leq N \leq N_0} e^{\rho_{N^*j}} \|f\|_{N,\rho_N}.
\]
From (10) we infer that
\[
e^{\rho_{n,j}} \|f\|_{k,r} \leq D e^{\rho_{N^*j}} \|f\|_{N^*,\rho_{N^*}} \quad \text{for } r \geq \rho_{n-1}.
\]
Hence
\[
\|f\|_{k,r} \leq D e^{(\rho_{N^*^j-\rho_{n,j}})} \|f\|_{N^*,\rho_{N^*}} \quad \|f\|_{N^*,\rho_{N^*}} < \|f\|_{N^*,\rho_{N^*}}
\]
holds for all $r > 0$. It is impossible.

Now from (10) we deduce
\[
e^{\rho_{n,j}} \|f\|_{k,r} \leq D e^{\rho_{N^*j}} \|f\|_{N^*,\rho_{N^*}}.
\]
or equivalently
\[ \| f \|_{k,r} \leq D e^{(\rho_{N^*} - \rho_n)j} \| f \|_{N^*, \rho_{N^*}} \]
\[ \leq D e^{\frac{j}{j-1}(\rho_{N^*} - \rho_n)(j-1)} \| f \|_{N^*, \rho_{N^*}} \]
where \( \theta = \frac{j}{j-1} \).

Put \( d = \theta \cdot \frac{\rho_{N^*} - \rho_n}{\rho_n - \rho_{k-1}} \). Then
\[ \| f \|_{k,r} \leq D \left( \frac{\| f \|_{p, \rho_{N^*}}}{\| f \|_{k,r}} \right)^d \| f \|_{N^*, \rho_{N^*}} \cdot \]

However
\[ d = \theta \cdot \frac{\rho_{N^*} - \rho_n}{\rho_n - \rho_{k-1}} \leq \frac{\theta}{\rho_n - \rho_{k-1}} \rho_{N^*} \leq \rho_{N^*} \]
for \( n \) sufficiently large such that \( \frac{\theta}{\rho_n - \rho_{k-1}} \leq 1 \). On the other hand,
\[ 1 \leq e^{(\rho_{n} - \rho_{k-1})(j-1)} \leq D \frac{\| f \|_{p, \rho_{N^*}}}{\| f \|_{k,r}} \cdot \]

By combining (12), (13) and (14) we obtain that
\[ \| f \|_{k,r} \leq D \left( D \frac{\| f \|_{p, \rho_{N^*}}}{\| f \|_{k,r}} \right)^{\rho_{N^*}} \| f \|_{N^*, \rho_{N^*}} \cdot \]
Thus
\[ \| f \|_{k,r}^{1 + \rho_{N^*}} \leq C \| f \|_{N^*, \rho_{N^*}} \| f \|_{p, \rho_{N^*}}^{\rho_{N^*}} \]
where \( C = D^{1 + \rho_{N^*}} \) which completes the proof of 4.1 Theorem.

5 – Fréchet-valued holomorphic mappings of uniformly bounded type and the linear topological invariants \((H_{ub})\) and \((DN)\)

Based on results obtained in Section 4 this section is devoted to study the connection between the uniform boundedness of Fréchet-valued holomorphic mappings and the linear topological invariants \((H_{ub})\) and \((DN)\). The main result of this section is the following
5.1. Theorem. Let $F$ be a Fréchet space. Then

\[ H_b(E, F) = H_{ub}(E, F) \]

holds for all Fréchet-Schwartz space $E$ having the property $(H_{ub})$ if and only if $F \in (DN)$.

Proof:
Necessity. Take $E = \Lambda_1(\beta)$ with $\beta = (\beta_n), \beta_n = n$. Then $\Lambda_1(\beta)$ is a nuclear Fréchet space and by the hypothesis $L(\Lambda_1(\beta), F) = LB(\Lambda_1(\beta), F)$. Hence by (Theorem 2.1 in [15]) $F \in (DN)$.

Sufficiency. By the hypothesis and (Theorem 2.6 in [17]) we have that $F$ is a subspace of $A_{\bar{s}}(B; A)$ where $A$ is a Banach space and $s = \Lambda_\infty(\alpha), \alpha = (\log(n+1))_{n\geq 1}$. Hence it suffices to show that

\[ H_b(E, \Lambda_\infty(B, A)) = H_{ub}(E, \Lambda_\infty(B, A)) \]

We shall show that the condition (ii) of 4.5 Proposition is satisfied. Indeed, take a sequence \( \{\gamma_n\} \) and \( \{\rho_n\} \) such that \( \lim_{n \to \infty} \frac{\rho_n}{n} = 0 \). As in (Theorem 3.2 in [15]) we may assume that \( \gamma(n) = n \) for all \( n \geq 1 \). By the hypothesis and by applying 4.1 Theorem for the sequence \( \{\rho_n\} \) we infer that there exists $k$ such that \( \forall p, s > 0, \forall r > 0, \forall n \) sufficiently large \( \exists N_0 > n, C > 0, \forall f \in H_b(E, A), \exists n \leq N^* \leq N_0 \):

\[
\|f\|_{1+\rho N^*} \leq C \|f\|_{N^*, \rho N^*} \|f\|_{p, \rho}^{\rho N^*}.
\]

Now take $p = 1, s = 1$. For given $n$ there exists $n_0$ sufficiently large such that for all $N \geq n_0$ we have

\[
\rho_N(n-1) \leq N - n.
\]

Applying (15) for $p = 1, s = 1$ and $n = n_0 \forall r > 0$ we can find $N_0 > n_0, C > 1 \forall f \in H_b(E, A), \exists n_0 \leq N^* \leq N_0$:

\[
\|f\|_{1+\rho N^*} \leq C \|f\|_{N^*, \rho N^*} \|f\|_{1, \rho}^{\rho N^*}.
\]

Now we need to prove

\[
e^{\alpha_j} \|f\|_{k,r} \leq C \max_{1 \leq N \leq N_0} e^{N_0} \|f\|_{N, \rho N} \quad \text{for all } j \geq 1.
\]

Given $j \geq 1$. Then either

\[
e^{\alpha_j} \|f\|_{k,r} \leq e^{\alpha_j} \|f\|_{1, \rho},
\]
or, in the converse case,

\[ e^{\alpha_j} \|f\|_{1,\rho_1} \leq e^{\alpha_j} \|f\|_{k,r} . \]

In the first case (17) obviously holds. We consider the second. Then we have

\[ \|f\|_{1,\rho_1} \leq e^{(n-1)\alpha_j} \|f\|_{k,r} . \]

From (16) we have

\[
\|f\|_{k,r}^{1+\rho_{N^*}} \leq C \|f\|_{N^*,\rho_{N^*}} e^{\rho_{N^*}(n-1)\alpha_j} \|f\|_{k,r}^{\rho_{N^*}} \\
\leq C \|f\|_{N^*,\rho_{N^*}} e^{(N^*-n)\alpha_j} \|f\|_{k,r}^{\rho_{N^*}} .
\]

Hence

\[ e^{\alpha_j} \|f\|_{k,r} \leq C e^{N^*\alpha_j} \|f\|_{N^*,\rho_{N^*}} . \]

Combining all these results we see that (17) is satisfied.

By 4.5 Proposition we have

\[ H_b(E, A \otimes_{\alpha} \Lambda_{\infty}(\alpha)) = H_{ub}(E, A \otimes_{\alpha} \Lambda_{\infty}(\alpha)) . \]

This completes the proof. \( \blacksquare \)

At the end of this paper we want to give an equivalent condition for which (1) holds in the case that \( E = \Lambda(B) \) is the space of Köthe sequences and \( F \) is a Fréchet space. With the notations used as above with \( B = (b_{j,k})_{j,k \geq 1} \) a matrix satisfying (*) we define the sequence space \( \Lambda(B) \) given by

\[ \Lambda(B) = \left\{ \xi = (\xi_1, \xi_2, \ldots): \|\xi\|_k = \sum_{j=1}^{\infty} |\xi_j| b_{j,k} < +\infty \text{ for all } k \geq 1 \right\} . \]

\( \Lambda(B) \) is a Fréchet space with the topology defined by the system of semi-norms \((\| \cdot \|_k)\). If we consider the Schauder basis \( \{e_j\}_{j \geq 1} \) in \( \Lambda(B) \) of the form

\[ e_j = \left( 0, 0, \ldots, 0, 1, 0, \ldots \right) \]

then \( \{e_j\}_{j \geq 1} \) is an absolute basis of \( \Lambda(B) \) and

\[ \|e_j\|_k = b_{j,k} \]

for \( j, k \geq 1. \)
Now we prove the following

5.2. Proposition. Let $\Lambda(B) \in (H_{ub})$ and $F$ be a Fréchet space. The following are equivalent

(i) $H_b(\Lambda(B), F) = H_{ub}(\Lambda(B), F)$;

(ii) $\forall \{\gamma(n)\} \uparrow \forall \{\rho_n\} \uparrow +\infty \exists k \forall r > 0 \forall n \exists N_0 > 0, C > 0$

\[
\frac{\|x\|_{n^p}}{b_{j_1,\gamma(k)} \cdots b_{j_p,\gamma(k)}} \leq C \max_{1 \leq N_\leq N_0} \frac{\|x\|_{N^p}}{b_{j_1,\gamma(N)} \cdots b_{j_p,\gamma(N)}}
\]

for $x \in F, j_1, \ldots, j_p \geq 1, p \geq 1$.

Proof: (i)⇒(ii) Let \{\gamma(n)\} \uparrow and \{\rho_n\} \uparrow +\infty be given. By 4.3 Proposition we can find $k$ satisfying (4). For $j_1, \ldots, j_p \geq 1, p \geq 1, x \in F$ we define $f \in H_b(\Lambda(B), F)$ given by

\[ f(\xi) = \xi_{j_1} \cdots \xi_{j_p} x \]

where $\xi = (\xi_1, \ldots, \xi_{j_1}, \ldots, \xi_{j_2}, \ldots, \xi_{j_p}, \ldots) \in \Lambda(B)$. Then

\[
\frac{\|x\|_{n^p}}{b_{j_1,\gamma(k)} \cdots b_{j_p,\gamma(k)}^p} = \|f\|_{n,\gamma(k), r} \leq C \max_{1 \leq N_\leq N_0} \|f\|_{N,\gamma(N), r^N} ,
\]

\[
\frac{\|x\|_{n^p}}{b_{j_1,\gamma(k)} \cdots b_{j_p,\gamma(k)}^p} \leq C \max_{1 \leq N_\leq N_0} \frac{\|x\|_{N^p}}{b_{j_1,\gamma(N)} \cdots b_{j_p,\gamma(N)}^p} .
\]

Hence we have (18).

(ii)⇒(i) Let $f \in H_b(E, F)$. Since $\Lambda(B) \in (H_{ub})$ it follows that for each $n \geq 1$ then exists $\gamma(n)$ such that

\[ M(n, \gamma(n), \rho) = \sup \left\{ \|f(\xi)\|_n : \|\xi\|_{\gamma(n)} \leq \rho \right\} < +\infty \]

for all $\rho > 0$. We may assume that \{\gamma(n)\} \uparrow. Fix a sequence \{\rho_n\} \uparrow. Write the Taylor expansion of $f$ at $0 \in \Lambda(B)$

\[ f(\xi) = \sum_{p \geq 0} P_p f(\xi) = \sum_{p \geq 0} \sum_{j_1, \ldots, j_p \geq 1} P_{j_1, \ldots, j_p} f(e_{j_1}, \ldots, e_{j_p}) \xi_{j_1} \cdots \xi_{j_p} . \]

Using (ii) for the sequence \{\gamma(n)\} \uparrow defined as above we can find $k$ such that (18)
holds. On the other hand, in (18) we can take \( r = 1 \). Now we have

\[
\|f(\xi)\|_n \leq \sum_{p \geq 0} \sum_{j_1, \ldots, j_p \geq 1} \left| \frac{P_p f(e_{j_1}, \ldots, e_{j_p})}{b_{j_1, \gamma(k)} \cdots b_{j_p, \gamma(k)}} \xi_{j_1} \cdots \xi_{j_p} \right|
\]

\[
\leq \sum_{p \geq 0} \sum_{j_1, \ldots, j_p \geq 1} \left| \frac{P_p f(e_{j_1}, \ldots, e_{j_p})}{b_{j_1, \gamma(k)} \cdots b_{j_p, \gamma(k)}} \xi_{j_1} \cdots b_{j_p, \gamma(k)} \xi_{j_p} \right|
\]

\[
\leq \sum_{p \geq 0} \sum_{j_1, \ldots, j_p \geq 1} \sup_{b_{j_1, \gamma(k)} \cdots b_{j_p, \gamma(k)}} \left| \frac{P_p f(e_{j_1}, \ldots, e_{j_p})}{b_{j_1, \gamma(k)} \cdots b_{j_p, \gamma(k)}} \right| \left| \xi_{j_1} \right| \cdots \left| b_{j_p, \gamma(k)} \xi_{j_p} \right|
\]

\[
\leq C \sum_{p \geq 0} \sup_{j_1, \ldots, j_p \geq 1} \left( \max_{1 \leq N \leq N_0} \frac{\|P_p f(e_{j_1}, \ldots, e_{j_p})\|_{N, \rho_N}}{b_{j_1, \gamma(N)} \cdots b_{j_p, \gamma(N)}} \right) \|\xi\|_n \|\xi\|_n
\]

\[
\leq C \sum_{p \geq 0} \frac{\rho_N}{\rho} \frac{1}{\rho} \sup_{j_1, \ldots, j_p \geq 1} \left( \max_{1 \leq N \leq N_0} \frac{\|P_p f(e_{j_1}, \ldots, e_{j_p})\|_{N, \rho_N}}{b_{j_1, \gamma(N)} \cdots b_{j_p, \gamma(N)}} \right) \|\xi\|_n \|\xi\|_n
\]

\[
\leq C \sum_{p \geq 0} \frac{\rho_N}{\rho} \frac{1}{\rho} \max_{1 \leq N \leq N_0} \left( \frac{\rho}{p!} \|f\|_{N, \gamma(N), \rho_N} \right) \|\xi\|_n \|\xi\|_n
\]

\[
\leq C \sum_{p \geq 0} \frac{\rho_N}{\rho} \frac{1}{\rho} \max_{1 \leq N \leq N_0} \left( \frac{\rho}{p!} \|f\|_{N, \gamma(N), \rho_N} \right) \|\xi\|_n \|\xi\|_n
\]

\[
\leq C \sum_{p \geq 0} \frac{\rho_N}{\rho} \frac{1}{\rho} \max_{1 \leq N \leq N_0} \left( \frac{\rho}{p!} \|f\|_{N, \gamma(N), \rho_N} \right) \|\xi\|_n \|\xi\|_n
\]

\[
\leq C \max_{1 \leq N \leq N_0} M(N, \gamma(N), \rho) \sum_{p \geq 0} \frac{\rho_N}{\rho} \frac{1}{p!} R^p < +\infty
\]

for \( \rho \) sufficiently large and the conclusion follows.

**REFERENCES**


Le Mau Hai. Nguyen Van Khue and Bui Quoc Hoan, Department of Mathematics, Pedagogical Institute Hanoi, Tuliem – Hanoi – VIETNAM