EXPLICIT UPPER BOUNDS FOR THE RESIDUES AT $s = 1$
OF THE DEDEKIND ZETA FUNCTIONS OF SOME
TOTALLY REAL NUMBER FIELDS

by

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Abstract. — We give an explicit upper bound for the residue at $s = 1$ of the Dedekind zeta function of a totally real number field $K$ for which $\zeta_K(s)/\zeta(s)$ is entire. Notice that this is conjecturally always the case, and that it holds true if $K/\mathbb{Q}$ is normal or if $K$ is cubic.

Résumé (Bornes supérieures explicites pour les résidus en $s = 1$ des fonctions zêta de Dedekind de corps de nombres totalement réels)

Nous donnons une borne supérieure explicite pour le résidu en $s = 1$ de la fonction zêta de Dedekind d’un corps de nombres $K$ totalement réel pour lequel $\zeta_K(s)/\zeta(s)$ est entière. On remarque que c’est conjecturalement toujours le cas, et que c’est vrai si $K/\mathbb{Q}$ est normale ou si $K$ est cubique.

1. Introduction

Let $d_K$ and $\zeta_K(s)$ denote the absolute value of the discriminant and the Dedekind zeta function of a number field $K$ of degree $m > 1$. It is important to have explicit upper bounds for the residue at $s = 1$ of $\zeta_K(s)$. As for the best general such bounds, we have (see [Lou01, Theorem 1]):

$$\text{Res}_{s=1}(\zeta_K(s)) \leq \left( \frac{e \log d_K}{2(m-1)} \right)^{m-1}.$$

However, for some totally real number fields an improvement on this bound is known (see [BL] and [Oka] for applications):

Theorem 1 (See [Lou01, Theorem 2]). — Let $K$ range over a family of totally real number fields of a given degree $m \geq 3$ for which $\zeta_K(s)/\zeta(s)$ is entire. There exists $C_m$
(computable) such that $d_K \geq C_m$ implies
\[
\text{Res}_{s=1}(\zeta_K(s)) \leq \frac{\log m - d_K}{2(m-1)!} \leq \frac{1}{\sqrt{2\pi(m-1)}} \left( \frac{e \log d_K}{2(m-1)} \right)^{m-1}.
\]
Moreover, for any non-normal totally real cubic field $K$ we have the slightly better bound
\[
\text{Res}_{s=1}(\zeta_K(s)) \leq \frac{1}{8}(\log d_K^2 - \kappa)^2
\]
where $\kappa := 2\log(4\pi) - 2 - 2\gamma = 1.90761\ldots$.

**Remark 2.** — If $K/\mathbb{Q}$ is normal or if $K$ is cubic, then $\zeta_K(s)/\zeta(s)$ is entire.

We will simplify our previous proof of Theorem 1 (by improving those of [Lou98, Theorem 5] and [Lou01, Theorem 2]) and we will give explicit constants $C_m$ for which Theorem 1 holds true:

**Theorem 3.** — There exists $C > 0$ (effective) such that for any totally real number field $K$ of degree $m \geq 3$ and root discriminant $\rho_K := d_K^{1/m} \geq C^m$ we have
\[
\text{Res}_{s=1}(\zeta_K(s)) \leq \frac{\log m - d_K}{2(m-1)!},
\]
provided that $\zeta_K(s)/\zeta(s)$ is entire. Moreover, $C = 3309$ will do for $m$ large enough.

This result is not the one we would have wished to prove. It would indeed have been much more satisfactory to prove that there exists $C > 0$ (effective) such that this bound is valid for such totally real number fields $K$ of root discriminants $\rho_K \geq C$ large enough. It would have been even more satisfactory to prove that this constant $C$ is small enough to obtain that our bound is valid for all totally real number fields $K$ for which $\zeta_K(s)/\zeta(s)$ is entire (e.g., see [Was, Page 224] for explicit lower bounds on root discriminants of totally real number fields $K$). Let us finally point out that, in the case that $K/\mathbb{Q}$ is abelian, we have an even better bound (see [Lou01, Corollary 8] and use [Ram, Corollary 1]):
\[
\text{Res}_{s=1}(\zeta_K(s)) \leq \left( \frac{\log d_K}{2(m-1)} \right)^{m-1}.
\]

**2. Proof of Theorem 1**

**Proposition 4.** — Let $K$ be a totally real number field of degree $m \geq 1$, set $d = \sqrt{d_K}$, and assume that $\zeta_K(s)/\zeta(s)$ is entire. Then, $\text{Res}_{s=1}(\zeta_K(s)) \leq \rho_{m-1}(d)$ where
\[
(1) \quad \rho_{m-1}(d) := \text{Res}_{s=1}\left\{ s \mapsto \left( \frac{\pi^{-s/2}\Gamma(s/2)}{\zeta(s)} \right)^{m-1} \left( \frac{1}{s} + \frac{1}{s-1} \right) \right\}.
\]
Proof. — To begin with, we set some notation: if $K$ is a totally real number field of degree $m \geq 1$, we set $A_K = \sqrt{d_K} / \pi^m$ and $F_K(s) = A_K \Gamma^m(s/2) \zeta_k(s)$. Hence, $F_K(s)$ is meromorphic, with only two poles, at $s = 1$ and $s = 0$, both simple, and it satisfies the functional equation $F_K(1-s) = F_K(s)$. We then set $F_{K/Q}(s) = F_K(s)/F_Q(s)$, which under our assumption is entire, and satisfies the functional equation $F_{K/Q}(1-s) = F_{K/Q}(s)$, and $A_{K/Q} := A_K / A_Q = \sqrt{d_K} / \pi^{m-1}$. Notice that $F_{K/Q}(1) = \sqrt{d_K} \text{Res}_{s=1}(\zeta_K(s))$. Let

$$S_{K/Q}(x) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_{K/Q}(s)x^{-s}ds \quad (c > 1 \text{ and } x > 0)$$

(2)

denote the Mellin transform of $F_{K/Q}(s)$. Since $F_{K/Q}(s)$ is entire, it follows that $S_{K/Q}(x)$ satisfies the functional equation

$$S_{K/Q}(x) = \frac{1}{x} S_{K/Q}\left(\frac{1}{x}\right)$$

(3)

(shift the vertical line of integration $\Re(s) = c > 1$ in (2) leftwards to the vertical line of integration $\Re(s) = 1-c < 0$, then use the functional equation $F_{K/Q}(1-s) = F_{K/Q}(s)$ to come back to the vertical line of integration $\Re(s) = c > 1$), and

$$F_{K/Q}(s) = \int_0^\infty S_{K/Q}(x)x^{s-1}dx = \int_1^\infty S_{K/Q}(x)(x^s + x^{1-s})dx$$

(4)

is the inverse Mellin transform of $S_{K/Q}(x)$. Now, set

$$F_{m-1}(s) = F_Q(s) = (\pi^{-s/2}\Gamma(s/2)\zeta(s))^{m-1},$$

$$A_{m-1} = A_Q^{m-1} = \pi^{-(m-1)/2}$$

(5)

and let

$$S_{m-1}(x) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_{m-1}(s)x^{-s}ds \quad (c > 1 \text{ and } x > 0)$$

(6)

denote the Mellin transform of $F_{m-1}(s)$. Here, $F_{m-1}(s)$ has two poles, at $s = 1$ and $s = 0$, the functional equation $F_{m-1}(1-s) = F_{m-1}(s)$ yields

$$\text{Res}_{s=0}(F_{m-1}(s)x^{-s}) = -\text{Res}_{s=1}(F_{m-1}(s)x^{s-1})$$

and

$$S_{m-1}(x) = \text{Res}_{s=1}\{F_{m-1}(s)(x^{-s} - x^{s-1})\} = \frac{1}{x} S_{m-1}\left(\frac{1}{x}\right)$$

(7)

(shift the vertical line of integration $\Re(s) = c > 1$ in (6) leftwards to the vertical line of integration $\Re(s) = 1-c < 0$, notice that you pick up residues at $s = 1$ and $s = 0$, then use the functional equation $F_{m-1}(1-s) = F_{m-1}(s)$ to come back to the vertical line of integration $\Re(s) = c > 1$). Finally, we set

$$H_{m-1}(x) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma^{m-1}(s/2)x^{-s}ds \quad (c > 1 \text{ and } x > 0).$$

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Notice that $0 < H_{m-1}(x)$ for $x > 0$ (see [Lou00, Proof of Theorem 2](1)). Now, write

$$\zeta_K(s)/\zeta(s) = \sum_{n \geq 1} a_{K/Q}(n)n^{-s}$$

and

$$\zeta^{m-1}(s) = \sum_{n \geq 1} a_{m-1}(n)n^{-s}.$$ 

Then, $|a_{K/Q}(n)| \leq a_{m-1}(n)$ for all $n \geq 1$ (see [Lou01, Lemma 26]). Since

$$S_{K/Q}(x) = \sum_{n \geq 1} a_{K/Q}(n)H_{m-1}(nx/A_{K/Q})$$

and

$$0 \leq S_{m-1}(x) = \sum_{n \geq 1} a_{m-1}(n)H_{m-1}(nx/A_{m-1}),$$

we obtain

$$S_{K/Q}(x) \leq S_{m-1}(x/d) \quad \text{with} \quad d := A_{K/Q}/A_{m-1} = \sqrt{d_{K}}.$$ 

We are now ready to proceed with the proof of Proposition 4. We have

$$d\text{Res}_{s=1}(\zeta_K) = F_{K/Q}(1) = \int_{1}^{\infty} S_{K/Q}(x)\left(1 + \frac{1}{x}\right)dx \quad \text{(by (4))}$$

$$\leq \int_{1}^{\infty} S_{m-1}(x/d)\left(1 + \frac{1}{x}\right)dx \quad \text{(by (8))}$$

$$= \int_{1/d}^{\infty} S_{m-1}(x)\left(d + \frac{1}{x}\right)dx$$

$$= \int_{1}^{\infty} S_{m-1}(x)\left(d + \frac{1}{x}\right)dx + \int_{1}^{d} \frac{1}{x}S_{m-1}(\frac{1}{x})\left(\frac{d}{x} + 1\right)dx$$

$$\leq (d + 1) \int_{1}^{\infty} S_{m-1}(x)\left(1 + \frac{1}{x}\right)dx$$

$$- \int_{1}^{d} \text{Res}_{s=1}\{F_{m-1}(s)(x^{-s} - x^{s-1})\}\left(\frac{d}{x} + 1\right)dx \quad \text{(by (7), and for} \ S_{m-1}(x) \geq 0 \text{for} \ x > 0)$$

(1) Notice the misprints in [Lou00, page 273, line 1] and [Lou01, Theorem 20] where one should read

$$(M_1 \ast M_2)(x) = \int_{0}^{\infty} M_1(x/t)M_2(t)\frac{dt}{t}.$$
\[ \begin{aligned}
&= (d + 1) \int_1^\infty S_{m-1}(x) \left( 1 + \frac{1}{x} \right) dx \\
&\quad - \text{Res}_{s=1} \left\{ F_{m-1}(s) \int_1^d (x^{-s} - x^{s-1}) \left( \frac{d}{x} + 1 \right) dx \right\} \\
&\quad \text{(compute these residues as contour integrals along a circle of center 1 and of small radius, and use Fubini’s theorem)} \\
&= (d + 1) \left( \int_1^\infty S_{m-1}(x) \left( 1 + \frac{1}{x} \right) dx - \text{Res}_{s=1} \left\{ F_{m-1}(s) \left( \frac{1}{s} + \frac{1}{s - 1} \right) \right\} \right) \\
&\quad + \text{Res}_{s=1} \left\{ F_{m-1}(s)(d^s + d^{1-s}) \left( \frac{1}{s} + \frac{1}{s - 1} \right) \right\}. 
\end{aligned} \]

The desired result now follows from Lemma 5 below. 

**Lemma 5.** — Set 
\[ G_{m-1}(s) := F_{m-1}(s) \left( \frac{1}{s} + \frac{1}{s - 1} \right). \]

Then, 
\[ I_{m-1} := \int_1^\infty S_{m-1}(x) \left( 1 + \frac{1}{x} \right) dx = \text{Res}_{s=1}(G_{m-1}(s)). \]

**Proof.** — By (6) and Fubini’s theorem, we have 
\[ I_{m-1} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_{m-1}(s) \left( \int_1^\infty (x^{-s} + x^{s-1}) dx \right) ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G_{m-1}(s) ds \]

The functional equation 
\[ G_{m-1}(1-s) = -G_{m-1}(s) \]

yields 
\[ I_{m-1} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G_{m-1}(s) ds \]

\[ = \text{Res}_{s=1}(G_{m-1}(s)) + \text{Res}_{s=0}(G_{m-1}(s)) + \frac{1}{2\pi i} \int_{1-c-i\infty}^{1-c+i\infty} G_{m-1}(s) ds \]

\[ = 2 \text{Res}_{s=1}(G_{m-1}(s)) - I_{m-1}, \]

from which the desired result follows. 

Let us now complete the proof of Theorem 1. Since 
\[ \pi^{-s/2} \Gamma(s/2) \zeta(s) = \frac{1}{s-1} - a + O(s-1), \]

with 
\[ a = (\log(4\pi) - \gamma)/2 = 0.97690 \ldots, \]

using (1) we obtain 
\[ \rho_{m-1}(d) = \frac{1}{(m-1)!} \log^{m-1} d - \frac{c_{m-1}}{(m-2)!} \log^{m-2} d + O(\log^{m-3} d) \]

with 
\[ c_{m-1} := (m-1)a - 1 > 0 \text{ for } m \geq 3, \]

and the desired first result follows. In the special case 
\[ m = 3, \]

in writing 
\[ \pi^{-s/2} \Gamma(s/2) \zeta(s) = \frac{1}{s-1} - a + b(s-1) + O((s-1)^2), \]
with \( b = 1.00024 \ldots \), and in setting \( \kappa' = \kappa/2 := 2a - 1 = \log(4\pi) - 1 - \gamma = 0.95380 \ldots \) and \( \kappa'' = 3 + 2a^2 - 4b = 0.90769 \ldots \), we have
\[
\rho_2(d) = \frac{1}{2}((\log d - \kappa')^2 - \kappa'') + \frac{1}{2d}((\log d + \kappa')^2 - \kappa'') \leq \frac{1}{2}(\log d - \kappa')^2
\]
for \((d + 1)\kappa'' \geq (\log d + \kappa')^2\), hence for \( d = \sqrt{d_K} \geq \sqrt{148}\) (notice that 148 is the least discriminant of a non-normal totally real cubic field).

3. Proof of Theorem 3

Set \( \kappa_K := \text{Res}_{s=1}(\zeta_K(s)), d = \sqrt{d_K}, g(t) = \sum_{n \geq 1} e^{-\pi n^2 t} \) for \( t > 0 \) and \( \Lambda(s) := s(s-1)\pi^{-s/2} \Gamma(s/2) \zeta(s) \). We have
\[
\Lambda(s) = 1 + s(s-1) \int_1^\infty g(t)(t^{s/2} + t^{(1-s)/2}) \frac{dt}{t}
\]
(see [Lan, Page 250]) and
\[
H_{m-1}(s) = \frac{2s-1}{sm} \Lambda^{m-1}(s).
\]

According to Proposition 4, we have
\[
\kappa_K \leq \rho_{m-1}(d) = \text{Res}_{s=1} \left\{ s \mapsto \frac{1}{(s-1)^m} (d^{s-1} + d^{-s}) \frac{2s-1}{sm} \Lambda^{m-1}(s) \right\}
\]
\[
= \frac{1}{(m-1)!} \frac{d^{m-1}}{ds^{m-1}} \left|_{s=1} (d^{s-1} + d^{-s}) H_{m-1}(s) \right.
\]
\[
= \frac{1}{(m-1)!} \sum_{k=0}^{m-1} \binom{m-1}{k} (\log^{m-1-k} d) (1 + \frac{(-1)^{m-1-k}}{d}) \frac{d^k H_{m-1}(s)}{ds^k} \bigg|_{s=1}.
\]

Now, \( H_{m-1}(1) = \Lambda(1) = 1 \) and
\[
H'_{m-1}(1) = 1 - (m-1)(1 - \Lambda'(1)) := -c_{m-1} < 0
\]
for \( m \geq 3 \) (for \( \Lambda'(1) = (2 + \gamma - \log(4\pi))/2 = 0.02309 \ldots \)). Using Lemma 6 below, we obtain
\[
((m-1)!) \cdot \kappa_K \leq (1 + \frac{(-1)^{m-1}}{d}) \log^{m-1} d
\]
\[
- (m-1)(1 + \frac{(-1)^{m-2}}{d}) c_{m-1} \log^{m-2} d
\]
\[
+ ((m-1)!) \cdot (1 + \frac{1}{d^r}) 2r + 1 \frac{\Lambda(1+r)}{1-r} r^{m-1} \frac{m-3}{k!} (r \log d)^k.
\]

(2) It follows that \( \Lambda(s) \) is positive and convex for \( s > 0 \) (see [SZ] for a different proof, and [Lou00, Lemma 9] for a stronger result), for (10) yields \( \Lambda^{(k)}(s) \geq 0 \) for \( s \geq 1/2 \) and \( k \geq 0 \), and the functional equation \( \Lambda(1-s) = \Lambda(s) \) then yields \( (-1)^k \Lambda^{(k)}(s) \geq 0 \) for \( s \leq 1/2 \).
Now, assume that $d \geq \exp(2(m - 3)/r)$. Then,
\[
\frac{(r \log d)^k}{k!} \leq \frac{(r \log d)^{m-3}}{(m-3)!} 2^{-(m-3-k)} \quad \text{for } 0 \leq k \leq m - 3
\]
and
\[
\frac{(m-1)!}{\log^{m-1} d} d^{\kappa_K} - 1 \leq 2 - \frac{(m-1)(1 - \frac{1}{d}) c_{m-1} \log^{-1} d}{d}
\]
\[
+ 2(m-1)(m-2)(1 + \frac{1}{d}) \frac{2r + 1}{(1 - r)^2} \left( \frac{\Lambda(1 + r)}{1 - r} \right)^{m-1} \log^{-2} d,
\]
and this right hand side is clearly negative for $m \geq 3$ and $d \geq d_m$ large enough. Now, we take
\[
r = \frac{2}{(m-1)(1 + \Lambda'(1))}
\]
(hence, $0 < r < 1$ for $m \geq 3$) and we still assume that
\[
d \geq \exp(2(m - 3)/r) = \exp((1 + \Lambda'(1))(m - 3)(m - 1)).
\]
We have
\[
\frac{2r + 1}{(1 - r)^2} \left( \frac{\Lambda(1 + r)}{1 - r} \right)^{m-1} = \frac{1}{4} (1 + \Lambda'(1))^2 c^2 m^2 + O(m)
\]
and for any
\[
C^* (1 + \Lambda'(1))^2 c^2 = \frac{4.052168162 \ldots}{2(1 - \Lambda'(1))}
\]
we obtain $(m-1)! \cdot \kappa_K \leq \log^{m-1} d$ for $d \geq \exp(C^*m^2)$ and $m$ large enough, which proves the desired result for any $C = \exp(2C^*) > 3308.78497 \ldots$.

**Lemma 6.** — For $k \geq 0$ and $0 < r < 1$, it holds that
\[
\left| \frac{d^k H_{m-1}(s)}{ds^k} \right|_{s=1} \leq \frac{2r + 1}{1 - r} \left( \frac{\Lambda(1 + r)}{1 - r} \right)^{m-1} \frac{k!}{r^k}
\]

*Proof.* — Since $H_{m-1}(s)$ is analytic in the half plane $\Re(s) > 0$, for any $r \in (0,1)$ we have
\[
\left| \frac{d^k H_{m-1}(s)}{ds^k} \right|_{s=1} = \left| k! \frac{1}{2\pi i} \int_{|z-1|=r} \frac{H_{m-1}(z)}{(z-1)^{k+1}} dz \right| \leq \frac{k!}{r^k} \sup_{|z-1|=r} |H_{m-1}(z)|.
\]
Since $(t > 0) \sigma \mapsto t^{\sigma/2} + t^{(1-\sigma)/2}$ is convex in $(0, \infty)$, we have
\[
|t^{\sigma/2} + t^{(1-\sigma)/2}| \leq t^{\sigma/2} + t^{(1-\sigma)/2} \leq \max(t^{(1-r)/2} + t^{r/2}, t^{(1+r)/2} + t^{-r/2}) = t^{(1+r)/2} + t^{-r/2}
\]
for $\sigma = \Re(z)$ and $0 < |1-z| = r < 1$ and $t \geq 1$, and using (10) we obtain
\[
|\Lambda(z)| \leq 1 + (1 + r) r \int_1^\infty g(t) (t^{(1+r)/2} + t^{-r/2}) dt = \Lambda(1 + r)
\]
and
\[
|H_{m-1}(z)| \leq \frac{2r + 1}{(1 - r)^m} \Lambda^{m-1}(1 + r)
\]
for $0 < |z - 1| = r < 1$. 

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References


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