A VIABILITY RESULT FOR A FIRST-ORDER DIFFERENTIAL INCLUSION

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Abstract: This paper deals with the existence of solutions of a first-order viability problem of the type

\[ \dot{x} \in f(t, x) + F(x), \quad x(t) \in K \]

where \( K \) a closed subset of \( \mathbb{R}^n \), \( F \) is upper semicontinuous with compact values contained in the subdifferential \( \partial V(x) \) of a convex proper lower semicontinuous function \( V \) and \( f \) is a Carathéodory single valued map.

1 – Introduction

Bressan, Cellina and Colombo [1] proved the existence of solutions of the problem \( \dot{x} \in F(x), \ x(0) = x_0 \in K \), where \( F \) is an upper semicontinuous multifunction contained in the subdifferential of a convex proper lower semicontinuous function in the finite dimensional space. This result has been generalized by Ancona and Colombo [2] by proving the existence of solutions of the perturbed problem \( \dot{x} \in F(x) + f(t, x), \ x(0) = x_0 \), with \( f \) satisfying the Carathéodory conditions.

The proof is based on approximate solutions; to overcome the weak convergence of derivatives of such solutions, the authors use the following basic relation:

\[ \frac{d}{dt} (V(x(t))) = \| \dot{x}(t) \|^2. \]

The aim of the present paper is to prove a viability result of the following problem:

\[
\begin{aligned}
\dot{x} &\in f(t, x) + F(x) \quad \text{a.e. } t \in [0, T], \\
x(0) &= x_0 \in K, \\
x(t) &\in K \quad \forall t \in [0, T].
\end{aligned}
\]
where \( F \) is an upper semicontinous with compact valued multifunction such that \( F(x) \subset \partial V(x) \), for some convex proper lower semicontinuous function \( V \) and \( f \) is a Carathéodory function.

This paper is a generalization of the work of Rossi [5]. Our argument is different from the one appearing in Rossi’s paper.

2 – The result

Let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space with scalar product \( \langle ., . \rangle \) and norm \( ||.|| \). Let \( K \) be a closed subset of \( \mathbb{R}^n \). Let \( F \) be a multifunction from \( \mathbb{R}^n \) into the set of all nonempty compact subsets of \( \mathbb{R}^n \). Let \( f \) be a function from \( \mathbb{R} \times \mathbb{R}^n \) into \( \mathbb{R}^n \). Assume that \( F \) and \( f \) satisfy the following conditions:

A1) \( F \) is upper semicontinuous, i.e. for all \( x \in \mathbb{R}^n \) and for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( ||x - x'|| \leq \delta \) then \( F(x') \subseteq F(x) + \varepsilon B \), where \( B \) is the unit ball of \( \mathbb{R}^n \).

A2) There exists a convex proper and lower semicontinuous function \( V : \mathbb{R}^n \to \mathbb{R} \) such that \( F(x) \subset \partial V(x) \), where \( \partial V \) is the subdifferential of the function \( V \).

A3) \( f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) is a Carathéodory function, i.e. for every \( x \in \mathbb{R}^n \), \( t \to f(t, x) \) is measurable and for all \( t \in \mathbb{R} \), \( x \to f(t, x) \) is continuous.

A4) There exists \( m \in L^2(\mathbb{R}) \) such that

\[
\| f(t, x) \| \leq m(t) \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^n .
\]

A5) (Tangential condition) \( \forall (t, x) \in \mathbb{R} \times K, \exists v \in F(x) \) such that

\[
\lim_{h \to 0^+} \inf \frac{1}{h} d_K \left( x + hv + \int_{t}^{t+h} f(s, x) \, ds \right) = 0 .
\]

Let \( x_0 \in K \), let \( f \) and \( F \) satisfying assumptions A1, ..., A5, then we shall prove the following result:

**Theorem 1.** There exist \( T > 0 \) and \( x : [0, T] \to \mathbb{R}^n \) such that

\[
\begin{cases}
  x(t) \in f(t, x(t)) + F(x(t)) & \text{a.e. on } [0, T] , \\
  x(0) = x_0 \in K , \\
  x(t) \in K & \forall t \in [0, T] .
\end{cases}
\]
3 – Proof of the main result

Lemma 2. Let $V$ be a convex proper lower semicontinuous function such that for all $x \in \mathbb{R}^n$, $F(x) \subset \partial V(x)$. Then there exist $r = r_x > 0$ and $M = M_x > 0$ such that $\|F(x)\| = \sup_{z \in F(x)} \|z\| \leq M$ and $V$ is Lipschitz continuous with constant $M$ on $B(x, r)$. ■

For the proof, see [1].

Let $r$ be the real given by Lemma 2 associated to $x_0$. Choose $T > 0$ such that

$$\int_0^T \left( m(s) + M + 1 \right) ds < \frac{r}{2}.$$ 

In all the sequel, denote by $K_0$ the compact subset $K \cap \overline{B}(x_0, r)$.

Lemma 3. Assume that $F$ and $f$ satisfy $A_1, \ldots, A_5$. Then for all $\varepsilon > 0$, there exists $\eta > 0$ ($\eta < \varepsilon$) with the following properties:

For all $(t, x) \in [0, T] \times K_0$, there exist $u \in F(x) + \frac{\varepsilon}{T} B$ and $h_{t, x} \in [\eta, \varepsilon]$ such that

$$x + h_{t, x} u + \int_t^{t + h_{t, x}} f(s, x) ds \in K.$$ 

Proof: Let $(t, x) \in [0, T] \times K_0$, let $\varepsilon > 0$. Since $F$ is upper semicontinuous, then there exists $\delta_x > 0$ such that

$$F(y) \subset F(x) + \varepsilon B, \quad \forall y \in B(x, \delta_x).$$

Let $(s, y) \in [0, T] \times K$. By the tangential condition there exist $h_{s, y} \in [0, \varepsilon]$ and $v \in F(y)$ such that

$$d_K \left( y + h_{s, y} v + \int_s^{s + h_{s, y}} f(\tau, y) d\tau \right) < h_{s, y} \frac{\varepsilon}{4T}.$$ 

Consider the subset

$$N(s, y) = \left\{ (t, z) \in \mathbb{R} \times \mathbb{R}^n / d_K \left( z + h_{s, y} v + \int_t^{t + h_{s, y}} f(\tau, z) d\tau \right) < h_{s, y} \frac{\varepsilon}{4T} \right\}.$$ 

Since

$$\|f(s, z)\| \leq m(s) \text{ a.e. on } [0, T], \quad \forall z \in \mathbb{R}^n$$
then, the dominated convergence theorem applied to the sequence of functions $(\chi_{[t,t+h_{s,y}]} f(\cdot,\cdot))_t$ shows that the function

$$(l, z) \rightarrow z + h_{s,y} v + \int_t^{t+h_{s,y}} f(\tau, z) \, d\tau$$

is continuous. So that, the function

$$(l, z) \rightarrow d_K \left( z + h_{s,y} v + \int_t^{t+h_{s,y}} f(\tau, z) \, d\tau \right)$$

is continuous and consequently the subset $N(s, y)$ is open.

Moreover, since $(s, y)$ belongs to $N(s, y)$, there exists a ball $B((s, y), \eta_{\tau,y})$ of radius $\eta_{\tau,y} < \delta_x$ contained in $N(s, y)$. Therefore, the compact subset $[0, T] \times K_0$ can be covered by $q$ such balls $B((s_i, y_i), \eta_{s_i,y_i})$. For simplicity, we set $h_{s_i,y_i} = h_i$, $i = 1, \ldots, q$. Put $\eta = \min_{i=1,\ldots,q} h_i > 0$.

Let $(t, x) \in [0, T] \times K_0$ be fixed. Since $(t, x) \in B((s_i, y_i), \eta_{s_i,y_i})$ which is contained in $N(s_i, y_i)$, then there exist $x_i \in K$ and $u_i \in F(y_i)$ such that

$$\left\| u_i - \frac{1}{h_i} \left( x_i - x - \int_t^{t+h_i} f(s, x) \, ds \right) \right\| \leq \frac{1}{h_i} d_K \left( x + h_i u_i + \int_t^{t+h_i} f(\tau, z) \, d\tau \right) + \frac{\varepsilon}{4T} \leq \frac{\varepsilon}{2T}.$$ 

Set

$$u = \frac{1}{h_i} \left( x_i - x - \int_t^{t+h_i} f(s, x) \, ds \right)$$

hence

$$x + h_i u + \int_t^{t+h_i} f(s, x) \, ds \in K$$

and

$$\| u_i - u \| \leq \frac{\varepsilon}{2T}.$$ 

Since

$$\| x - y_i \| < \eta_{\tau,y} < \delta_x$$

then

$$F(y_i) \subset F(x) + \frac{\varepsilon}{2T} B$$

so that

$$u \in F(x) + \frac{\varepsilon}{T} B.$$ 

Hence the Lemma 3 is proved. $\blacksquare$

Now, our purpose is to define on $[0, T]$ a family of approximate solutions and show that a subsequence converges to a solution of the problem (1.1).
4 – Construction of approximate solutions

Let $x_0 \in K_0$ and $\varepsilon < T$. By Lemma 3, there exist $\eta > 0$, $h_0 \in [\eta, \varepsilon]$ and $u_0 \in F(x_0) + \frac{\varepsilon}{T} B$ such that

$$x_1 = x_0 + h_0 u_0 + \int_0^{h_0} f(s, x_0) \, ds \in K$$

then if $h_0 \leq T$ we have

$$\|x_1 - x_0\| = \left\|h_0 u_0 + \int_0^{h_0} f(s, x_0) \, ds\right\| \leq \frac{r}{2}$$

and thus $x_1 \in K_0$. Hence for $(h_0, x_1)$ there exist $h_1 \in [\eta, \varepsilon]$ and $u_1 \in F(x_1) + \varepsilon T B$ such that

$$x_2 = x_1 + h_1 u_1 + \int_{h_0}^{h_0 + h_1} f(s, x_1) \, ds \in K$$

we have

$$\|x_2 - x_0\| = \left\|h_0 u_0 + \int_0^{h_0} f(s, x_0) \, ds + h_1 u_1 + \int_{h_0}^{h_0 + h_1} f(s, x_1) \, ds\right\|$$

then if $h_0 + h_1 < T$ we have

$$\|x_2 - x_0\| \leq \frac{r}{2}$$

thus $x_2 \in K_0$.

Set $h_{-1} = 0$, by induction, since $h_i$ belongs to $[\eta, \varepsilon]$, then there exists an integer $s$ such that $\sum_{i=0}^{s-1} h_i < T \leq \sum_{i=0}^{s} h_i$. Hence we construct the sequences $(h_p)_p \subset [\eta, \varepsilon]$, $(x_p)_p \subset K_0$, and $(u_p)_p$ such that for every $p = 0, \ldots, s-1$, we have

$$\begin{cases} x_{p+1} = x_p + h_p u_p + \int_{h_p}^{h_p + h_p} f(s, x_p) \, ds \in K \\ u_p \in F(x_p) + \frac{\varepsilon}{T} B \end{cases}$$

By induction, for all $p \geq 2$ we have

$$\begin{align*} x_p &= x_0 + \sum_{i=0}^{i=p-1} h_i u_i + \sum_{i=1}^{i=p-1} \int_{j=0}^{i} h_j f(\tau, x_\tau) \, d\tau \\ u_p &\in F(x_p) + \frac{\varepsilon}{T} B \end{align*}$$
and the estimates

\[ \| x_p - x_0 \| = \left\| \sum_{i=0}^{i=p-1} h_i u_i + \sum_{i=0}^{i=p-1} \int_{\tau_{i-1}}^{\tau_i} f(\tau, x_i) d\tau \right\| \]

\[ \leq (M + 1) \sum_{i=1}^{i=p-1} h_i + \int_0^T m(\tau) d\tau . \]

Since \( \sum_{i=0}^{i=p-1} h_i \leq T \), then we obtain \( \| x_p - x_0 \| \leq \frac{r}{2} \).

For any nonzero integer \( k \) and for every integer \( q = 0, \ldots, s \), denote by \( h^k_q \) a real associated to \( \varepsilon = \frac{1}{k} \) and \( x = x_q \) given by Lemma 3, consider the sequence \( (\tau^q_k)_k \)

\[ \begin{cases} \tau^0_k = 0, & \tau^s_k = T \\ \tau^q_k = h^k_0 + \cdots + h^k_{q-1} \end{cases} \]

and define on \([0, T]\) the sequence of functions \( (x_k(.)_k) \) by

\[ x_k(t) = x_{q-1} + (t - \tau^q_k) u_{q-1} + \int_{\tau^q_{k-1}}^{t} f(s, x_{q-1}) ds \quad \forall t \in [\tau^q_{k-1}, \tau^q_k] \]

\[ x_k(0) = x_0 \]

then for all \( t \in [\tau^q_{k-1}, \tau^q_k] \)

\[ \dot{x}_k(t) = u_{q-1} + f(t, x_{q-1}) . \]

5 – Convergence of approximate solutions

Observe that the sequence \( (x_k(.)_k) \) satisfies the following relations

1) \( \| \dot{x}_k(t) \| \leq \| u_{q-1} \| + \| f(t, x_{q-1}) \| \leq M + 1 + m(t) , \)

2) \( \| x_k(t) \| = \left\| x_k(\tau^q_k) + \int_{\tau^q_{k-1}}^{t} \dot{x}_k(\tau) d\tau \right\| \)

\[ \leq \| x_{q-1} \| + \int_0^T (M + 1 + m(t)) d\tau \]

\[ \leq \| x_0 \| + \frac{r}{2} + \frac{r}{2} \leq \| x_0 \| + r . \)
Hence

\[ \int_0^T \| \dot{x}_k(t) \|^2 \, dt \leq \int_0^T (M + 1 + m(t))^2 \, dt \]

the sequence \((\dot{x}_k(.))\) is bounded in \(L^2([0,T], \mathbb{R}^n)\) and therefore \((x_k(.))\) is equi-uniformly continuous. Hence there exists a subsequence, still denoted by \((x_k(.))\) and an absolutely continuous function \(x(.): [0, T] \rightarrow \mathbb{R}^n\) such that \(x_k(.)\) converges to \(x(.)\) uniformly and \(\dot{x}_k(.)\) converges weakly in \(L^2([0, T], \mathbb{R}^n)\) to \(\dot{x}(.).\)

The family of approximate solutions \(x_k(.)\) has the following property:

**Proposition 4.** For every \(t \in [0, T]\) there exists \(q \in \{1, ..., s\}\) such that

\[ \lim_{k \to \infty} d_{gr}F \left( x_k(t), \dot{x}_k(t) - f(t, x_k(\tau^q_k-1)) \right) = 0. \]

**Proof:** Let \(t \in [0, T]\). By construction of \(\tau^q_k\) there exists \(q\) such that \(t \in [\tau^q_k-1, \tau^q_k]\) and \((\tau^q_k)\) converges to \(t\).

Since

\[ \dot{x}_k(t) - f(t, x_k(\tau^q_k-1)) = u_{q-1} \in F(x_k(\tau^q_k-1)) + \frac{1}{kT} \]

then

\[ \lim_{k \to \infty} d_{gr(F)} \left( x_k(t), \dot{x}_k(t) - f(t, x_k(\tau^q_k-1)) \right) \leq \lim_{k \to \infty} \left( \| x_k(t) - x_k(\tau^q_k-1) \| + \frac{1}{kT} \right) \]

hence

\[ \lim_{k \to \infty} d_{gr(F)} \left( x_k(t), \dot{x}_k(t) - f(t, x_k(\tau^q_k-1)) \right) = 0. \]

This completes the proof. 

Since the sequences \(x_k(.) \rightarrow x(.)\) uniformly, \(\dot{x}_k(.) \rightarrow \dot{x}(.)\) weakly in \(L^2([0,T], \mathbb{R}^n)\), \((f(., x_k(\tau^q_k)))\) converges to \(f(., x(.))\) in \(L^2([0,T], \mathbb{R}^n)\) and \(F\) is upper semi-continuous, then by theorem 1.4.1 in [3], \(x(.)\) is a solution of the following convexified problem:

\[
\begin{cases}
\dot{x}(t) \in f(t, x(t)) + \text{co} F(x(t)) \\
x(0) = x_0.
\end{cases}
\]

Consequently, for all \(t \in [0, T]\) we have that

\[ \dot{x}(t) - f(t, x(t)) \in \partial V(x(t)). \]

**Proposition 5.** The application \(x(.)\) is a solution of the problem (1.1).
Proof: To begin with, we prove that \(|\|\dot{x}_k\|_2\|_k\) converges to \(|\|\dot{x}\|_2\). Since the map \(x(.)\) and \(V(x(.)\) are absolutely continuous, we obtain from (5.1) by applying Lemma 3.3 in [4] that
\[
\frac{d}{dt} V(x(t)) = \left\langle \dot{x}(t), \dot{x}(t) - f(t, x(t)) \right\rangle \quad \text{a.e. on } [0,T]
\]
therefore
\[
V(x(T)) - V(x_0) = \int_0^T \|\dot{x}(s)\|^2 \, ds - \int_0^T \left\langle \dot{x}(s), f(s, x(s)) \right\rangle \, ds.
\]
On the other hand, since for all \(q = 1, \ldots, s\)
\[
\dot{x}_k(t) - f(t, x_k(t)) = \dot{x}_k(t) - f(t, x_{q-1}) \in \partial V(x_k(t)) + \frac{1}{kT} B.
\]
there exists \(b_q \in B\) such that
\[
\dot{x}_k(t) - f(t, x_{q-1}) + \frac{1}{kT} b_q \in \partial V(x_k(t)).
\]
Moreover the subdifferential properties of a convex function imply that for every \(z \in \partial V(x_k(t))\)
\[
V(x_k(t)) - V(x_k(t_{q-1})) \geq \left\langle x_k(t) - x_k(t_{q-1}), z \right\rangle
\]
particularly, for
\[
z = \dot{x}_k(t) - f(t, x_{q-1}) + \frac{1}{kT} b_q
\]
we have
\[
V(x_k(t)) - V(x_k(t_{q-1})) \geq \int_{t_{q-1}}^{t_q} \left\langle \dot{x}_k(s), \dot{x}_k(t) - f(t, x_{q-1}) + \frac{1}{kT} b_q \right\rangle \, ds.
\]
thus
\[
V(x_k(t)) - V(x_k(t_{q-1})) \geq \int_{t_{q-1}}^{t_q} \left\langle \dot{x}_k(s), \dot{x}_k(s) \right\rangle \, ds + \int_{t_{q-1}}^{t_q} \left\langle \dot{x}_k(s), f(s, x_k(t_{q-1})) \right\rangle \, ds
\]
\[
- \int_{t_{q-1}}^{t_q} \left\langle \dot{x}_k(s), f(s, x_k(t_{q-1})) \right\rangle \, ds,
\]
therefore, it is clear that
\[
V(x_k(T)) - V(x_0) \geq \int_0^T \|\dot{x}_k(s)\|^2 \, ds - \sum_{q=1}^{s} \int_{t_{q-1}}^{t_q} \left\langle \dot{x}_k(s), f(s, x_k(t_{q-1})) \right\rangle \, ds
\]
\[
+ \sum_{q=1}^{s} \frac{1}{kT} \int_{t_{q-1}}^{t_q} \left\langle \dot{x}_k(s), b_q \right\rangle \, ds.
\]
Claim. The sequence
\[
\left( \sum_{q=1}^{s} \int_{\tau_{k-1}^{q}}^{\tau_{k}^{q}} \left\langle \dot{x}_k(s), f(s, x_k(\tau_{k}^{q-1})) \right\rangle \, ds \right)_k
\]
converges to
\[
\int_{0}^{T} \left\langle \dot{x}(s), f(s, x(s)) \right\rangle \, ds .
\]

Proof: We have
\[
\left\| \sum_{q=1}^{s} \int_{\tau_{k-1}^{q}}^{\tau_{k}^{q}} \left( \left\langle \dot{x}_k(s), f(s, x_k(\tau_{k}^{q-1})) \right\rangle - \left\langle \dot{x}(s), f(s, x(s)) \right\rangle \right) \, ds \right\| =
\]
\[
= \left\| \sum_{q=1}^{s} \int_{\tau_{k-1}^{q}}^{\tau_{k}^{q}} \left( \left\langle \dot{x}_k(s), f(s, x_k(\tau_{k}^{q-1})) \right\rangle - \left\langle \dot{x}_k(s), f(s, x_k(s)) \right\rangle \right) \, ds \right\|
\]
\[
\leq \sum_{q=1}^{s} \int_{\tau_{k-1}^{q}}^{\tau_{k}^{q}} \left\| \left\langle \dot{x}_k(s), f(s, x_k(\tau_{k}^{q-1})) \right\rangle - \left\langle \dot{x}_k(s), f(s, x_k(s)) \right\rangle \right\| \, ds
\]
\[
+ \sum_{q=1}^{s} \int_{\tau_{k-1}^{q}}^{\tau_{k}^{q}} \left\| \left\langle \dot{x}_k(s), f(s, x_k(s)) \right\rangle - \left\langle \dot{x}_k(s), f(s, x_k(s)) \right\rangle \right\| \, ds
\]
\[
+ \sum_{q=1}^{s} \int_{\tau_{k-1}^{q}}^{\tau_{k}^{q}} \left\| \left\langle \dot{x}(s), f(s, x(s)) \right\rangle - \left\langle \dot{x}(s), f(s, x(s)) \right\rangle \right\| \, ds
\]
\[
= \sum_{q=1}^{s} \int_{\tau_{k-1}^{q}}^{\tau_{k}^{q}} \left\| \left\langle \dot{x}_k(s), f(s, x_k(\tau_{k}^{q-1})) \right\rangle - \left\langle \dot{x}_k(s), f(s, x_k(s)) \right\rangle \right\| \, ds
\]
\[
+ \int_{0}^{T} \left\| \left\langle \dot{x}_k(s), f(s, x_k(s)) \right\rangle - \left\langle \dot{x}_k(s), f(s, x_k(s)) \right\rangle \right\| \, ds
\]
\[
+ \int_{0}^{T} \left\| \left\langle \dot{x}(s), f(s, x(s)) \right\rangle - \left\langle \dot{x}(s), f(s, x(s)) \right\rangle \right\| \, ds .
\]

Since \( f \) is a Carathéodory function, \( x_k(.) \to x(.) \) uniformly, \( \|\dot{x}_k(.)\| \leq M + 1 + m(.) \), \( m(.) \in L^2([0, T], \mathbb{R}^n) \) and \( \dot{x}_k(.) \to \dot{x}(.) \) weakly in \( L^2([0, T], \mathbb{R}^n) \) then the last term converges to 0. This completes the proof of the claim.
Since
\[ \lim_{k \to \infty} \sum_{q=1}^{s} \frac{1}{k} \int_{\tau_{q-1}^{k}}^{\tau_{q}^{k}} \langle \dot{x}_{k}(s), b_{q} \rangle \, ds = 0 \]
then by passing to the limit for \( k \to \infty \) in (5.4) and using the continuity of the function \( V \) on the ball \( B(x_{0}, r) \), we obtain the following inequality
\[ V(x(T) - V(x_{0})) \geq \lim_{k \to \infty} \sup_{s} \int_{0}^{T} \| \dot{x}_{k}(s) \|^{2} \, ds - \int_{0}^{T} \langle \dot{x}(s), f(s, x(s)) \rangle \, ds . \]
Moreover, by the equality (5.2) we have
\[ \| \dot{x} \|_{2}^{2} \geq \lim_{k \to \infty} \sup_{s} \| \dot{x}_{k} \|_{2}^{2} \]
and by the weak lower semicontinuity of the norm, it follows that
\[ \| \dot{x} \|_{2}^{2} \leq \lim_{k \to \infty} \inf_{s} \| \dot{x}_{k} \|_{2}^{2} . \]
Finally, since \((\dot{x}_{k})_{k}\) converges to \( \dot{x}(.) \) strongly in \( L^{2}([0, T], \mathbb{R}^{n}) \), then there exists a subsequence denoted by \( \dot{x}_{k}(.) \) which converges pointwisely to \( \dot{x}(.) \). Therefore, we conclude, in view of Proposition 4, that
\[ d_{gr}F \left( x(t), \dot{x}(t) - f(t, x(t)) \right) = 0 \quad \text{a.e. on } [0, T] . \]
Since the graph of \( F \) is closed we have
\[ \dot{x}(t) \in f(t, x(t)) + F(x(t)) \quad \text{a.e. on } [0, T] . \]
Finally, let \( t \in [0, T] \). Recall that there exists \((\tau_{k}^{0})_{k}\) such that \( \lim_{k \to \infty} \tau_{k}^{0} = t \) for all \( t \in [0, T] \). Since
\[ \lim_{k \to \infty} \| x(t) - x_{k}(\tau_{k}^{0}) \| \to 0 \]
\( x_{k}(\tau_{k}^{0}) \in K, K \) is closed, by passing to the limit we obtain \( x(t) \in K \).
This completes the proof. ■

REFERENCES

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