ON A CLASS OF MONGE–AMPÈRE PROBLEMS WITH
NON-HOMOGENEOUS DIRICHLET BOUNDARY CONDITION

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Abstract: We assume in the plane that $\Omega$ is a strictly convex domain, with its boundary $\partial\Omega$ sufficiently regular. We consider the Monge–Ampère equations in its general form $\det u_{ij} = g(|\nabla u|^2)h(u)$, where $u_{ij}$ denotes the Hessian of $u$, and $g$, $h$ are some given functions. This equation is subject to the non-homogeneous Dirichlet boundary condition $u = f$. A sharp necessary condition of solvability for this equation is given using the maximum principle in $\mathbb{R}^2$. We note that this maximum principle is extended to the N-dimensional case and two different applications have been given to illustrate this principle.

1 – Introduction

Let $u$ be a classical solution of the following Monge–Ampère equations

$$\det(u_{ij}) = F(x, u, |\nabla u|^2) \quad \text{in } \Omega,$$

where $\Omega$ is assumed to be a bounded domain, strictly convex. In this note, we derive a new maximum principle for the general Monge–Ampère equations (1) with $F(x, u, |\nabla u|^2) = g(|\nabla u|^2)h(u)$ in $\mathbb{R}^N$, $N \geq 2$, which generalizes a recent result of Ma [11] (the particular case when $g, h = \text{const. in } \Omega$).

In order to prove this maximum principle, we assume in the sequel that the functions $g$ and $h$ are subject to some appropriate conditions. These conditions lead to some differential inequality, which will be investigated in Section 2.

Received: July 4, 2003; Revised: February 6, 2004.
AMS Subject Classification: 28D10, 35B05, 35B50, 35J25, 35J60, 35J65.
Keywords: Monge–Ampère equations; maximum principle.
Then employing the second maximum principle of E. Hopf [10], we conclude that the corresponding maximum value is attained on the boundary $\partial \Omega$ of $\Omega$.

For the first application, we shall treat the following non-homogeneous Dirichlet boundary condition

$$u = f \quad \text{on} \quad \partial \Omega,$$

where $\partial \Omega$ denotes the boundary of $\Omega$ sufficiently regular and $f$ is a positive function of class $C^1$. Monge–Ampère equations in conjunction with Dirichlet and Neumann conditions were investigated in [1,2,4,5,6,7,8]. For the second application, we consider the particular Dirichlet case $f = 0$. Ma [11] showed that the combination $P = |\nabla u|^2 - 2\sqrt{c}u$ is a constant, $u$ is radial and $\Omega$ is a ball. We extend this result for more general combination and prove for some particular values of $g$, that $\Omega$ is an N-ball and $u$ is radial.

Some applications are given involving different situations, where various bounds for $u$ and its gradient $|\nabla u|$ are obtained. The maximum principle for Monge–Ampère equations was already used by Ma [11, 12] and Safoui [13].

In the case of the Neumann boundary condition

$$\frac{\partial u}{\partial n} = \cos(\theta(x, u))(1 + |\nabla u|^2)^{1/2} \quad \text{on} \quad \partial \Omega,$$

where $n$ is the outward normal vector and the wetting angle $\theta$ is an element of $(0, \frac{\pi}{2})$, Ma in [11], proved the following result, by assuming that the bounded domain $\Omega$ is strictly convex, the constant $c$ is positive and, the angle $\theta$ is an element of $(0, \frac{\pi}{2})$

**Theorem 1.** Under the above hypotheses on $\Omega, c, \theta_0$, if $u$ is a strictly convex solution of (1), (3) then the following relation is satisfied

$$k_0 \leq \max \left\{ c_1 \cos(\theta_0), c_2 \tan(\theta_0) \right\},$$

where $k_0 := \min_{x \in \partial \Omega} k(x)$ and $k(x)$ is the curvature of the boundary $\partial \Omega$ of $\Omega$ at $x$.

In the case when $F := $ const. and $f(x) = 0$, he showed the following theorem (see [11])

**Theorem 2.** Under the above hypotheses on the domain $\Omega$ and constant $c$, if $u$ is a strictly convex solution for the boundary value problems (1)–(2) then we have the following estimates

$$\max_{x \in \Gamma} |\nabla u|^2 \leq \frac{c}{k_0^2},$$
\[ -\frac{\sqrt{c}}{2k_0} \leq u \leq 0 \quad \text{in} \quad \Omega , \]

where \( k_0 := \min_{x \in \partial \Omega} k(x) \), \( k(x) \) is the curvature of \( \partial \Omega \) at \( x \).

For the proof of Theorem 2, he used the maximum principle \([9,10]\) in \( \mathbb{R}^2 \) for the following combination

\[ \Phi := |\nabla u|^2 - 2c^2 u , \]

and the expression of the Monge–Ampère equations (1) in normal coordinates (see Section 3, (40)).

The purpose of this paper is, firstly, to generalize this maximum principle in \( \mathbb{R}^N \) for a general combination of the form

\[ \Phi := g(|\nabla u|^2) + h(u) , \]

where \( g \) and \( h \) are supposed to be positive. Secondly, to consider a more general equation

\[ \det(u_{ij}) = g(|\nabla u|^2) h(u) \quad \text{in} \quad \Omega , \]

with non-homogeneous boundary condition (2). This generalization gives us an upper bound for \( u \) and its gradient \( |\nabla u| \) in function of the geometry of \( \Omega \) and the first and second derivatives of \( f \).

Throughout the paper, we shall be concerned with a bounded domain \( \Omega \) of \( \mathbb{R}^N \), strictly convex. A comma will be used to denote differentiation. We make use the summation convention with repeated Latin indices running from 1 to \( N \).

\[ u_{,i} := \frac{\partial u}{\partial x_i} , \]

\[ u_{,ij} := \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ \frac{\partial^2 u}{\partial x_i \partial x_j} \right] , \]

\[ u_{,n} := \frac{\partial u}{\partial n} , \]

\[ u_{,s} := \frac{\partial u}{\partial s} , \]

\[ (u_{,s})_n := \frac{\partial}{\partial n} \left( \frac{\partial u}{\partial s} \right) , \]

\[ (u_{,s})_n := (u_{,n})_s - Ku_{,s} , \]

\[ u_{,nn} := \frac{\partial}{\partial n} \left( \frac{\partial u}{\partial n} \right) , \]

\[ u_{,ss} := \frac{\partial}{\partial s} \left( \frac{\partial u}{\partial s} \right) . \]
2 – On a maximum principle

Hereafter, we shall assume that the solution \( u \) of the Monge–Ampère equations defined by (8) is at least of class \( C^2(\Omega) \cap C^3(\Omega) \) in a bounded domain \( \Omega \) described in Section 1. In this Section, we will show, that the maximum principle of the combination \( \Phi \) defined by the following equation

\[
\Phi := g(u, u_i) + h(u) \quad \text{in} \quad \bar{\Omega},
\]

attains its maximum value on the boundary \( \partial \Omega \), where the functions \( h \) and \( g \) are subject to some conditions. For the differential equation of the form

\[
\Delta u + f(u) = 0 \quad \text{in} \quad \bar{\Omega},
\]

the corresponding function constructed for this type of equation depends essentially on the dimension \( N \) and the imposed boundary conditions, for which in general the treatment in \( \mathbb{R}^2 \) differs from that of \( \mathbb{R}^N \), where \( N \geq 3 \), since some differential equalities are valid in \( \mathbb{R}^2 \) and unfortunately not valid in \( \mathbb{R}^N \), as

\[
|\nabla u|^2 u_{ij} u_{ij} = |\nabla u|^2 (\Delta u)^2 + u_{ij} u_{ik} u_{jk} - 2 (\Delta u) u_{ij} u_{ij}.
\]

It is already known that, the combination \( \Phi \) attains its maximum principle at three different places for an arbitrary \( g \) and \( f \) (see R. Sperb [14]). Assuming that \( \Phi \) is nonconstant, the corresponding maximum is attained on the boundary \( \partial \Omega \) as first possibility, at a critical point as second possibility and finally at an interior point of the domain \( \Omega \). In our context, we choose \( g \) and \( h \) such that, the elliptic differential inequality formed is strictly positive.

**Theorem 3.** Let \( u \) be a strictly convex solution of (9) and \( \Phi \) the combination defined by (8), then

\[
\frac{1}{2} u_{ij} \Phi_{,ij} + \cdots = g'(|\nabla u|^2) \left( -\frac{h'}{g} + \frac{h'}{h} - \frac{h'g''}{(g')^2} - \frac{h''}{h'} \right) + 2 g' \Delta u + Nh'
\]

where the dots stand for terms of the form \( V_{,k} \Phi_{,k} \) with specific vector fields \( V_{,k} \) which are bounded except at critical points of \( u \).

To start the proof of Theorem 3, we construct an appropriate differential inequality for \( \Phi \), except at a critical value of the solution \( u \). Let \( \Phi \) defined by (8) then

\[
\Phi_{,i} = 2 u_{,ik} u_{,k} g' + u_{,i} h',
\]

\[
\Phi_{,ij} = 2 g'(u_{,ijk} u_{,k} + u_{,jk} u_{,ik}) + 4(u_{,jl} u_{,il} u_{,ik} u_{,k}) g'' + u_{,ij} h' + u_{,i} u_{,j} h''.
\]
Let $u_{ij}$ be the inverse of the Hessian matrix $H := u^{ij}$. As $u$ is strictly convex solution of (9), the matrix $u^{ij}$ is positive definite and consequently by computing

$$u^{ij}_{,i} = 2g'(u^{ij}_{,ijk} u_{ij} + u^{ij}_{,jk} u_{,ik}) + 4(u^{ij}_{,jkl} u_{,i} u_{,ik} u_{,jk} + u^{ij}_{,jk} u_{,ik} u_{,ik}) + 4(u^{ij}_{,jl} u_{,i} u_{,ik} u_{,k} + u^{ij}_{,jl} u_{,i} u_{,jk}) g''(15)$$

we claim that $u^{ij}_{,ij}$ is strictly positive in $\Omega$. Knowing that the following identities $u^{ij}_{,ij} u_{,ij} u_{,jk} = 0$, $u^{ij}_{,ij} u_{,ij} = N$, $u^{ij}_{,i} u_{,i} u_{,i} u_{,i} = u_{,i} u_{,i} h'$ and $(gh)[u^{ij}_{,i} u_{,k}] = (gh)_{,j} u_{,j}$ are valid in $\mathbb{R}^N$, then we are able to prove that $\Phi$ satisfies an appropriate differential inequality. For this, we compute

$$u^{ij}_{,i} = 2g'' u_{,i} u_{,ik} u_{,k} + u_{,i} u_{,i} h'. (17)$$

From (16) and (17), we obtain

$$-u^{ij}_{,i} u_{,j} h' + u^{ij}_{,i} u_{,j} = 2g'' u^{ij}_{,i} u_{,ik} u_{,k} = 2 u_{,i} u_{,i} g', (18)$$

$$2u_{,ij} u_{,i} g' - u_{,i} u_{,i} = -u_{,i} u_{,i} h'. (19)$$

Hence by (18) and (19), we conclude that

$$u^{ij}_{,i} + \cdots = 2g'(|\nabla u|^2) \left( -\frac{h'}{g} + \frac{h'}{h} - \frac{h' g''}{h'} \right) + 2g' \Delta u + Nh'. \quad (20)$$

Using the following arithmetic-geometric inequality

$$\Delta u \geq N (gh)^{\frac{1}{N}}$$

(or simply $\Delta u > 0$ since $g'$ is positive), we obtain

$$u^{ij}_{,i} + \cdots \geq 2g'(|\nabla u|^2) \left( -\frac{h'}{g} + \frac{h'}{h} - \frac{h' g''}{h'} \right) + 2g' N (gh)^{\frac{1}{N}} + Nh', \quad (21)$$

where $g$, $h$, $g'$ and $h'$ satisfy the following conditions

$$g' > 0, \quad h' > 0, \quad (22)$$

and

$$-\frac{h'}{g} + \frac{h'}{h} - \frac{h' g''}{h'} > 0. \quad (23)$$

Then the maximum of $\Phi$ is attained on the boundary $\partial \Omega$ of $\Omega$ at some point $P$. If inequalities (22) and (23) are reversed, then we conclude that the minimum value of $\Phi$ occurs on the boundary $\partial \Omega$, or at a critical point of $u$.

We have then established the following theorem which extends the result of Ma [11,12] to the $N$-dimensional case.
Theorem 4. Let $\Phi$ be defined by (8) where $g$, $h$, $g'$ and $h'$ satisfy (22), (23) and $u$ supposed to be strictly convex. Then the maximum principle of the combination $\Phi$ is attained on the boundary $\partial \Omega$ of $\Omega$ at some point $P$.

3 – Estimates of the solution $u$ and its gradient $|\nabla u|$ for the Dirichlet boundary condition

In this Section, we investigate in dimension 2 the following result which illustrates Theorem 4. The bounds obtained for $u$ and its gradient $|\nabla u|$ seems appear for the first time in the non-homogeneous Dirichlet case.

Theorem 5. We assume that $u$ is a classical solution of the non-homogeneous Dirichlet problem (2), (9), strictly convex, at least of class $C^2(\Omega) \cap C^3(\Omega)$. Let $\Omega$ be a bounded domain, convex in $\mathbb{R}^2$. Then we have

$$
\max_{\Omega} |\nabla u|^2 \leq \frac{1}{K} \left\{ \frac{h'}{2g'} + M + \frac{1}{2} \tilde{M}^2 + \frac{1}{2} f_s^2 + |f_{ss}| \right\},
$$

$$
-h(u_{\text{min}}) + h(f) \leq g\left( \frac{1}{K} \left\{ \frac{h'}{2g'} + M + \frac{1}{2} \tilde{M}^2 + \frac{1}{2} f_s^2 + |f_{ss}| \right\} \right) \quad \text{in} \quad \Omega.
$$

For the proof of Theorem 5, we need to use in dimension 2 some differential equality valid in $\mathbb{R}^2$. This fact consists on the computation of the normal derivative of the combination $\Phi$ in function of the mean curvature $K$, the first $u_s$ and second $u_{ss}$ tangential derivatives of $u$ in the plane. This result will be established as follows.

We begin by computing the normal derivative of $\Phi$ in $\mathbb{R}^2$

$$
\frac{\partial \Phi}{\partial n} = 2g' \left\{ u_1 (u_{11} n_1 + u_{12} n_2) + u_2 (u_{21} n_1 + u_{22} n_2) \right\} + h'u_n
$$

$$
= 2g' u_n \left\{ \Delta u + u_2 \frac{\partial}{\partial s} u_1 - u_1 \frac{\partial}{\partial s} u_2 \right\} + h'u_n
$$

$$
= 2g' u_n \left\{ \Delta u + u_s u_{ns} - u_{ss} - K |\nabla u|^2 \right\} + h'u_n,
$$

where $s$ denotes differentiation in the tangential direction on the boundary $\partial \Omega$ and $K$ stands for curvature of $\partial \Omega$ at some point $\hat{P}$.

In the terms $\frac{\partial}{\partial s} u_1$ and $\frac{\partial}{\partial s} u_2$ we have broken $u_1$ and $u_2$ into normal and tangential derivative components and used the identities

$$
\frac{\partial u_1}{\partial s} = -Kn_2 \quad \text{and} \quad \frac{\partial u_2}{\partial s} = Kn_1.
$$
Since the maximum of the combination $\Phi$ defined by (11) is attained on the boundary $\partial \Omega$ at $\hat{P}$, we must have
\[
\frac{\partial \Phi}{\partial s}(\hat{P}) = g'(\nabla u)^2 + h'u_s \\
= 2g'(u_n u_{ns} + u_s u_{ss}) + h'u_s = 0.
\] (28)

Now we need to use the differential equality (28) in order to eliminate the product $u_n u_{ns}$ in (26). In fact, involving (28) we deduce
\[
u_n u_{ns} = \frac{h'u_s}{2g'} - u_s u_{ss}.
\] (29)

The Monge–Ampère equations (9) can be rewritten in $\mathbb{R}^2$ as
\[
u_{nn}(K u_n + u_{ss}) = gh + [u_{sn}]^2.
\] (30)

In this case, by using (26), (28), (29), and making use of the following inequality
\[
u_{ns} = u_{sn} - Ku_s,
\] (31)
we obtain
\[
\max_{\Omega} |\nabla u|^2 \leq \frac{1}{K} \left\{ \frac{h'}{2g'} + M + \frac{1}{2} \tilde{M}^2 + \frac{1}{2} f_s^2 + |f_{ss}| \right\},
\] (32)
where $M$ and $\tilde{M}$ are two positive bounds of Laplace $u$ and the mixed derivative $u_{sn}$ since $u$ is assumed to be of class $C^2$.

For this last differential inequality (32), we have only considered the case when the normal derivative of the solution $u$ is non-equal to zero. Since for the nullity case, we are conducted to the triviality of the solution $u$. We are concerned now with the estimation of the solution $u$, which will be illustrated by applying the statement of Theorem 4. We know that
\[
-h(u) \leq g(A) - h(u) + g(|\nabla u|^2),
\] (33)
where $A$ is defined by
\[
A := \frac{1}{K} \left\{ \frac{h'}{2g'} + M + \frac{1}{2} \tilde{M}^2 + \frac{1}{2} f_s^2 + |f_{ss}| \right\},
\] (34)
At a critical point of $u$, we obtain
\[
0 < -h(u_{\min}) \leq g(A) - h(f),
\] (35)
from which we deduce

\[ -h(u_{\min}) + h(f) \leq g(A). \]

Finally, we have explicitly

\[ -h(u_{\min}) + h(f) \leq g\left(\frac{1}{K}\left\{ \frac{h'}{2g} + M + \frac{1}{2}M^2 + \frac{1}{2}f_s^2 + |fss| \right\} \right). \]

4 – On an over-determined Monge–Ampère problem

Ma in [11] proved in \( \mathbb{R}^2 \) the following result

**Theorem 6.** Under the same hypothesis of \( c, \Omega \) and \( u(x) \) as in Theorem 2, if \( P(x) := |\nabla u|^2 - 2\sqrt{cu} \) attains its maximum in \( \Omega \), then

\[ \Omega = B_R(0), \]

\[ u = \frac{\sqrt{c}(x_1^2 + x_2^2)}{2} - \frac{\sqrt{c}R^2}{2}, \]

\[ P = cR^2, \]

where \( R \) is a positive constant.

Our goal is to extend this result to the \( N \)-dimensional space for more general Monge–Ampère equations (9). In the next theorem, we establish our result.

**Theorem 7.** We assume that \( u \) is a classical solution of (2), (9) with \( f = 0 \).
If \( \Phi = g(|\nabla u|^2) + h(u) \), where \( gg' = 1 \) and \( h(0) > 0 \), attains its maximum on the boundary \( \partial\Omega \), then

\[ \Omega = B_r(0), \]

\[ u = \left( \frac{h'(0)}{2h(0)} \right)^{\frac{N-1}{2}} \left( |x|^2 - h(0) \right)^{\frac{N-1}{2}} - \left( \frac{h'(0)}{2h(0)} \right)^{\frac{N-1}{2}} \left( r^2 - h(0) \right)^{\frac{N-1}{2}}, \]

\[ \Phi = \left( \frac{2h(0)}{h'(0)K} \right)^{\frac{2}{N-1}} + h(0) = \text{const. on } \partial\Omega, \]

where \( r \) is a positive constant.
In order to establish this statement, we compute the normal and tangential derivatives of $\Phi$ and we use the fact that its maximum is attained on the boundary $\partial \Omega$, we then obtain

\begin{equation}
\frac{\partial \Phi}{\partial n} = 2 g'\left(u_n u_{nn} + (\nabla_s u)(\nabla_s u)_n\right) + h'u_n = 0 ,
\end{equation}

where $\nabla_s u$ denotes the tangential gradient of $u$ on the boundary $\partial \Omega$.

From (44), we deduce that the second normal derivative of $u$ can be evaluated explicitly on the boundary $\partial \Omega$ as

\begin{equation}
u_{nn} = -\frac{h'}{2g'} ,
\end{equation}

which is non-positive by the hypothesis on $h'$ and $g'$ (see (22)).

The general Monge Ampère equations (9) with $f = 0$ takes the form

\begin{equation}
k(P)|\nabla u|^{N-1} u_{nn}(P) = g(u_n^2) h(0) .
\end{equation}

In fact, this is due to the following Lemma investigated by Safoui in [13]

**Lemma 1** (Lemma 1.5 p:16 [13]). Let $u$ be a function of class $C^2$ strictly convex in $\bar{\Omega}$ and constant on $\partial \Omega$, and let $P_0$ be an element of $\partial \Omega$ where $|\nabla u|^2$ realizes its maximum.

We have then at this point the relation

\[ \det(u_{ij}) = \Gamma(P_0) u_n^{N-1} u_{nn} , \]

where $\Gamma(P_0)$ denotes the curvature of Gauss of $\partial \Omega$ at the point $P_0$.

This last differential equality (46) becomes in view of Lemma 2

\begin{equation}
u_{nn} = \frac{g(u_n^2) h(0)}{K|\nabla u|^{N-1}} .
\end{equation}

Combining (45) and (47), we get

\begin{equation}
\frac{\partial u}{\partial n} = \left(\frac{2h(0)}{h'(0)K}\right)^{\frac{1}{N-1}} .
\end{equation}

Then we have

\begin{equation}
\Phi = \left(\frac{2h(0)}{h'(0)K}\right)^{\frac{N}{N-1}} + h(0) = \text{const. on } \partial \Omega .
\end{equation}
From (49), we obtain the value of the mean curvature $K$ of the boundary $\partial \Omega$ as follows
\[ K = \left( \frac{h'(0)}{2h(0)} \right)^{\frac{N-1}{2}} \left( r^2 - h(0) \right)^{\frac{N-1}{2}}, \]
where $r$ is a positive constant.

To this end, the solution $u$ takes the form
\[ u = \left( \frac{h'(0)}{2h(0)} \right)^{\frac{N-1}{2}} \left( |x|^2 - h(0) \right)^{\frac{N-1}{2}} - \left( \frac{h'(0)}{2h(0)} \right)^{\frac{N-1}{2}} \left( r^2 - h(0) \right)^{\frac{N-1}{2}}, \]
which achieves the proof of our theorem.

We remark that the statement of Theorem 7 is also valid if we have $gg' = A|\nabla u|^{2N}$, where $A$ is a positive constant. In the special case when $A = N = 1$, we obtain the result of Ma [11] in $\mathbb{R}^2$.

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