EXISTENCE OF SOLUTIONS FOR SOME NONLINEAR BEAM EQUATIONS *

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Abstract: We study the existence of solutions for some nonlinear ordinary differential equations under a nonlinear boundary condition which arise on beam theory. Assuming suitable conditions we prove the existence of at least one solution applying topological methods.

1 – Introduction

This work is devoted to the study of the existence of solutions for some nonlinear ordinary differential equations under a nonlinear boundary condition. In 1995 Rebelo and Sanchez [9] have considered the second order problem

\[
\begin{align*}
    u'' + g(t, u) &= 0 & 0 < t < T \\
    u'(0) &= -f(u(0)) \\
    u'(T) &= f(u(T))
\end{align*}
\]

with \( g : [0, T] \times \mathbb{R} \to \mathbb{R} \) for \( g \) satisfying a sign condition or either nondecreasing with respect to \( u \), and \( f \in C(\mathbb{R}, \mathbb{R}) \) continuous and strictly nondecreasing. This equation may be regarded as a mathematical model for the axial deformation of a nonlinear elastic beam, with two nonlinear elastic springs acting at the extremities according to the law \( u'(0) = -f(u(0)) \), \( u'(\pi) = f(u(\pi)) \), and the total force exerted by the nonlinear spring undergoing the displacement \( u \) given by \( g(t, u) \).

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On the other hand, the following fourth order problem for the deflection of a beam resting on elastic bearings was considered, among other authors, by Grossinho and Ma (see [3], [6], and also [4] for asymmetric boundary conditions):

$$
\begin{align*}
\begin{cases}
    u^{(4)}(t,u) + g(t,u) &= 0 & 0 < t < T \\
    u''(0) &= u''(T) = 0 \\
    u'''(0) &= -f(u(0)) \\
    u'''(T) &= f(u(T)) .
\end{cases}
\end{align*}
$$

(2)

In section 2 we study (1) for $g = g(t,u,u')$. We remark that in this more general situation the problem is no longer variational; for this reason we shall apply instead topological methods. On the other hand, in order to find a priori bounds for the derivative we shall assume as in [2] the following Nagumo type condition:

$$
|g(t,u,v)| \leq \psi(|v|) \quad \forall (t,u,v) \in E .
$$

(3)

Here $E$ is a subset of $[0,T] \times \mathbb{R}^2$ to be specified, and $\psi : [0, +\infty) → (0, +\infty)$ is a continuous function satisfying the inequality

$$
\int_r^M \frac{1}{\psi(s)} \, ds > T
$$

for some constants $M$ and $r$ to be specified. Under these assumptions we shall prove the existence of solutions by the method of upper and lower solutions.

Moreover, in section 3 we obtain an existence result under Landesman–Lazer type conditions (see e.g. [8]) applying topological degree methods [7].

Finally, in section 4 we consider the fourth order problem (2) for $g = g(t,u,u',u'',u'''$. More precisely, we prove the existence of symmetric solutions, i.e. such that $u(t) = u(T - t)$, under appropriate Landesman–Lazer and Nagumo type conditions.

2 – The second order case. Upper and lower solutions

In this section we prove an existence result for the following second order problem:

$$
\begin{align*}
\begin{cases}
    u'' + g(t,u,u') &= 0 & 0 < t < T \\
    u'(0) &= -f(u(0)) \\
    u'(T) &= f(u(T)) .
\end{cases}
\end{align*}
$$

(4)
We shall assume the existence of an ordered couple of a lower and an upper solution. Namely, we shall assume there exist \( \alpha, \beta : [0, T] \to \mathbb{R} \) such that \( \alpha(t) \leq \beta(t) \),
\[
\alpha''(t) + g(t, \alpha, \alpha') \geq 0 ,
\]
\[
\beta''(t) + g(t, \beta, \beta') \leq 0 ,
\]
and
\[
\begin{align*}
\alpha'(0) & \geq -f(\alpha(0)), & \alpha'(T) & \leq f(\alpha(T)) \\
\beta'(0) & \leq -f(\beta(0)), & \beta'(T) & \geq f(\beta(T)).
\end{align*}
\]

In this context, set
\[
r = \min \left\{ \max \left\{ \frac{|\alpha(0) - \beta(T)|}{T}, \frac{|\alpha(T) - \beta(0)|}{T} \right\}, \max_{\alpha(0), \alpha(T) \leq \beta(0), \beta(T)} |f(s)| \right\},
\]
fix a constant \( M > r \) such that
\[
M \geq \max \left\{ \|\alpha'\|_{C([0, T])}, \|\beta'\|_{C([0, T])} \right\}
\]
and define
\[
\mathcal{E} = \left\{ (t, u, v) \in [0, T] \times \mathbb{R}^2 : \alpha(t) \leq u \leq \beta(t), \ |v| \leq M \right\}.
\]

**Theorem 2.1.** With the previous notations, assume there exists an ordered couple of a lower and an upper solution of (4). Furthermore, assume that \( g \) satisfies the Nagumo condition (3). Then the boundary value problem (4) admits at least one solution \( u \), with
\[
\alpha(t) \leq u(t) \leq \beta(t), \quad |u'(t)| < M \quad \forall t \in [0, T].
\]

**Proof:** Set \( \lambda > 0 \) and consider the functions \( P : [0, T] \times \mathbb{R} \to \mathbb{R}, Q : \mathbb{R} \to \mathbb{R} \) given by
\[
P(t, x) = \begin{cases} 
x & \alpha(t) \leq x \leq \beta(t) \\
\beta(t) & x > \beta(t) \\
\alpha(t) & x < \alpha(t),
\end{cases}
\]
\[
Q(x) = \begin{cases} 
x & -M \leq x \leq M \\
M & x > M \\
-M & x < -M.
\end{cases}
\]
We define a compact fixed point operator \( \phi : C^1([0,T]) \rightarrow C^1([0,T]) \) in the following way: for each \( v \in C^1([0,T]) \), let \( u = \phi(v) \) be the unique solution of the linear Neumann problem

\[
\begin{align*}
u'' - \lambda u &= g(t, P(t,v), Q(v')) - \lambda P(t, v), \\
u'(0) &= -f(P(0, v(0))), \quad u'(T) = f(P(T, v(T))).
\end{align*}
\]

By standard results, \( \phi \) is well defined and compact. Moreover, multiplying the previous equation by \( u \) it follows that

\[
- \int_0^T (u'' - \lambda u) u \leq C \|u\|_{L^2}
\]

for some constant \( C \). Hence

\[
\|u''\|_{L^2} + \lambda \|u\|_{L^2} \leq C \|u\|_{L^2} + f(P(T, v(T))) u(T) + f(P(0, v(0))) u(0),
\]

and it follows that \( \|u\|_{H^1} \leq C \) for some constant \( C \). We conclude that \( \|u\|_{C^1} \leq C \) for some constant \( C \), and by a straightforward application of Schauder Theorem it follows that \( \phi \) has a fixed point \( u \). We claim that

\[
\alpha(t) \leq u(t) \leq \beta(t), \quad |u'(t)| < M \quad \forall t \in [0,T],
\]

and hence \( u \) is a solution of the problem. Indeed, if for example \( (u - \beta)(t_0) > 0 \) for some \( t_0 \in (0,T) \) maximum, then \( P(t_0, u(t_0)) = \beta(t_0), u'(t_0) = \beta'(t_0), \) and

\[
(u - \beta)''(t_0) - \lambda (u - \beta)(t_0) \geq g(t_0, P(t_0, u(t_0)), Q(u'(t_0))) - \lambda P(t_0, u(t_0))
\]

\[
- \left[g(t_0, \beta(t_0), \beta'(t_0)) - \lambda \beta(t_0)\right] = 0,
\]

a contradiction. Now, if \( u - \beta \) attains an absolute positive maximum for example at \( t = 0 \), then \((u - \beta)'(0) \leq 0\). Moreover, as \( P(0, u(0)) = \beta(0) \) we deduce that \((u - \beta)'(0) = -f(P(0, u(0))) - \beta'(0) \geq 0\), and hence \((u - \beta)'(0) = 0\). On the other hand, in a neighborhood of 0 we have that \( u(t) > \beta(t) \) and then

\[
(u - \beta)'' - \lambda (u - \beta) \geq g(t, P(t,u), Q(u')) - \lambda P(t,u) - \left[g(t, \beta, \beta') - \lambda \beta\right]
\]

\[
= g(t, \beta, Q(u')) - g(t, \beta, \beta').
\]

As \( u'(0) = \beta'(0) \in [-M, M] \), the right-hand term vanishes at \( t = 0 \), meanwhile \( u(0) - \beta(0) > 0 \). It follows that \((u - \beta)'' \geq \lambda (u - \beta) + g(t, \beta, Q(u')) - g(t, \beta, \beta') > 0\) in \((0, \delta)\) for some \( \delta > 0 \), which contradicts the fact that 0 is an absolute maximum
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of \( u - \beta \). In the same way, it follows that \( u - \beta \) cannot attain a positive absolute maximum at \( T \). We deduce in a similar way that \( u(t) \geq \alpha(t) \) for every \( t \in [0, T] \).

Next, assume for example that \( u'(t_0) = M \) for some \( t_0 \).

If \( r = \max_{\alpha(0), \alpha(T) \leq s \leq \beta(0), \beta(T)} |f(s)| \), then \( |u'(0)|, |u'(T)| \leq r \); otherwise there exists \( \tilde{t} \) such that

\[
u'(\tilde{t}) = \frac{u(T) - u(0)}{T} \leq \frac{\beta(T) - \alpha(0)}{T} \leq r.
\]

In both cases, we deduce the existence of \( t_1 \) such that \( u'(t_1) = r \). We may assume that \( r < u'(t) < M \) for any \( t \) between \( t_1 \) and \( t_0 \), and hence

\[
T < \int_r^M \frac{1}{\psi(s)} \, ds = \int_{t_1}^{t_0} \frac{u''(t)}{\psi(u'(t))} \, dt \leq \left| \int_{t_1}^{t_0} \frac{g(t, u, u')}{\psi(u'(t))} \, dt \right| \leq |t_0 - t_1|,
\]

a contradiction. The proof is analogous if \( u'(t_0) = -M \). \( \blacksquare \)

Remark 2.2. In particular, the conditions of the previous theorem hold if there exist two constants \( \alpha < \beta \) such that

\[g(t, \alpha, 0) \geq 0 \geq g(t, \beta, 0)\]

and

\[f(\alpha) \geq 0 \geq f(\beta)\]

provided that \( g \) satisfies \( |g(t, u, v)| \leq \psi(|v|) \) for \( \alpha \leq u \leq \beta \), \( |v| < M \) and \( \int_0^M \frac{1}{\psi(s)} \, ds > T \). \( \square \)

Remark 2.3. When \( f \) is nondecreasing, a more general result is proved in [1]. \( \square \)

3 – Landesman–Lazer type conditions

In this section we prove the existence of solutions of (4) under Landesman–Lazer type conditions. We shall assume that \( f \) is one-side globally bounded, i.e. \( f \leq r \) or \( f \geq -r \) for some positive constant \( r \), and that \( g \) satisfies the Nagumo condition (3) over the set

\[
\mathcal{E} = \left\{ (t, u, v) \in [0, T] \times \mathbb{R}^2 : |v| \leq M \right\}
\]

for some \( M > r \).
Moreover, we shall assume that the limits
\[ \limsup_{u \to \pm \infty} g(t, u, v) := g_s^\pm(t) \]
and
\[ \liminf_{u \to \pm \infty} g(t, u, v) := g_i^\pm(t) \]
exist, and that they are uniform for \(|v| < M\). We also define the (possibly infinite) quantities
\[ \limsup_{u \to \pm \infty} f(u) := f_s^\pm \]
and
\[ \liminf_{u \to \pm \infty} f(u) := f_i^\pm. \]

Then we have:

**Theorem 3.1.** Under the previous assumptions, problem (4) admits at least one solution, provided that one of the following conditions holds:

(8) \[ 2 f_s^+ + \int_0^T g_s^+(t) \, dt < 0 < 2 f_i^- + \int_0^T g_i^-(t) \, dt \]

(9) \[ 2 f_s^- + \int_0^T g_s^-(t) \, dt < 0 < 2 f_i^+ + \int_0^T g_i^+(t) \, dt. \]

**Remark 3.2.** Conditions of this kind are known in the literature as Landesman–Lazer type conditions after the pioneering paper of E. Landesman and A. Lazer [5]. In particular, taking \( f = 0 \) in Theorem 3.1 we obtain standard Landesman–Lazer conditions for the Neumann problem. 

For the sake of completeness, we summarize the main aspects of coincidence degree theory. Let \( \mathcal{V} \) and \( \mathcal{W} \) be real normed spaces, \( L : \Dom(L) \subset \mathcal{V} \to \mathcal{W} \) a linear Fredholm mapping of index 0, and \( N : \mathcal{V} \to \mathcal{W} \) continuous. Moreover, set two continuous projectors \( \pi_\mathcal{V} : \mathcal{V} \to \mathcal{V} \) and \( \pi_\mathcal{W} : \mathcal{W} \to \mathcal{W} \) such that \( \Dom(\pi_\mathcal{V}) = \Ker(L) \) and \( \Ker(\pi_\mathcal{W}) = \Dom(L) \). And an isomorphism \( J : \Dom(\pi_\mathcal{W}) \to \Ker(L) \). It is readily seen that
\[ L\pi_\mathcal{V} := L|_{\Dom(L) \cap \Ker(\pi_\mathcal{V})} : \Dom(L) \cap \Ker(\pi_\mathcal{V}) \to \Dom(L) \]
is one-to-one; denote its inverse by $K_{\pi\Psi}$. If $\Omega$ is a bounded open subset of \( \mathcal{V} \), $N$ is called $L$-compact on $\Omega$ if $\pi\Psi N(\Omega)$ is bounded and $K_{\pi\Psi}(I - \pi\Psi)N : \Omega \to \mathcal{V}$ is compact.

The following continuation theorem is due to Mawhin [7]:

**Theorem 3.3.** Let $L$ be a Fredholm mapping of index zero and $N$ be $L$-compact on a bounded domain $\Omega \subset \mathcal{V}$. Suppose that:

1. $Lx \neq \lambda Nx$ for each $\lambda \in (0, 1]$ and each $x \in \partial \Omega$.
2. $\pi\Psi N x \neq 0$ for each $x \in \text{Ker}(L) \cap \partial \Omega$.
3. $d(J\pi\Psi N, \Omega \cap \text{Ker}(L), 0) \neq 0$, where $d$ denotes the Brouwer degree.

Then the equation $Lx = Nx$ has at least one solution in $\text{Dom}(L) \cap \Omega$.

**Proof of Theorem 3.1:** Set $\mathcal{V} = C([0, T])$, $\mathcal{W} = L^2(0, T) \times \mathbb{R}^2$, and the operators $L : H^2(0, T) \to \mathcal{W}$, $N : \mathcal{V} \to \mathcal{W}$ given by

$$Lu = (u'', u'(0), u'(T)),$$

$$Nu = (-g(\cdot, u, u'), -f(u(0), f(u(T))).$$

It is easy to verify that

$$\text{Ker}(L) = \mathbb{R}, \quad R(L) = \left\{ (\varphi, A, B) \in \mathcal{W} : \varphi = \frac{B-A}{T} \right\},$$

where $\varphi$ denotes the usual average given by $\varphi = \frac{1}{T} \int_0^T \varphi(t) dt$. Then, we may define $\pi\Psi(X) = \pi\Psi(\varphi, A, B) = (\varphi - \frac{B-A}{T}, 0)$, and $J : R(\pi\Psi) \to \mathbb{R}$ given by $J(C, 0, 0) = C$. In this case, for $(\varphi, A, B) \in R(L)$, the function $U = K_{\pi\Psi}(\varphi, A, B)$ is defined as the unique solution of the problem

$$U'' = \varphi, \quad U'(0) = A, \quad U'(T) = B$$

that satisfies $U = 0$. Writing $U'(t) = A + \int_0^t \varphi$ and using Wirtinger inequality, $L$-compactness of $N$ follows.

We claim there exists a constant $R$ such that if $Lu = \lambda Nu$ with $0 < \lambda \leq 1$ then $\|u\|_{C^1} \leq R$. Indeed, suppose by contradiction that $Lu_n = \lambda_n Nu_n$, with $0 < \lambda_n \leq 1$ and $\|u_n\|_{C^1} \to \infty$. As $u_n'' = -\lambda_n g(t, u_n, u_n')$ and $u_n'(0) = -\lambda_n f(u_n(0))$, $u_n'(T) = \lambda_n f(u_n(T))$, by the Nagumo condition and using the fact that

$$\min \{u_n'(0), u_n'(T)\} \leq r \quad \text{and} \quad \max \{u_n'(0), u_n'(T)\} \geq -r,$$

it follows as in the previous section that $\|u_n'\|_{C([0, T])} < M$ for every $n$. Hence $\|u_n\|_{C([0, T])} \to \infty$, and $\|u_n - \pi\Psi_n\|_{C([0, T])} \leq C$ for some constant $C$. Taking
a subsequence, assume for example that \( P_{n} \to +\infty \) and that (8) holds; then

integrating the equation we obtain the equality

\[
f(u_{n}(T)) + f(u_{n}(0)) = -\int_{0}^{T} g(t, u_{n}, u'_{n}) \, dt,
\]

and thus

\[
0 \leq \limsup_{n \to \infty} f(u_{n}(T)) + \limsup_{n \to \infty} f(u_{n}(0)) + \int_{0}^{T} g_{+}(t) \, dt < 0
\]
a contradiction. The proof is similar for the other cases; hence, taking \( \Omega = B_{R}(0) \)
for \( R \) large enough, the first condition in Theorem 3.3 is fulfilled.

Further, the function \( J_{\pi W}N|_{\pi Ker(L)} = [-R, R] \) is given by

\[
J_{\pi W}N(s) = \frac{1}{T} \left( \int_{0}^{T} g(t, s, 0) \, dt + 2f(s) \right),
\]

and in the same way as before it follows that for \( R \) large enough

\[
J_{\pi W}N(R) J_{\pi W}N(-R) < 0.
\]

Thus, \( \deg(J_{\pi W}N, \Omega \cap Ker(L), 0) = \pm 1 \), and the proof is complete.

4 – Symmetric solutions for the general fourth order case

In this section we study the existence of symmetric solutions for the problem

\[
\begin{align*}
\begin{cases}
    u^{(4)} + g(t, u, u', u'', u''') = 0 & 0 < t < T \\
    u''(0) = u''(T) = 0 \\
    u'''(0) = -f(u(0)) \\
    u'''(T) = f(u(T)).
\end{cases}
\end{align*}
\]

(10)

We shall assume that \( g \) is symmetric with respect to \( t \), namely:

\[
g(t, u, v, w, x) = g(T - t, u, v, w, x).
\]

(11)

Our Nagumo condition for this problem reads:

\[
|g(t, u, v, w, x)| \leq \psi(|x|) \quad \forall (t, u, v, w, x) \in \mathcal{E}
\]

(12)
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with \( \mathcal{E} = [0, T] \times \mathbb{R}^3 \times [-M, M] \), and \( \psi : [0, +\infty) \to (0, +\infty) \) continuous, with
\[
\int_0^M \frac{1}{\psi(s)} \, ds > T.
\]
Moreover, assume that the limits
\[
\limsup_{s \to \pm \infty} g(t, s, v, w, x) := g^\pm_s(t)
\]
and
\[
\liminf_{s \to \pm \infty} g(t, s, v, w, x) := g^\pm_i(t)
\]
exist, and that they are uniform over the set
\[
\mathcal{C} = \{(v, w, x) \in \mathbb{R}^3 : |v| < \frac{T^2}{4} M, |w| < \frac{T}{2} M \text{ and } |x| < M \}.
\]
The quantities \( f^\pm_s \) and \( f^\pm_i \) are defined as before. Then we have:

**Theorem 4.1.** Under the previous assumptions, problem (10) admits at least one symmetric solution, provided that one of the conditions (8) or (9) holds.

**Proof:** We proceed as in the proof of Theorem 3.1. Let
\[
\mathcal{V} = \left\{ u \in C^3([0, T]) : u(t) = u(T - t), u''(0) = 0 \right\},
\]
\[
\mathcal{W} = \left\{ u \in L^2(0, T) : u(t) = u(T - t) \right\} \times \mathbb{R}
\]
and define the operators \( L : H^1(0, T) \cap \mathcal{V} \to \mathcal{W}, N : \mathcal{V} \to \mathcal{W} \) by
\[
Lu = (u(4), u'''(0)), \quad Nu = -\left(g(\cdot, u, u', u'', u'''), f(u(0))\right).
\]
Again, it is easy to verify that
\[
\ker(L) = \mathbb{R}, \quad \text{R}(L) = \left\{ (\varphi, c) \in \mathcal{W} : \int_0^T \varphi(t) \, dt + 2c = 0 \right\}.
\]
Then, we may define \( \pi_\mathcal{V}(u) = \pi, \pi_\mathcal{W}(\varphi, c) = (\varphi + 2c, 0) \), and \( J : \text{R}(\pi_\mathcal{W}) \to \mathbb{R} \) given by \( J(C, 0) = C \). For \( (\varphi, c) \in \text{R}(L) \), the function \( U = K_{\pi_\mathcal{V}}(\varphi, c) \) is defined as the unique solution of the problem
\[
\begin{cases}
U(4) = \varphi \\
U''(0) = 0, \quad U'''(0) = c \\
U(t) = U(T - t) \\
U = 0.
\end{cases}
\]
As before, it is easy to prove that $N$ is $L$-compact. Next, if $Lu_n = \lambda_n Nu_n$, with $0 < \lambda_n \leq 1$ and $\|u_n\|_{C^3} \to \infty$, by the Nagumo condition and using the fact that $u_n'''(T) = 0$, it follows that $\|u_n'''\|_{C([0,T])} < M$ for every $n$. Moreover, for $t \leq \frac{T}{2}$ we have:

$$|u_n''| \leq \left| \int_0^T |u_n'''| \right| < \frac{T}{2} M$$

and

$$|u_n'| \leq \left| \int_t^\frac{T}{2} u_n'' \right| < \frac{T^2}{4} M .$$

As $u_n$ is symmetric, we conclude that $(u_n'(t), u_n''(t), u_n'''(t)) \in \mathcal{C}$ for every $t \in [0, T]$. Then $\|u_n\|_{C([0,T])} \to \infty$, and $\|u_n - \bar{u}_n\|_{C^3([0,T])} \leq C$ for some constant $C$. The rest of the proof follows as in the second order case.

Some examples and remarks

**Example 4.2.** As an example of Theorem 4.1 we may consider a symmetric function $g$ such that

$$g(t, u, v, w, x) = g_0(t, u) + \gamma(u) g_1(t, u, v, w, x) ,$$

where $g_0$ is bounded, $|g_1(t, u, v, w, x)| \leq A + B|x|$ and $\gamma(u) \to 0$ as $|u| \to \infty$.

Then $|g(t, u, v, w, x)| \leq C + D|x|$ for some positive constants $C$ and $D$ and the Nagumo condition is satisfied taking $\psi(x) = C + Dx$ and $M$ large enough. Moreover,

$$\limsup_{u \to \pm \infty} g_0(t, u) = g_s^\pm(t) , \quad \liminf_{u \to \pm \infty} g_0(t, u) = g_i^\pm(t) ,$$

and the assumptions of Theorem 4.1 are fulfilled if (8) or (9) holds. For example, it suffices to assume that

$$\lim_{|u| \to \infty} f(u) \text{ sgn}(u) = +\infty \quad \text{or} \quad \lim_{|u| \to \infty} f(u) \text{ sgn}(u) = -\infty .$$

**Remark 4.3.** In the situation of Theorem 4.1, if $g_s^\pm = g_i^\pm := g^\pm$ and $f_s^\pm = f_i^\pm := f^\pm$, integrating the equation it follows that if for example

$$g^+(t) \leq g \leq g^-(t) \quad \text{and} \quad f^+ < f < f^-$$

or

$$g^-(t) \leq g \leq g^+(t) \quad \text{and} \quad f^- < f < f^+$$

then the respective conditions (8) and (9) are also necessary.
Remark 4.4. The Nagumo condition (12) can be dropped if we assume that $g$ has a linear growth of the type

$$|g(t, u, v, w, x)| \leq A + B|u| + C|v| + D|w| + E|x|$$

(with $B, C, D$ and $E$ small enough), and that the limits $g^+_t$ and $g^+_s$ are uniform on $\mathbb{R}^3$. Indeed, in this case if $Lu_n = \lambda_n Nu_n$, with $0 < \lambda_n \leq 1$, then using the fact that $u''''_n = \lambda_n \int_0^T g(s, u_n, u'_n, u''_n, u'''_n) ds$, we deduce:

$$(1 - \frac{TE}{2}) \|u''''_n\|_{C([0,T])} \leq \frac{T}{2} \left( A + B\|u_n\|_{C([0,T])} + C\|u'_n\|_{C([0,T])} + D\|u''_n\|_{C([0,T])} \right).$$

Integrating twice, as $E, D$ and $C$ are small, we obtain:

$$\|u''_n\|_{C([0,T])} \leq \delta (A + B\|u_n\|_{C([0,T])})$$

for some constant $\delta$. By the mean value theorem, for $B < \delta$ we conclude that if for example $u_n \to +\infty$ then $\inf_{t \in [0,T]} u_n(t) \to +\infty$, and the rest of the proof follows as before. In particular, for $g = g(t, u)$ it suffices to take $B < \frac{16}{T^4}$. \(\square\)

Remark 4.5. In [3], Theorem 2, it is proved by variational methods that if $g = g(t, u)$ is symmetric on $t$, and $f, g(t, \cdot)$ are nondecreasing, then problem (10) admits a symmetric solution if and only if

$$2f(a) + \int_0^T g(t, a) dt = 0 \quad \text{for some } a \in \mathbb{R}.\$$

By monotonicity, this condition is equivalent to (9), unless $f(u) \equiv f(a)$ and $g(t, u) \equiv g(t, a)$ for all $u \geq a$ or for all $u \leq a$. Note that, in this last case, existence of solutions can be easily proved; thus, taking into account the previous remarks 4.3 and 4.4, when $|g(t, u)| \leq A + B|u|$ (with $B < \frac{16}{T^4}$) we may conclude that Theorem 4.1 is essentially equivalent to Theorem 2 in [3].

Moreover, without the monotonicity condition the authors prove (see [3], Theorem 5) the existence of a symmetric solution of (10) for $g$ and $f$ sublinear, i.e.

$$\frac{g(t, u)}{u} \to 0 \quad \text{as } |u| \to \infty$$

uniformly in $t$, and

$$\frac{f(u)}{u} \to 0 \quad \text{as } |u| \to \infty,$$

assuming a growth condition for $f$ and $g$, and that one of the following hypotheses holds:
i) $g(t, u) \to \pm \infty$ as $u \to \pm \infty$ uniformly in $t$ and $f$ bounded by below.

ii) $f(u) \to \pm \infty$ as $u \to \pm \infty$ and $g$ bounded by below.

It is clear that the sublinearity condition implies that $|g(t, u)| \leq A + B|u|$ for some $B < \frac{16}{T^4}$ and some $A$, and that if i) or ii) holds then the second inequality in condition (9) is fulfilled. Thus, some cases of Theorem 5 in [3] are covered by Theorem 4.1; in particular, if $f$ is bounded by above for $u < 0$ in i) or if $g$ is bounded by above for $u < 0$ in ii).

However, the first inequality in (9) does not necessarily hold under assumptions i) or ii): one may consider for instance the (sublinear) functions $f(u) = |u|^{1/2}$ and $g(t, u) = u^{1/3}$.

\[ \Box \]

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