EVERY FUNCTION IS THE REPRESENTATION FUNCTION OF AN ADDITIVE BASIS FOR THE INTEGERS

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Abstract: Let $A$ be a set of integers. For every integer $n$, let $r_{A,h}(n)$ denote the number of representations of $n$ in the form $n = a_1 + a_2 + \cdots + a_h$, where $a_1, a_2, \ldots, a_h \in A$ and $a_1 \leq a_2 \leq \cdots \leq a_h$. The function

$$r_{A,h} : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$$

is the representation function of order $h$ for $A$. The set $A$ is called an asymptotic basis of order $h$ if $r_{A,h}^{-1}(0)$ is finite, that is, if every integer with at most a finite number of exceptions can be represented as the sum of exactly $h$ not necessarily distinct elements of $A$. It is proved that every function is a representation function, that is, if $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ is any function such that $f^{-1}(0)$ is finite, then there exists a set $A$ of integers such that $f(n) = r_{A,h}(n)$ for all $n \in \mathbb{Z}$. Moreover, the set $A$ can be arbitrarily sparse in the sense that, if $\varphi(x) \geq 0$ for $x \geq 0$ and $\varphi(x) \to \infty$, then there exists a set $A$ with $f(n) = r_{A,h}(n)$ and $\text{card}\{a \in A : |a| \leq x\} < \varphi(x)$ for all $x$.

It is an open problem to construct dense sets of integers with a prescribed representation function. Other open problems are also discussed.

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1. Additive bases and the Erdős–Turán conjecture

Let $\mathbb{N}, \mathbb{N}_0,$ and $\mathbb{Z}$ denote the positive integers, nonnegative integers, and integers, respectively. Let $A$ be a set of integers. For every positive integer $h$, we define the sumset

$$hA = \left\{ a_1 + \cdots + a_h : a_i \in A \text{ for all } i = 1, ..., h \right\}.$$

We denote by $r_{A,h}(n)$ the number of representations of $n$ in the form $n = a_1 + a_2 + \cdots + a_h$, where $a_1, a_2, ..., a_h \in A$ and $a_1 \leq a_2 \leq \cdots \leq a_h$. The function $r_{A,h}$ is called the representation function of order $h$ of the set $A$.

In this paper we consider additive bases for the set of all integers. The set $A$ of integers is called a basis of order $h$ for $\mathbb{Z}$ if every integer can be represented as the sum of $h$ not necessarily distinct elements of $A$. The set $A$ of integers is called an asymptotic basis of order $h$ for $\mathbb{Z}$ if every integer with at most a finite number of exceptions can be represented as the sum of $h$ not necessarily distinct elements of $A$. Equivalently, the set $A$ is an asymptotic basis of order $h$ if the representation function $r_{A,h} : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ satisfies the condition

$$\text{card} \left( r_{A,h}^{-1}(0) \right) < \infty.$$

For any set $X$, let $\mathcal{F}_0(X)$ denote the set of all functions

$$f : X \to \mathbb{N}_0 \cup \{\infty\}$$

such that

$$\text{card} \left( f^{-1}(0) \right) < \infty.$$

We ask: Which functions in $\mathcal{F}_0(\mathbb{Z})$ are representation functions of asymptotic bases for the integers? This question has a remarkably simple and surprising answer. In the case $h = 1$ we observe that $f \in \mathcal{F}_0(\mathbb{Z})$ is a representation function if and only if $f(n) = 1$ for all integers $n \notin f^{-1}(0)$. For $h \geq 2$ we shall prove that every function in $\mathcal{F}_0(\mathbb{Z})$ is a representation function. Indeed, if $f \in \mathcal{F}_0(\mathbb{Z})$ and $h \geq 2$, then there exist infinitely many sets $A$ such that $f(n) = r_{A,h}(n)$ for every $n \in \mathbb{Z}$. Moreover, we shall prove that we can construct arbitrarily sparse asymptotic bases $A$ with this property. Nathanson [7] previously proved this theorem for $h = 2$ and the function $f(n) = 1$ for all $n \in \mathbb{Z}$.

This result about asymptotic bases for the integers contrasts sharply with the case of the nonnegative integers. The set $A$ of nonnegative integers is called an asymptotic basis of order $h$ for $\mathbb{N}_0$ if every sufficiently large integer can be
represented as the sum of \( h \) not necessarily distinct elements of \( A \). Very little is known about the class of representation functions of asymptotic bases for \( \mathbb{N}_0 \). However, if \( f \in \mathcal{F}_0(\mathbb{N}_0) \), then Nathanson [5] proved that there exists at most one set \( A \) such that \( r_{A,h}(n) = f(n) \).

Many of the results that have been proved about asymptotic bases for \( \mathbb{N}_0 \) are negative. For example, Dirac [2] showed that the representation function of an asymptotic basis of order 2 cannot be eventually constant. Erdős and Fuchs [4] proved that the average value of a representation function of order 2 cannot even be approximately constant, in the sense that, for every infinite set \( A \) of nonnegative integers and every real number \( c > 0 \),

\[
\sum_{n \leq N} r_{A,2}(n) \neq cN + o \left( N^{1/4} \log^{-1/2} N \right).
\]

Erdős and Turán [3] conjectured that if \( A \) is an asymptotic basis of order \( h \) for the nonnegative integers, then the representation function \( r_{A,h}(n) \) must be unbounded, that is,

\[
\limsup_{n \to \infty} r_{A,h}(n) = \infty.
\]

This famous unsolved problem in additive number theory is only a special case of the general problem of classifying the representation functions of asymptotic bases of finite order for the nonnegative integers.

2 – Two lemmas

We use the following notation. For sets \( A \) and \( B \) of integers and for any integer \( t \), we define the sumset

\[
A + B = \{ a + b: a \in A, b \in B \},
\]

the translation

\[
A + t = \{ a + t: a \in A \},
\]

and the difference set

\[
A - B = \{ a - b: a \in A, b \in B \}.
\]

For every nonnegative integer \( h \) we define the \( h \)-fold sumset \( hA \) by induction:

\[
0A = \{ 0 \}, \quad hA = A + (h-1)A = \{ a_1 + a_2 + \cdots + a_h: a_1, a_2, \ldots, a_h \in A \}.
\]
We denote the cardinality of a set $S$ by $\text{card}(S)$. The counting function for the set $A$ is

$$A(y, x) = \text{card}\left( \{a \in A: y \leq a \leq x \} \right).$$

In particular, $A(-x, x)$ counts the number of integers $a \in A$ with $|a| \leq x$. If $A$ is a finite set of integers, we denote the maximum element of $A$ by $\max(A)$.

Let $[x]$ denote the integer part of the real number $x$.

**Lemma 1.** Let $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ be a function such that $f^{-1}(0)$ is finite. Let $\Delta$ denote the cardinality of the set $f^{-1}(0)$. Then there exists a sequence $U = \{u_k\}_{k=1}^{\infty}$ of integers such that, for every $n \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$f(n) = \text{card}\left( \{k \geq 1: u_k = n \} \right)$$

and

$$|u_k| \leq \left\lfloor \frac{k + \Delta}{2} \right\rfloor.$$

**Proof:** Every positive integer $m$ can be written uniquely in the form

$$m = s^2 + s + 1 + r,$$

where $s$ is a nonnegative integer and $|r| \leq s$. We construct the sequence

$$V = \{0, -1, 0, 1, -2, -1, 0, 1, 2, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

$$= \{v_m\}_{m=1}^{\infty},$$

where

$$v_{s^2+s+1+r} = r \quad \text{for} \quad |r| \leq s.$$

For every nonnegative integer $k$, the first occurrence of $-k$ in this sequence is $v_{k^2+1} = -k$, and the first occurrence of $k$ in this sequence is $v_{(k+1)^2} = k$.

The sequence $U$ will be the unique subsequence of $V$ constructed as follows. Let $n \in \mathbb{Z}$. If $f(n) = \infty$, then $U$ will contain the terms $v_{s^2+s+1+n}$ for every $s \geq |n|$. If $f(n) = \ell < \infty$, then $U$ will contain the $\ell$ terms $v_{s^2+s+1+n}$ for $s = |n|, |n| + 1, \ldots, |n| + \ell - 1$ in the subsequence $U$, but not the terms $v_{s^2+s+1+n}$ for $s \geq |n| + \ell$. Let $m_1 < m_2 < m_3 < \cdots$ be the strictly increasing sequence of positive integers such that $\{v_{m_k}\}_{k=1}^{\infty}$ is the resulting subsequence of $V$. Let $U = \{u_k\}_{k=1}^{\infty}$, where $u_k = v_{m_k}$. Then

$$f(n) = \text{card}\left( \{k \geq 1: u_k = n \} \right).$$
Let $\text{card} \ (f^{-1}(0)) = \Delta$. The sequence $U$ also has the following property: If $|u_k| = n$, then for every integer $m \notin f^{-1}(0)$ with $|m| < n$ there is a positive integer $j < k$ with $u_j = m$. It follows that

$$
\{0, 1, -1, 2, -2, ..., n - 1, -(n - 1)\} \setminus f^{-1}(0) \subseteq \{u_1, u_2, ..., u_{k-1}\},
$$

and so

$$
k - 1 \geq 2(n - 1) + 1 - \Delta.
$$

This implies that

$$
|u_k| = n \leq \frac{k + \Delta}{2}.
$$

Since $u_k$ is an integer, we have

$$
|u_k| \leq \left\lfloor \frac{k + \Delta}{2} \right\rfloor.
$$

This completes the proof. \(\Box\)

Lemma 1 is best possible in the sense that for every nonnegative integer $\Delta$ there is a function $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ with $\text{card} \ (f^{-1}(0)) = \Delta$ and a sequence $U = \{u_k\}_{k=1}^\infty$ of integers such that

(1) \quad $|u_k| = \left\lfloor \frac{k + \Delta}{2} \right\rfloor$ \quad for all $k \geq 1$.

For example, if $\Delta = 2\delta + 1$ is odd, define the function $f$ by

$$
f(n) = \begin{cases} 
0 & \text{if } |n| \leq \delta \\
1 & \text{if } |n| \geq \delta + 1 
\end{cases}
$$

and the sequence $U$ by

$$
u_{2i-1} = \delta + i,
$$

$$
u_{2i} = -(\delta + i)
$$

for all $i \geq 1$.

If $\Delta = 2\delta$ is even, define $f$ by

$$
f(n) = \begin{cases} 
0 & \text{if } -\delta \leq n \leq \delta - 1 \\
1 & \text{if } n \geq \delta \text{ or } n \leq -\delta - 1 
\end{cases}
$$

and the sequence $U$ by $u_1 = \delta$ and

$$
u_{2i} = \delta + i,
$$

$$
u_{2i+1} = -(\delta + i)
$$

for all $i \geq 1$. In both cases the sequence $U$ satisfies (1).
The set $A$ is called a Sidon set of order $h$ if $r_{A,h}(n) = 0$ or 1 for every integer $n$. If $A$ is a Sidon set of order $h$, then $A$ is a Sidon set of order $j$ for all $j = 1, 2, \ldots, h$.

**Lemma 2.** Let $A$ be a finite Sidon set of order $h$ and $d = \max(||a| : a \in A||)$. If $|c| > (2h - 1)d$, then $A \cup \{c\}$ is also a Sidon set of order $h$.

**Proof:** Let $n \in h(A \cup \{c\})$. Suppose that
\[
n = a_1 + \cdots + a_j + (h - j)c = a'_1 + \cdots + a'_\ell + (h - \ell)c,
\]
where
\[
0 \leq j \leq \ell \leq h,
\]
\[
a_1, \ldots, a_j, a'_1, \ldots, a'_\ell \in A,
\]
and
\[
a_1 \leq \cdots \leq a_j \quad \text{and} \quad a'_1 \leq \cdots \leq a'_\ell.
\]
If $j < \ell$, then
\[
|c| \leq |(\ell - j)c| = |a'_1 + \cdots + a'_\ell - (a_1 + \cdots + a_j)| \leq (\ell + j)d \leq (2h - 1)d < |c|,
\]
which is absurd. Therefore, $j = \ell$ and $a_1 + \cdots + a_j = a'_1 + \cdots + a'_\ell$. Since $A$ is a Sidon set of order $j$, it follows that $a_i = a'_i$ for all $i = 1, \ldots, j$. Consequently, $A \cup \{c\}$ is a Sidon set of order $h$.

**3 – Construction of asymptotic bases**

We can now construct asymptotic bases of order $h$ for the integers with arbitrary representation functions.

**Theorem 1.** Let $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ be a function such that the set $f^{-1}(0)$ is finite. Let $\varphi : \mathbb{N}_0 \to \mathbb{R}$ be a nonnegative function such that $\lim_{x \to \infty} \varphi(x) = \infty$. For every $h \geq 2$ there exist infinitely many asymptotic bases $A$ of order $h$ for the integers such that
\[
r_{A,h}(n) = f(n) \quad \text{for all} \quad n \in \mathbb{Z},
\]
and
\[ A(-x, x) \leq \varphi(x) \]
for all \( x \geq 0 \).

**Proof:** By Lemma 1, there is a sequence \( U = \{u_k\}_{k=1}^\infty \) of integers such that
\[ f(n) = \text{card}\left( \{k \geq 1: u_k = n\} \right) \]
for every integer \( n \).

Let \( h \geq 2 \). We shall construct an ascending sequence of finite sets \( A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \) such that, for all positive integers \( k \) and for all integers \( n \),

(i) \[ r_{A_k,h}(n) \leq f(n) \ , \]

(ii) \[ r_{A_k,h}(n) \geq \text{card}\left( \{i: 1 \leq i \leq k \text{ and } u_i = n\} \right) \ , \]

(iii) \[ \text{card}(A_k) \leq 2k \ , \]

(iv) \( A_k \) is a Sidon set of order \( h - 1 \).

Conditions (i) and (ii) imply that the infinite set
\[ A = \bigcup_{k=1}^\infty A_k \]
is an asymptotic basis of order \( h \) for the integers such that \( r_{A,h}(n) = f(n) \) for all \( n \in \mathbb{Z} \).

We construct the sets \( A_k \) by induction. Since the set \( f^{-1}(0) \) is finite, there exists a nonnegative integer \( d_0 \) such that \( f(n) \geq 1 \) for all integers \( n \) with \( |n| \geq d_0 \). If \( u_1 \geq 0 \), choose a positive integer \( c_1 > 2hd_0 \). If \( u_1 < 0 \), choose a negative integer \( c_1 < -2hd_0 \). Then
\[ |c_1| > 2hd_0 \ . \]

Let
\[ A_1 = \{ -c_1, (h-1)c_1 + u_1 \} \ . \]
The sumset $hA_1$ is the finite arithmetic progression
\[
hA_1 = \{-hc_1 + (hc_1 + u_1)i: \ i = 0, 1, ..., h\} = \{-hc_1, u_1, hc_1 + 2u_1, 2hc_1 + 3u_1, ..., (h-1)hc_1 + hu_1\}.
\]
Then $|n| \geq h|c_1| > d_0$ for all $n \in hA_1 \setminus \{u_1\}$. Since $f(u_1) \geq 1$, we have $r_{A_1,h}(n) = 1 \leq f(n)$ for all $n \in hA_1$. Similarly, since $r_{A_1,h}(n) = 0$ for all $n \notin hA_1$, it follows that
\[
r_{A_1,h}(n) \leq f(n)
\]
for all $n \in \mathbb{Z}$. The set $A_1$ is a Sidon set of order $h$, hence also a Sidon set of order $h-1$. Thus, the set $A_1$ satisfies conditions (i)--(iv).

We assume that for some integer $k \geq 2$ we have constructed a set $A_{k-1}$ satisfying conditions (i)--(iv). If
\[
r_{A_{k-1},h}(u_k) = \max\{|i|: 1 \leq i \leq k \text{ and } u_i = u_k\}.
\]
and
\[
r_{A_{k,1},h}(n) = \begin{cases} r_{A_{k-1},h}(n) + 1 & \text{if } n = u_k \\ r_{A_{k-1},h}(n) & \text{if } n \in hA_{k-1} \setminus \{u_k\} \\ 1 & \text{if } n \in hA_k \setminus (hA_{k-1} \cup \{u_k\}) \end{cases}
\]
Define the nonnegative integer
\[
d_{k-1} = \max\{|a|: a \in A_{k-1} \cup \{u_k\}\}.
\]
Then
\[
A_{k-1} \subseteq [-d_{k-1}, d_{k-1}].
\]
If $u_k \geq 0$, choose a positive integer $c_k$ such that $c_k > 2hd_{k-1}$. If $u_k < 0$, choose a negative integer $c_k$ such that $c_k < -2hd_{k-1}$. Then
\[
|c_k| > 2hd_{k-1}.
\]
Let
\[
A_k = A_{k-1} \cup \{-c_k, (h-1)c_k + u_k\}.
\]
Then
\[
\text{card}(A_k) = \text{card}(A_{k-1}) + 2 \leq 2k .
\]

We shall assume that \( u_k \geq 0 \), hence \( c_k > 0 \). (The argument in the case \( u_k < 0 \) is similar.) We decompose the sumset \( hA_k \) as follows:
\[
hA_k = \bigcup_{r+i+j=h} \left( r(h-1)c_k + ru_k - ic_k + jA_{k-1} \right) = \bigcup_{r=0}^h B_r ,
\]
where
\[
B_r = r(h-1)c_k + ru_k + \bigcup_{i=0}^{h-r} (-ic_k + (h-r-i)A_{k-1}) .
\]
If \( n \in B_r \), then there exist integers \( i \in \{0,1,...,h-r\} \) and \( y \in (h-r-i)A_{k-1} \) such that
\[
n = r(h-1)c_k + ru_k - ic_k + y .
\]
Since
\[
|y| \leq (h-r-i)d_{k-1} ,
\]
it follows that
\[
(5) \quad n \geq r(h-1)c_k + ru_k - ic_k - (h-r-i)d_{k-1}
\]
and
\[
n \leq r(h-1)c_k + ru_k - ic_k + (h-r-i)d_{k-1} .
\]

Let \( m \in B_{r-1} \) and \( n \in B_r \) for some \( r \in \{1,...,h\} \). There exist nonnegative integers \( i \leq h-r \) and \( j \leq h-r+1 \) such that
\[
n - m \geq \left( r(h-1)c_k + ru_k - ic_k - (h-r-i)d_{k-1} \right)
\]
\[
- \left( (r-1)(h-1)c_k + (r-1)u_k - jc_k + (h-r+1-j)d_{k-1} \right)
\]
\[
= (h-1+j-i)c_k + u_k - (2h-2r-i-j+1)d_{k-1}
\]
\[
\geq (h-1-i)c_k - 2hd_{k-1} .
\]
If \( r \geq 2 \), then \( i \leq h-2 \) and inequality (4) implies that
\[
n - m \geq c_k - 2hd_{k-1} > 0 .
\]
Therefore, if \( m \in B_{r-1} \) and \( n \in B_r \) for some \( r \in \{2,...,h\} \), then \( m < n \).

In the case \( r = 1 \) we have \( m \in B_0 \) and \( n \in B_1 \). If \( i \leq h-2 \), then (4) implies that
\[
n - m \geq (h-1-i)c_k - 2hd_{k-1} \geq c_k - 2hd_{k-1} > 0
\]
and (5) implies that

\[ n \geq (h-1-i)c_k + u_k - (h-1-i)d_{k-1} > c_k - hd_{k-1} > d_0 . \]

If \( r = 1 \) and \( i = h - 1 \), then \( n = u_k \). Therefore, if \( m \in B_0 \) and \( n \in B_1 \), then \( m < n \) unless \( m = n = u_k \). It follows that the sets \( B_0, B_1 \setminus \{u_k\}, B_2, \ldots, B_h \) are pairwise disjoint.

Let \( n \in B_r \) for some \( r \geq 1 \). Suppose that \( 0 \leq i \leq j \leq h - r \), and that

\[ n = r(h-1)c_k + ru_k - ic_k + y \quad \text{for some} \quad y \in (h-r-i)A_{k-1} \]

and

\[ n = r(h-1)c_k + ru_k - jc_k + z \quad \text{for some} \quad z \in (h-r-j)A_{k-1} . \]

Subtracting these equations, we obtain

\[ z - y = (j - i)c_k . \]

Recall that \( |a| \leq d_{k-1} \) for all \( a \in A_{k-1} \). If \( i < j \), then

\[ c_k \leq (j - i)c_k = z - y \]
\[ \leq |y| + |z| \leq (2h - 2r - i - j)d_{k-1} \]
\[ < 2hd_{k-1} < c_k , \]

which is impossible. Therefore, \( i = j \) and \( y = z \). Since \( 0 \leq h - r - i \leq h - 1 \) and \( A_{k-1} \) is a Sidon set of order \( h - 1 \), it follows that

\[ r_{A_{k-1}, h-r-i}(y) = 1 \]

and so

\[ r_{A_k, h}(n) = 1 \leq f(n) \quad \text{for all} \quad n \in (B_1 \setminus \{u_k\}) \cup \bigcup_{r=2}^{h} B_r . \]

Next we consider the set

\[ B_0 = hA_{k-1} \cup \bigcup_{i=1}^{h} \left( -ic_k + (h-i)A_{k-1} \right) . \]

For \( i = 1, \ldots, h \), we have

\[ c_k > 2hd_{k-1} \geq (2h - 2i + 1)d_{k-1} \]
and so
\[
\max\left(-ic_k + (h-i)A_{k-1}\right) \leq -ic_k + (h-i)d_{k-1} < -(i-1)c_k - (h-i+1)d_{k-1} \leq \min\left(-(i-1)c_k + (h-i+1)A_{k-1}\right).
\]
Therefore, the sets $-ic_k + (h-i)A_{k-1}$ are pairwise disjoint for $i = 0, 1, ..., h$.

In particular, if $n \in B_0 \setminus hA_{k-1}$, then
\[
n \leq \max\left(-c_k + (h-1)A_{k-1}\right) \leq -c_k + (h-1)d_{k-1} < -d_{k-1} \leq -d_0
\]
and $f(n) \geq 1$. Since $A_{k-1}$ is a Sidon set of order $h-1$, it follows that
\[
r_{A_k,h}(n) = 1 \leq f(n)
\]
for all
\[
n \in \bigcup_{i=1}^{h} \left(-ic_k + (h-i)A_{k-1}\right) = B_0 \setminus hA_{k-1}.
\]
It follows from (3) that for any $n \in B_0 \setminus hA_{k-1}$ we have
\[
n < -d_{k-1} \leq u_k,
\]
and so $u_k \notin B_0 \setminus hA_{k-1}$. Therefore,
\[
r_{A_k,h}(u_k) = r_{A_{k-1,h}}(u_k) + 1,
\]
and the representation function $r_{A_k,h}$ satisfies the three requirements of (2).

We shall prove that
\[
A_k = A_{k-1} \cup \{-c_k, (h-1)c_k + u_k\}
\]
is a Sidon set of order $h-1$. Since $A_{k-1}$ is a Sidon set of order $h-1$ with $d_{k-1} \geq \max\{|a| : a \in A_{k-1}\}$, and since
\[
c_k > 2hd_{k-1} > (2(h-1) - 1)d_{k-1},
\]
Lemma 2 implies that $A_{k-1} \cup \{-c_k\}$ is a Sidon set of order $h-1$.

Let $n \in (h-1)A_k$. Suppose that
\[
n = r(h-1)c_k + ru_k - ic_k + x
\]
\[
= s(h-1)c_k + su_k - jc_k + y,
\]
where
\[ 0 \leq r \leq s \leq h - 1, \]
\[ 0 \leq i \leq h - 1 - r, \]
\[ 0 \leq j \leq h - 1 - s, \]
\[ x \in (h - 1 - r - i)A_{k-1}, \]
and
\[ y \in (h - 1 - s - j)A_{k-1}. \]

Then
\[ |x| \leq (h - 1 - r - i)d_{k-1} \]
and
\[ |y| \leq (h - 1 - s - j)d_{k-1}. \]

If \( r < s \), then \( j \leq h - 2 \) and
\[
(h - 1)c_k \leq (s - r)(h - 1)c_k + (s - r)u_k \\
= (j - i)c_k + x - y \\
\leq (j - i)c_k + (2h - 2 - r - s - i - j)d_{k-1} \\
\leq (h - 2)c_k + 2hd_{k-1} \\
< (h - 1)c_k,
\]
which is absurd. Therefore, \( r = s \) and
\[ -ic_k + x = -jc_k + y \in (h - 1 - r)(A_k \cup \{-c_k\}). \]

Since \( A_k \cup \{-c_k\} \) is a Sidon set of order \( h - 1 \), it follows that \( i = j \) and that \( x \)
has a unique representation as the sum of \( h - 1 - r - i \) elements of \( A_k \). Thus, \( A_k \)
is a Sidon set of order \( h - 1 \).

The set \( A_k \) satisfies conditions (i)–(iv). It follows by induction that there exists
an infinite increasing sequence \( A_1 \subseteq A_2 \subseteq \cdots \) of finite sets with these properties,
and that \( A = \bigcup_{k=1}^\infty A_k \) is an asymptotic basis of order \( h \) with representation
function \( r_{A,h}(n) = f(n) \) for all \( n \in \mathbb{Z} \).

Finally, we shall prove that, for every nonnegative function \( \varphi(x) \) with
\( \lim_{x \to -\infty} \varphi(x) = \infty \), there exist infinitely many asymptotic bases \( A \) of order \( h \)
such that \( r_{A,h}(n) = f(n) \) for all \( n \in \mathbb{Z} \) and \( A(-x, x) \leq \varphi(x) \) for all \( x \in N_0 \). Let
\( A_0 = \emptyset \), and let \( K' \) be the set of all positive integers \( k \) such that \( A_k \neq A_{k-1} \).
Then \( 1 \in K' \) and
\[
A = \bigcup_{k \in K'} A_k = \bigcup_{k \in K'} \{-c_k, (h-1)c_k\}. 
\]
For each $k \in K'$, the only constraints on the choice of the number $c_k$ in the construction of the set $A_k$ were the sign of $c_k$ and the growth condition (4)

$$|c_k| > 2hd_{k-1}.$$ 

Since $\varphi(x) \to \infty$ as $x \to \infty$, for every integer $k \geq 0$ there exists an integer $w_k$ such that

$$\varphi(x) \geq 2k \quad \text{for all } x \geq w_k.$$

We now impose the following additional constraint: Choose $c_k$ such that

$$|c_k| \geq w_k \quad \text{for all integers } k \in K'.$$

Then

$$A_1(-x, x) = 0 \leq \varphi(x) \quad \text{for } 0 \leq x < |c_1|$$

and

$$A_1(-x, x) \leq 2 \leq \varphi(x) \quad \text{for } x \geq |c_1| \geq w_1.$$

Suppose that $k \geq 2$ and the set $A_{k-1}$ satisfies $A_{k-1}(-x, x) \leq \varphi(x)$ for all $x \geq 0$. If $k \notin K'$, then $A_k = A_{k-1}$ and $A_k(-x, x) \leq \varphi(x)$ for all $x \geq 0$. If $k \in K$, then

$$A_k \cap (-|c_k|, |c_k|) = A_{k-1} \cap (-|c_k|, |c_k|) = A_{k-1},$$

and so

$$A_k(-x, x) = A_{k-1}(-x, x) \leq \varphi(x) \quad \text{for } 0 \leq x < |c_k|$$

and

$$A_k(-x, x) \leq 2k \leq \varphi(x) \quad \text{for } x \geq |c_k| \geq w_k.$$

It follows by induction that the finite sets $A_k$ satisfy $A_k(-x, x) \leq \varphi(x)$ for all $k$ and $x$. The infinite set $A = \cup_{k \in K'} A_k$ is an asymptotic basis with $r_{A,h}(n) = f(n)$ for all $n \in \mathbb{Z}$. Since $\lim_{k \to \infty} |c_k| = \infty$, for every nonnegative integer $x$ we can choose $k \in K'$ such that $|c_k| > x$. It follows that

$$A(-x, x) = A_k(-x, x) \leq \varphi(x).$$

For every integer $k \in K'$ we had infinitely many choices for the integer $c_k$ to use in the construction of the set $A_k$, and so there are infinitely many asymptotic bases $A$ with the property that $r_A(n) = f(n)$ for all $n \in \mathbb{Z}$ and $A(-x, x) \leq \varphi(x)$ for all $x \in \mathbb{N}_0$. This completes the proof. \ \framebox{ }
4 – Sums of pairwise distinct integers

Let \( A \) be a set of integers and \( h \) a positive integer. We define the sumset \( h \cdot A \) as the set consisting of all sums of \( h \) pairwise distinct elements of \( A \), and the restricted representation function

\[
\hat{r}_{A,h} : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}
\]

by

\[
\hat{r}_{A,h}(n) = \text{card}\left(\{a_1, \ldots, a_h \subseteq A: a_1 + \cdots + a_h = n \text{ and } a_1 < \cdots < a_h\}\right).
\]

The set \( A \) of integers is called a restricted asymptotic basis of order \( h \) if \( h \cdot A \) contains all but finitely many integers, or, equivalently, if \( \hat{r}_{A,h}^{-1}(0) \) is a finite subset of \( \mathbb{Z} \).

We can obtain the following result by the same method used to prove Theorem 1.

**Theorem 2.** Let \( f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\} \) be a function such that \( f^{-1}(0) \) is a finite set of integers. Let \( \varphi : \mathbb{N}_0 \to \mathbb{R} \) be a nonnegative function such that \( \lim_{x \to \infty} \varphi(x) = \infty \). For every \( h \geq 2 \) there exist infinitely many sets \( A \) of integers such that

\[
\hat{r}_{A,h}(n) = f(n) \quad \text{for all} \quad n \in \mathbb{Z}
\]

and

\[
A(-x, x) \leq \varphi(x)
\]

for all \( x \geq 0 \).  

5 – Open problems

Let \( X \) be an abelian semigroup, written additively, and let \( A \) be a subset of \( X \). We define the \( h \)-fold sumset \( hA \) as the set consisting of all sums of \( h \) not necessarily distinct elements of \( A \). The set \( A \) is called an asymptotic basis of order \( h \) for \( X \) if the sumset \( hA \) consists of all but at most finitely many elements of \( X \). We also define the \( h \)-fold restricted sumset \( h \cdot A \) as the set consisting of all sums of \( h \) pairwise distinct elements of \( A \). The set \( A \) is called a restricted asymptotic basis of order \( h \) for \( X \) if the restricted sumset \( h \cdot A \) consists of all but at most
...REPRESENTATION FUNCTION OF AN ADDITIVE BASIS...

finitely many elements of $X$. The classical problems of additive number theory concern the semigroups $\mathbb{N}_0$ and $\mathbb{Z}$.

There are four different representation functions that we can associate to every subset $A$ of $X$ and every positive integer $h$. Let $(a_1, ..., a_h)$ and $(a'_1, ..., a'_h)$ be $h$-tuples of elements of $X$. We call these $h$-tuples equivalent if there is a permutation $\sigma$ of the set $\{1, ..., h\}$ such that $a'_{\sigma(i)} = a_i$ for all $i = 1, ..., h$. For every $x \in X$, let $r_{A,h}(x)$ denote the number of equivalence classes of $h$-tuples $(a_1, ..., a_h)$ of elements of $A$ such that $a_1 + \cdots + a_h = x$. The function $r_{A,h}$ is called the unordered representation function of $A$. This is the function that we studied in this paper.

The set $A$ is an asymptotic basis of order $h$ if $r^{(-1)}_{A,h}(0)$ is a finite subset of $X$.

Let $R_{A,h}(x)$ denote the number of $h$-tuples $(a_1, ..., a_h)$ of elements of $A$ such that $a_1 + \cdots + a_h = x$. The function $R_{A,h}$ is called the ordered representation function of $A$.

Let $\hat{r}_{A,h}(x)$ denote the number of equivalence classes of $h$-tuples $(a_1, ..., a_h)$ of pairwise distinct elements of $A$ such that $a_1 + \cdots + a_h = x$, and let $\hat{R}_{A,h}(x)$ denote the number of $h$-tuples $(a_1, ..., a_h)$ of pairwise distinct elements of $A$ such that $a_1 + \cdots + a_h = x$. These functions are called the unordered restricted representation function of $A$ and the ordered restricted representation function of $A$, respectively. The two restricted representation functions are essentially identical, since $\hat{R}_{A,h}(x) = h!\hat{r}_{A,h}(x)$ for all $x \in X$.

In the discussion below, we use only the unordered representation function $r_{A,h}$, but each of the problems can be reformulated in terms of the other representation functions.

For every countable abelian semigroup $X$, let $\mathcal{F}(X)$ denote the set of all functions $f : X \to \mathbb{N}_0 \cup \{\infty\}$, and let $\mathcal{F}_0(X)$ denote the set of all functions $f : X \to \mathbb{N}_0 \cup \{\infty\}$ such that $f^{-1}(0)$ is a finite subset of $X$. Let $\mathcal{F}_c(X)$ denote the set of all functions $f : X \to \mathbb{N}_0 \cup \{\infty\}$ such that $f^{-1}(0)$ is a cofinite subset of $X$, that is, $f(x) \neq 0$ for only finitely many $x \in X$, or, equivalently,

$$\text{card} \left( f^{-1}(\mathbb{N} \cup \{\infty\}) \right) < \infty.$$ 

Let $\mathcal{R}(X,h)$ denote the set of all $h$-fold representation functions of subsets $A$ of $X$. If $r_{A,h}$ is the representation function of an asymptotic basis $A$ of order $h$ for $X$, then $r^{-1}_{A,h}(0)$ is a finite subset of $X$, and so $r_{A,h} \in \mathcal{F}_0(X)$. Let $\mathcal{R}_0(X,h)$ denote the set of all $h$-fold representation functions of asymptotic bases $A$ of order $h$ for $X$. Let $\mathcal{R}_c(X,h)$ denote the set of all $h$-fold representation functions of finite subsets of $X$. We have

$$\mathcal{R}(X,h) \subseteq \mathcal{F}(X),$$
\[ \mathcal{R}_0(X, h) \subseteq \mathcal{F}_0(X) , \]

and

\[ \mathcal{R}_c(X, h) \subseteq \mathcal{F}_c(X) . \]

In the case \( h = 1 \), we have, for every set \( A \subseteq X \),
\[ r_{A,1}(x) = \begin{cases} 
1 & \text{if } x \in A , \\
0 & \text{if } x \notin A , 
\end{cases} \]

and so
\[ \mathcal{R}(X, 1) = \left\{ f: X \to \{0, 1\} \right\} , \]
\[ \mathcal{R}_0(X, 1) = \left\{ f: X \to \{0, 1\} : \text{card}(f^{-1}(0)) < \infty \right\} , \]

and
\[ \mathcal{R}_c(X, 1) = \left\{ f: X \to \{0, 1\} : \text{card} \left( f^{-1}(\mathbb{N} \cup \{\infty\}) \right) < \infty \right\} . \]

In this paper we proved that
\[ \mathcal{R}_0(\mathbb{Z}, h) = \mathcal{F}_0(\mathbb{Z}) \quad \text{for all } h \geq 2 . \]

Nathanson [8] has extended this result to certain countably infinite groups and semigroups. Let \( G \) be any countably infinite abelian group such that \( \{2g : g \in G\} \) is infinite. For the unordered restricted representation function \( r_{A,2} \), we have
\[ \mathcal{R}_0(G, 2) = \mathcal{F}_0(G) . \]

More generally, let \( S \) is any countable abelian semigroup such that for every \( s \in S \) there exist \( s', s'' \in S \) with \( s = s' + s'' \). In the abelian semigroup \( X = S \oplus G \), we have
\[ \mathcal{R}_0(X, 2) = \mathcal{F}_0(X) . \]

If \( \{12g : g \in G\} \) is infinite, then \( \mathcal{R}_0(X, 2) = \mathcal{F}_0(X) \) for the unordered representation function \( r_{A,2} \).

The following problems are open for all \( h \geq 2 \):

1. Determine \( \mathcal{R}_0(\mathbb{N}_0, h) \). Equivalently, describe the representation functions of additive bases for the nonnegative integers. This is a major unsolved problem in additive number theory, of which the Erdős–Turán conjecture is only a special case.
2. Determine $R(Z, h)$. In this paper we computed $R_0(Z, h)$, the set of representation functions of additive bases for the integers, but it is not known under what conditions a function $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ with $f^{-1}(0)$ infinite is the representation function of a subset $A$ of $X$. It can be proved that if $f^{-1}(0)$ is infinite but sufficiently sparse, then $f \in R(Z, h)$.

3. Determine $R(N_0, h)$. Is there a simple list of necessary and sufficient conditions for a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ with $f^{-1}(0)$ infinite but sufficiently sparse, then $f \in R(N_0, h)$?

4. Determine $R_c(Z, h)$. Equivalently, describe the representation functions of finite sets of integers, and identify the functions $f \in F_c(Z)$ such that $f(n) = r_{A,h}(n)$ for some finite set $A$ of integers. If $A$ is a set of integers and $t$ is an integer, then for the translated set $t + A$ we have

$$r_{t+A,h}(n) = r_{A,h}(n - ht)$$

for all integers $n$. This implies that if $f(n) \in R_c(Z, h)$, then $f(n - ht) \in R_c(Z, h)$ for every integer $t$, so it suffices to consider only finite sets $A$ of nonnegative integers with $0 \in A$. Similarly, if $\gcd(A) = d$, then $r_{A,h}(n) > 0$ only if $d$ divides $n$. Setting $B = \{a/d : d \in A\}$, we have $r_{h,A}(n) = r_{B,h}(n/d)$. It follows that we need to consider only finite sets $A$ of relatively prime nonnegative integers with $0 \in A$.

5. Determine $R_0(G, 2)$, $R(G, 2)$, and $R_c(G, 2)$ for the infinite abelian group $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$. Note that $\{2g : g \in G\} = \{0\}$ for this group.

6. Determine $R_0(G, h)$ and $R(G, h)$, where $G$ is an arbitrary countably infinite abelian group and $h \geq 2$.

7. There is a class of problems of the following type. Do there exist integers $h$ and $k$ with $2 \leq h < k$ such that

$$R(Z, h) \neq R(Z, k) ?$$

We can easily find sets of integers to show that $R_0(\mathbb{N}_0, h) \neq R_0(\mathbb{N}_0, k)$. For example, let $A = \mathbb{N}$ be the set of all positive integers, and let $h \geq 1$. Then $r_{\mathbb{N},h}(0) = 0$ and $r_{\mathbb{N},h}(h) = 1$. If $B$ is any set of nonnegative integers and $k > h$, then $r_{B,k}(h) = 0$, and so $r_{\mathbb{N},h} \not\in R_0(\mathbb{N}_0, k)$. Is it true that

$$R_0(\mathbb{N}_0, h) \cap R_0(\mathbb{N}_0, k) = \emptyset$$

for all $h \neq k$?
8. By Theorem 1, for every $h \geq 2$ and every function $f \in \mathcal{F}_0(\mathbb{Z})$, there exist arbitrarily sparse sets $A$ of integers such that $r_{A,h}(n) = f(n)$ for all $n$. It is an open problem to determine how dense the sets $A$ can be. For example, in the special case $h = 2$ and $f(n) = 1$, Nathanson [7] proved that there exists a set $A$ such that $r_{A,2}(n) = 1$ for all $n$, and $\log x \ll A(-x,x) \ll \log x$. For an arbitrary representation function $f \in \mathcal{F}_0(\mathbb{Z})$, Nathanson [6] constructed an asymptotic basis of order $h$ with $A(-x,x) \gg x^{1/(2h-1)}$. In the case $h = 2$, Cilleruelo and Nathanson [1] improved this to $A(-x,x) \gg x^{\sqrt{2}-1+o(1)}$.

**REFERENCES**


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