ON THE CONTRACTED $l^1$-ALGEBRA OF A POLYCYCLIC MONOID

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Abstract: Let $P(X)$ denote the polycyclic monoid (Cuntz semigroup) on a nonempty set $X$ and let $A$ denote the Banach algebra $l^1(P(X))/Z$, where $Z$ is the (closed) ideal spanned by the zero of $P(X)$. Then $A$ is primitive. Moreover, $A$ is simple if and only if $X$ is infinite.

The $l^1$-algebra $l^1(S)$ of a semigroup $S$ consists of all functions $a: S \to \mathbb{C}$ (the complex field) of finite or countably infinite support and such that $\sum_{x \in S} |a(x)| < \infty$, where addition and scalar multiplication are defined pointwise and multiplication is taken to be convolution. As noted in [1], $l^1(S)$ is a Banach algebra with respect to the norm $\|a\| := \sum_{x \in S} |a(x)|$. By identifying each $x \in S$ with its characteristic function, we can write a typical element of $l^1(S)$ in the form $\sum_{x \in S} \alpha_x x$, where $\sum_{x \in S} |\alpha_x| < \infty$, ($\alpha_x \in \mathbb{C}$).

The semigroup algebra $\mathbb{C}[S]$ is the subalgebra consisting of all functions $a: S \to \mathbb{C}$ of finite support. When $S$ is a nontrivial semigroup with zero $z$, it is often helpful to replace $\mathbb{C}[S]$ by $\mathbb{C}[S]/\mathbb{C}z$, where $\mathbb{C}z$ is the ideal $\{\alpha z : \alpha \in \mathbb{C}\}$. We have thus, in effect, simply identified $z$ with the zero of the algebra. In [4, Chapter 5], $\mathbb{C}[S]/\mathbb{C}z$ is called the ‘contracted semigroup algebra’ of $S$ over $\mathbb{C}$ and is denoted by $\mathbb{C}_0[S]$. With this in mind, we call the Banach algebra $l^1(S)/\mathbb{C}z$ the contracted $l^1$-algebra of $S$ and denote it by $l^1_0(S)$. A typical element $u$ of $l^1_0(S)$ can be written in the form $u = \sum_{x \in S \setminus 0} \alpha_x x$, where $\sum_{x \in S \setminus 0} |\alpha_x| < \infty$, and we define its support, $\text{supp}(u)$, to be $\{x \in S \setminus 0 : \alpha_x \neq 0\}$.

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In this paper, we study \( l^1_0(S) \) for the case in which \( S \) is the polycyclic monoid \( P(X) \) on nonempty a set \( X \) [13]. It is shown that \( l^1_0(S) \) is primitive for all choices of \( X \) (Theorem 1) and is simple if and only if \( X \) is infinite (Theorem 2).

We begin by recalling the definition of \( P(X) \). Let \( M(X) \) denote the free monoid on \( X \). For \( w = x_1x_2 \ldots x_n \in M(X) \), where each \( x_i \in X \), we define the length \( l(w) \) and the content \( c(w) \) of \( w \) by \( l(w) := n \) and \( c(w) := \{ x_1, x_2, \ldots, x_n \} \).

In addition, we take \( l(1) := 0 \) and \( c(1) := \emptyset \), where \( 1 \) denotes the identity of \( M(X) \) (the empty word). We say that \( u \in M(X) \) is an initial segment of \( v \in M(X) \), written \( u \preceq v \), if and only if \( v = uw \) for some \( w \in M(X) \). For \( u, v \in M(X) \), we write \( u \parallel v \) if and only if \( u \nleq v \) and \( v \nleq u \).

Let \( P(X) := (M(X) \times M(X)) \cup \{0\} \) and define a multiplication in \( P(X) \) by

\[
(a, b)(c, d) = \begin{cases} 
(au, d) & \text{if } c = bu \text{ for some } u \in M(X) \\
(a, dv) & \text{if } b = cv \text{ for some } v \in M(X) \\
0 & \text{if } b \parallel c 
\end{cases}
\]

\[0(a, b) = (a, b)0 = 0^2 = 0 \,.
\]

Then \( P(X) \) is a monoid with identity \( (1, 1) \) and zero \( 0 \); further, it admits an involution \( * \) given by

\[(a, b)^* = (b, a) \, , \quad 0^* = 0 \,.
\]

(In fact, \( P(X) \) is an example of a 0-bisimple inverse semigroup in which \( * \) denotes inversion and in which each subgroup is trivial.) Note that \( (a, b)^2 = (a, b) \) if and only if \( a = b \). Thus the set \( E(X) \) of idempotents of \( P(X) \) is

\[
\{(a, a) : a \in M(X)\} \cup \{0\} \,.
\]

Clearly \( E(X) \) is a commutative submonoid of \( P(X) \) (the ‘semilattice’ of \( P(X) \)) and it is easily seen to be partially ordered by

\[(a, a) \succeq (b, b) \iff a \preceq b \, , \quad (a, a) > 0 \,.
\]

Observe that \( (a, a) \succeq (b, b) \) if and only if \( (a, a)(b, b) = (b, b)(a, a) \).

An alternative approach to the monoid described above is as follows. Let \( FI(X) \) denote the free monoid with involution* on a nonempty set \( X \). Adjoin a zero \( 0 \) to \( FI(X) \), take \( 0^* = 0 \) and write \( Q(X) := (FI(X) \cup \{0\})/\rho \), where \( \rho \) is the congruence determined by the relations \( x^*x = 1 \) \( (x \in X) \) and \( x^*y = 0 \) \( (x, y \in X \text{ and } x \neq y) \). This monoid is termed the Cuntz semigroup on \( X \). Note that every nonzero \( \rho \)-class has a unique representative of the form \( ab^* \) \( (a, b \in M(X)) \). We identify this element with its \( \rho \)-class and so can write
$Q(X) = \{ab^*: a, b \in M(X)\} \cup \{0\}$. It is routine to verify that $\theta: P(X) \to Q(X)$ is an isomorphism. Various aspects of algebras associated with $Q(X)$ have been studied in [5], [6] and [2]; see also [14]. For an extended discussion of polycyclic monoids, see [9, §9.3].

Next, we review the concept of primitivity. Let $A$ be a complex algebra and let $V$ be a nonzero right $A$-module under the action $\circ$. A vector $v \in V \setminus 0$ is called cyclic if and only if $v \circ A = V$. Recall that $V$ is termed

(i) faithful if and only if, for all $a \in A$, $V \circ a = 0$ implies $a = 0$,

(ii) strictly irreducible if and only if every nonzero vector in $V$ is cyclic.

We say that $A$ is (right) primitive if and only if there exists a faithful strictly irreducible right $A$-module.

For the case in which $A$ is a Banach algebra, $V$ a Banach space with norm $\|\cdot\|_V$ and $\circ$ a right action of $A$ on $V$ with $\|v \circ a\|_V \leq \|v\|_V \|a\|$ ($v \in V$, $a \in A$), we make a further definition. We say that $V$ is topologically irreducible if and only if, for all $v \in V \setminus 0$, all $u \in V$ and a given positive real number $\epsilon$, there exists $a \in A$ such that

$$\|v \circ a - u\|_V < \epsilon .$$

The following result ([8], [10]) is required below. For convenience, we include a proof.

**Lemma.** Let $A$ and $V$ be as in the preceding paragraph. If $V$ is topologically irreducible and possesses a cyclic vector then $V$ is strictly irreducible.

**Proof:** Let $V$ be topologically irreducible, with a cyclic vector $v_1$. Since the mapping $f: A \to V$ defined by $f(a) = v_1 \circ a$ is continuous, the open mapping theorem shows that, for some positive real number $\delta$,

$$\left\{ v \in V: \|v\|_V < \delta \right\} \subseteq \left\{ f(a): a \in A \text{ and } \|a\| < 1 \right\} .$$

Let $v \in V \setminus 0$. Since $V$ is topologically irreducible, there exists $b \in A$ such that $\|v_1 - v \circ b\|_V < \delta$. Hence there exists $a \in A$ with $\|a\| < 1$ such that $v_1 - v \circ b = v_1 \circ a$. Consider $c \in A$ defined by $c = -\sum_{r=1}^{\infty} a^r$. Then $a + c - ac = 0$. Hence

$$v \circ (b - bc) = (v_1 - v_1 \circ a) - (v_1 - v_1 \circ a) \circ c$$

$$= v_1 - v_1 \circ (a + c - ac) = v_1 .$$

Consequently, $v$ is cyclic. Thus $V$ is strictly irreducible. ■
We now come to our first result. Note that since the polycyclic monoid on $X$ admits an involution, so also does its contracted $l^1$-algebra. Thus the term ‘primitive’ can be used without qualification.

**Theorem 1.** For every nonempty set $X$, $l_0^1(P(X))$ is primitive.

**Proof:** For a given nonempty set $X$ write $S := P(X)$, $E := E(X)$ and $V := l_0^1(E)$.

We begin by defining a right action of $l_0^1(S)$ on $V$. First note that, if $x \in S$ and $e \in E$ then $xx^*, x^*ex \in E$. Now define $\circ : E \times S \to E$ by the rule that

$$(\forall e \in E) \quad (\forall x \in S) \quad e \circ x = \begin{cases} x^*ex & \text{if } e \leq xx^*, \\ 0 & \text{otherwise.} \end{cases}$$

Let $e \in E$ and let $x, y \in S$. A straightforward calculation shows that

$$(1) \quad e \leq xx^* \land x^*ex \leq yy^* \iff e \leq xy(xy)^*.$$ Using this, we now prove that

$$(2) \quad (e \circ x) \circ y = e \circ (xy).$$

Suppose that $e \leq xy(xy)^*$. Then $e \circ (xy) = (xy)^*exy$. But, by $(1)$, $e \leq xx^*$ and $x^*ex \leq yy^*$. Hence $(e \circ x) \circ y = (x^*ex) \circ y = y^*(x^*ex)y = (xy)^*exy$. Thus $(2)$ holds in this case. Now suppose that $e \not\leq xy(xy)^*$. Then $e \circ (xy) = 0$. But, by $(1)$, either $e \not\leq xx^*$ or $x^*ex \not\leq yy^*$. If $e \leq xx^*$ and $x^*ex \not\leq yy^*$ then $(e \circ x) \circ y = (x^*ex) \circ y = 0$, while if $e \not\leq xx^*$ then $e \circ x = 0$ and so again $(e \circ x) \circ y = 0$. Thus $(2)$ holds in this case also. Since, for all $e \in E$ and $x \in S$, $||e \circ x|| \leq ||x^*ex|| \leq 1$ we can extend $\circ$ to a right action, also denoted by $\circ$, of $l_0^1(S)$ on $V$; and, clearly, for all $v \in V$ and all $u \in l_0^1(S)$, $||v \circ u|| \leq ||v|| ||u||$.

We show next that $V$ is faithful. Let $S'$ and $E'$ denote $S \setminus 0$ and $E \setminus 0$, respectively. Observe first that $E'$ satisfies the maximal condition with respect to $\leq$; for if $T$ is a nonempty subset of $M(X)$ and $s \in T$ is chosen such that $l(s) \leq l(t)$ for all $t \in T$ then $(s, s)$ is maximal in the subset $\{(t, t) : t \in T\}$ of $E'$. Let $u \in l_0^1(S) \setminus 0$, say $u = \sum_{x \in S'} \alpha_x x$, with $\sum_{x \in S'} |\alpha_x| < \infty$ and not all $\alpha_x = 0$. Choose $e \in E'$ maximal in $\{xx^* : x \in \text{supp}(u)\}$. Then

$$(3) \quad e \circ u = \sum_{xx^* = e} \alpha_x (x^*ex).$$

Now let $x, y \in S'$ be such that $xx^* = yy^* = e$ and $x^*ex = y^*ey$. We have that $x = (a, b)$ and $y = (c, d)$ for some $a, b, c, d \in M(X)$. Thus $(a, a) = e = (c, c)$ and $(b, b) = x^*ex = y^*ey = (d, d)$. Hence $a = c = b = d$ and so $x = y$. It follows from $(3)$ that $e \circ u \neq 0$. This shows that $V$ is faithful.
To complete the proof, we show that \( V \) is strictly irreducible. Let \( v \in V \setminus 0 \) and let \( e \in \text{supp}(v) \), with coefficient \( \alpha \in \mathbb{C} \setminus 0 \). We prove first that, for a given positive real number \( \epsilon \), there exist \( v' \in V \) and \( u \in l_0^1(E) (\subseteq l_0^1(S)) \) such that 

\[
(4) \quad v \circ u = \alpha e + v', \quad \|v'\| < \epsilon .
\]

Note that if \( e \) is minimal in \( \text{supp}(v) \) then \( v \circ e = \alpha e \) and so (4) holds with \( u = e \) and \( v' = 0 \). Suppose, therefore, that \( e \) is not minimal in \( \text{supp}(v) \). Write \( v = w + w' \), where \( w, w' \in V \) are such that 

\[
(5) \quad e \in \text{supp}(w), \quad \text{supp}(w) \text{ is finite}, \quad \text{supp}(w) \cap \text{supp}(w') = \emptyset, \quad \|w'\| < \epsilon .
\]

Without loss of generality, we may assume that \( e \) is not minimal in \( \text{supp}(w) \). (If need be, transfer a term from \( w' \) to \( w \).) Let \( F := \{ f \in \text{supp}(w) : f < e \} \) and define \( u \in l_0^1(E) \) by 

\[
u := \prod_{f \in F} (e - f) .
\]

We now show that 

\[
(6) \quad (\forall g \in E') \quad g \circ u = \begin{cases} g & \text{if } g \leq e \text{ and, for all } f \in F, \ g \napprox f, \\ 0 & \text{if } g \leq e \text{ and, for some } f \in F, \ g \leq f, \\ 0 & \text{if } g \napprox e. \end{cases}
\]

Suppose first that \( g \in E' \) is such that \( g \leq e \) and that, for all \( f \in F \), \( g \napprox f \). Then, for all \( f \in F \), \( g \circ (e - f) = g \) and so \( g \circ u = g \). Next, suppose that \( g \in E' \) is such that \( g \leq e \) and that there exists \( f \in F \) with \( g \leq f \). Then \( g \circ (e - f) = g - g = 0 \) and so \( g \circ u = 0 \). Finally, suppose that \( g \in E' \) is such that \( g \napprox e \). Then, for any \( f \in F \), \( g \napprox f \) and so \( g \circ (e - f) = 0 \). Hence again \( g \circ u = 0 \). This establishes (6).

It follows from (6) that \( w \circ u = \alpha e \). Write \( v' := w' \circ u \). Since, by (6), for all \( g \in \text{supp}(w') \), \( g \circ u \) is either \( g \) or 0 we have that \( \|v'\| \leq \|w'\| \). Thus, from (5), we see that (4) holds.

Next, let \( f \in E' \). There exist \( a, b \in M(X) \) such that \( e = (a, a) \) and \( f = (b, b) \). Write \( x := (a, b) \). Then \( xx^a = e \) and 

\[
(7) \quad e \circ x = f .
\]

Hence, from (4), \( v \circ (ux) = \alpha f + (v' \circ x) \) and, in addition, \( \|v' \circ x\| \leq \|v'\| < \epsilon \). Thus 

\[
\|v \circ (ux) - \alpha f\| < \epsilon ,
\]

from which we deduce that \( V \) is topologically irreducible. But, from (7), it follows that \( e \) is a cyclic vector in \( V \). Hence, by the Lemma, \( V \) is strictly irreducible. ■
The corresponding result for $C_0[P(X)]$ is a consequence of a theorem of Domanov [7]. A short proof is given in [12]. As already remarked, $P(X)$ is a special case of a 0-bisimple inverse semigroup with only trivial subgroups. In [3], we show that if $S$ is a 0-bisimple inverse semigroup with a nonzero maximal subgroup $G$ such that $l^1(G)$ is primitive then $l^0_1(S)$ is primitive. This generalises Theorem 1 above, but is harder to prove since we have to allow for the presence of nontrivial subgroups and cannot assume that the semilattice of $S$ satisfies the maximal condition under the natural partial ordering.

Our second result gives a necessary and sufficient condition for $l^0_1(P(X))$ to be a simple algebra.

**Theorem 2.** Let $X$ be a nonempty set. Then $l^0_1(P(X))$ is simple if and only if $X$ is infinite.

**Proof:** Write $S := P(X)$ and $S' := S\setminus 0$. Assume first that $X$ is infinite. Let $T$ be a nonzero ideal of $l^0_1(S)$. We show that $T = l^0_1(S)$.

Let $t \in T \setminus 0$. Choose $a \in M(X)$ such that $a$ has minimal length amongst the first components of the elements of supp$(t)$; and choose $b \in M(X)$ such that $(a, b) \in$ supp$(t)$. Then, for some positive integer $n$, we may write $t$ in the form

\[ t = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n + v, \]

where $u_1, u_2, \ldots, u_n$ are distinct elements of supp$(t)$ with $u_1 = (a, b)$, $\alpha_i \in C \setminus 0$ ($i = 1, 2, \ldots, n$) and $v \in l^0_1(S)$ is such that $\|v\| < |\alpha_1|$. Write $u_i = (a_i, b_i) \in M(X) \times M(X)$ ($i = 1, 2, \ldots, n$) and assume, without loss of generality, that for some $k \in \{1, 2, \ldots, n\}$, $(a =) a_1 = a_2 = \cdots = a_k$, while $a_i \neq a$ if $k < i \leq n$. Since $u_1, u_2, \ldots, u_k$ are distinct, it follows that $(b =) b_1, b_2, \ldots, b_k$ are distinct.

Let $Y$ denote $\bigcup_{i=1}^{n} (c(a_i) \cup c(b_i))$. Since $Y$ is a finite subset of the infinite set $X$, there exists $x \in X \setminus Y$. Write

\[ e := (ax, ax), \quad f := (bx, bx). \]

We shall show that

\[ eu_1f = \begin{cases} (ax, bx) & \text{if } i = 1, \\ 0 & \text{if } 2 \leq i \leq n. \end{cases} \]

Suppose first that $1 \leq i \leq k$. Then $eu_1f = (ax, ax)(a, b_i)(bx, bx) = (ax, b_ix)(bx, bx)$. In particular, $eu_1f = (ax, bx)$. Now consider the case where $2 \leq i \leq k$. Here $b_ix \neq bx$; for otherwise, since $x \notin c(b)$, we would have $b_i = b$. Similarly, $bx \neq b_ix$. 

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Hence $eu_i f = 0$. Next, suppose that $k < i \leq n$. Then, by the choice of $x$, $ax \not\parallel a_i$. Further, $a_i \not\parallel ax$; for otherwise $a_i \leq a$, which is impossible since $l(a_i) \not\parallel l(a)$ and $a_i \neq a$. Hence $ax \parallel a_i$ and so $eu_i = (ax, ax)(a_i, b_i) = 0$, which gives $eu_i f = 0$. Thus we have established (2).

Take $p := (1, ax)$ and $q := (bx, 1)$. Then, from (1) and (2),

$$pe tf q = \alpha_1(1, 1) + pe vf q .$$

But, since $p, e, f, q \in S'$, we have that $\|pe vf q\| \leq \|v\| < |\alpha_1|$. Thus

$$\|\alpha_1^{-1}(pe tf q) - (1, 1)\| < 1 .$$

Consequently, $\alpha_1^{-1}(pe tf q)$ is invertible in $l_0^1(S)$; thus there exists $r \in l_0^1(S)$ such that $\alpha_1^{-1}(pe tf q r) = (1, 1)$. Since $t \in T$, it follows that $(1, 1) \in T$ and so $T = l_0^1(S)$.

This shows that $l_0^1(S)$ is simple.

Now assume that $X$ is finite, with elements $x_1, x_2, ..., x_n$. For $(a, b) \in S'$ define $w_{a,b} \in l_0^1(S)$ by

$$w_{a,b} := (a, b) - \sum_{i=1}^{n} (ax_i, bx_i) .$$

Then $\|w_{a,b}\| = n + 1$. Define a subspace $T$ of $l_0^1(S)$ by

$$T := \left\{ \sum_{(a,b) \in S'} \alpha_{a,b} w_{a,b} : \alpha_{a,b} \in \mathbb{C} \text{ and } \sum_{(a,b) \in S'} |\alpha_{a,b}| < \infty \right\} .$$

Let $(a, b), (c, d) \in S'$ and consider the product $w_{a,b}(c,d)$. If $b = cu$ for some $u \in M(X)$ then $w_{a,b}(c,d) = (a, du) - \sum_{i=1}^{n} (ax_i, du x_i) = w_{a,du} \in T$. If $c = bx_r v$ for some $r$ and some $v \in M(X)$ then $w_{a,b}(c,d) = (ax_r v, d) - (ax_r v, d) = 0$. If $b \parallel c$ then $w_{a,b}(c,d) = 0$. Thus $T(c,d) \subseteq T$. This shows that $T$ is a right ideal of $l_0^1(S)$.

A similar argument shows that it is a left ideal.

Finally, we prove that the ideal $T$ is proper. Define $\phi : S' \to \mathbb{C}$ by $\phi((a, b)) = n^{-(1/2)(l(a) + l(b))}$. Since $|\phi((a, b))| \leq 1$, $\phi$ extends to a continuous linear functional on $l_0^1(S)$. Now, for all $(a, b) \in S'$,

$$\phi(w_{a,b}) = \phi((a, b)) - \sum_{i=1}^{n} \phi((ax_i, bx_i))$$

$$= n^{-(1/2)(l(a) + l(b))} - n \cdot n^{-(1/2)(l(a) + l(b) + 2)} = 0 .$$

Hence, by continuity, $\phi(t) = 0$ for all $t \in T$. But $\phi((1, 1)) = 1$ and so $(1, 1) \not\in T$. Thus $T$ is proper.

The corresponding result for $\mathbb{C}_0[P(X)]$ was obtained in [11].
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