On Free Field Realizations of $W(2, 2)$-Modules

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Abstract. The aim of the paper is to study modules for the twisted Heisenberg–Virasoro algebra $\mathcal{H}$ at level zero as modules for the $W(2, 2)$-algebra by using construction from [J. Pure Appl. Algebra 219 (2015), 4322–4342, arXiv:1405.1707]. We prove that the irreducible highest weight $\mathcal{H}$-module is irreducible as $W(2, 2)$-module if and only if it has a typical highest weight. Finally, we construct a screening operator acting on the Heisenberg–Virasoro vertex algebra whose kernel is exactly $W(2, 2)$ vertex algebra.

Key words: Heisenberg–Virasoro Lie algebra; vertex algebra; $W(2, 2)$ algebra; screening-operators

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1 Introduction

Lie algebra $W(2, 2)$ was first introduced by W. Zhang and C. Dong in [20] as part of a classification of certain simple vertex operator algebras. Its representation theory has been studied in [14, 15, 18, 19] and several other papers. Although $W(2, 2)$ is an extension of the Virasoro algebra, its representation theory is very different. This is most notable with highest weight representations. It was shown in [19] that some Verma modules contain a cosingular vector.

Highest weight representation theory of the twisted Heisenberg–Virasoro Lie algebra has also been studied recently. Representations with nontrivial action of $C_I$ have been developed in [6]. Representations at level zero, i.e., with trivial action of $C_I$ were studied in [8] due to their importance in some constructions over the toroidal Lie algebras (see [7, 9]). In this case, a free field realization of highest weight modules along with the fusion rules for a suitable category of modules were obtained in [4].

Irreducible highest weight modules of highest weights $(0, 0)$ over these algebras carry the structure of simple vertex operator algebras. Let us denote these vertex operator algebras as $L^{W(2, 2)}(c_L, c_W)$ and $L^{\mathcal{H}}(c_L, c_{L,I})$. It was proved in [4] that simple vertex operator algebra $L^{W(2, 2)}(c_L, c_W)$ embeds into Heisenberg–Virasoro vertex operator algebra $L^{\mathcal{H}}(c_L, c_{L,I})$ so that $c_W = -24 c_{L,I}^2$. As a result each highest weight module over $\mathcal{H}$ is also a $W(2, 2)$-module. In this paper we shall completely described the structure of the irreducible highest weight $\mathcal{H}$-modules as $W(2, 2)$-modules. We show that in generic case the resulting $W(2, 2)$-module is irreducible. However, in case of a module of highest weight such that associated Verma module over $W(2, 2)$ contains cosingular vectors (we shall call this kind of weight atypical), irreducible $\mathcal{H}$-module is reducible over $W(2, 2)$. We shall denote the irreducible highest weight $\mathcal{H}$-module...
Our construction uses an extension which commutes with the action of screening operator \( M \) operator inside the Heisenberg vertex algebra \( L^2 \). Adamović and G. Radobolja for operators and logarithmic modules. We present an explicit realization of module for the Heisenberg–Virasoro vertex algebra. In our forthcoming paper [5], we shall Feigin–Fuchs modules for the Virasoro algebra (cf. Remark 3.8).

We recall some aspects of representation theories of infinite-dimensional Lie algebras \( H \) and \( W(2, 2) \) in Section 2. The main results of the branching rules will be proved in Section 3. From the free field realization in [4] follows that irreducible \( W(2, 2) \)-modules are pairwise contragredient. For half of these modules, proofs rely on a contragredient. For the rest is then proved elegantly by passing to contragredients. We also prove a very interesting result that the Verma module for \( H \) with typical highest weight is an infinite direct sum of irreducible \( W(2, 2) \)-modules (cf. Theorem 3.7). This result presents a \( W(2, 2) \)-analogue of certain Feigin–Fuchs modules for the Virasoro algebra (cf. Remark 3.8).

From the results in the paper, we see that the vertex algebra \( L^{W(2,2)}(c_L, c_W) \) has many properties similar to the \( W \)-algebras appearing in logarithmic conformal field theory (LCFT):

- \( L^{W(2,2)}(c_L, c_W) \) admits a free field realization inside of the Heisenberg–Virasoro vertex algebra \( L^H(c_L, c_L, I) \).
- Typical modules are realized as irreducible modules for \( L^H(c_L, c_L, I) \).
- In the atypical case, irreducible \( L^H(c_L, c_L, I) \)-modules as \( L^{W(2,2)}(c_L, c_W) \)-modules have semi–simple rank two.

The singlet vertex algebra \( \overline{M}(1) \) has similar properties. \( \overline{M}(1) \) is realized as kernel of a screening operator inside the Heisenberg vertex algebra \( M(1) \) (cf. [1]). In Section 4 we construct the screening operator

\[
S_1: L^H(c_L, c_L, I) \to L^H(1, 0),
\]

which commutes with the action of \( W(2, 2) \)-algebra such that

\[
\text{Ker}_{L^H(c_L, c_L, I)} S_1 \cong L^{W(2,2)}(c_L, c_W).
\]

Our construction uses an extension \( V_{ext} \) of the vertex algebra \( L^H(c_L, c_L, I) \) by a non-weight module for the Heisenberg–Virasoro vertex algebra. In our forthcoming paper [5], we shall present an explicit realization of \( V_{ext} \) and apply this construction to the study of intertwining operators and logarithmic modules.

\footnote{We emphasise a term \( \frac{p-1}{24} \) for its importance in a free field realization of \( H \) (see [4] for details).}
2 Lie algebra $W(2, 2)$ and the twisted Heisenberg–Virasoro
Lie algebra at level zero

$W(2, 2)$ is a Lie algebra with basis $\{L(n), W(n), C_L, C_W : n \in \mathbb{Z}\}$ over $\mathbb{C}$, and a Lie bracket

\[
[L(n), L(m)] = (n - m)L(n + m) + \delta_{n, -m} \frac{n^3 - n}{12} C_L,
\]

\[
[L(n), W(m)] = (n - m)W(n + m) + \delta_{n, -m} \frac{n^3 - n}{12} C_W,
\]

\[
[W(n), W(m)] = [\cdot, C_L] = [\cdot, C_W] = 0.
\]

Highest weight representation theory over $W(2, 2)$ was studied in [14, 19]. However, representations treated in these papers have equal central charges $C_L = C_W$. These results have recently been generalised to $C_L \neq C_W$ in [15]. Here we state the most important results. Verma module with central charge $(c_L, c_W)$ and highest weight $(h, h_W)$ is denoted by $V^{W(2, 2)}(c_L, c_W, h, h_W)$, its highest weight vector by $v_{h, h_W}$ and irreducible quotient module by $L^{W(2, 2)}(c_L, c_W, h, h_W)$.

Recall the definition of a cosingular vector. Homogeneous vector $v \in M$ is called cosingular (or subsingular) if it is not singular in $M$ and if there is a proper submodule $N \subset M$ such that $v + N$ is a singular vector in $M/N$.

**Theorem 2.1** ([15, 19]). Let $c_W \neq 0$.

(i) Verma module $V^{W(2, 2)}(c_L, c_W, h, h_W)$ is reducible if and only if $h_W = \frac{1-p^2}{24} c_W$ for some $p \in \mathbb{Z}_{>0}$. In that case, there exists a singular vector $u'_p \in \mathbb{C}[W(-1), \ldots, W(-p)]v_{h, h_W}$ such that $U(W(2, 2))u'_p \cong V^{W(2, 2)}(c_L, c_W, h + p, h_W)$.

(ii) A quotient module\(^2\)

\[
V^{W(2, 2)}(c_L, c_W, h, h_W)/U(W(2, 2))u'_p =: \tilde{L}^{W(2, 2)}(c_L, c_W, h_p, h_W)
\]

is reducible if and only if $h = h_p$ for some $r \in \mathbb{Z}_{>0}$. In that case, there is a cosingular vector $u_{p, r} \in V^{W(2, 2)}(c_L, c_W, h, h_W)_{h + rp}$ such that the short sequence

\[
\begin{align*}
0 &\rightarrow L^{W(2, 2)}(c_L, c_W, h_p, h_W + rp, h_W) \rightarrow \tilde{L}^{W(2, 2)}(c_L, c_W, h_p, h_W) \\
&\rightarrow L^{W(2, 2)}(c_L, c_W, h_p, h_W) \rightarrow 0
\end{align*}
\]

is exact.

Define

\[
\mathcal{AT}_{W(2, 2)}(c_L, c_W) = \left\{ h_{p, r}, \frac{1-p^2}{24} c_W \mid p, r \in \mathbb{Z}_{>0} \right\}.
\]

**Remark 2.2.** We will refer to the (modules of) highest weights $(h, h_W) \in \mathcal{AT}_{W(2, 2)}(c_L, c_W)$ as *atypical* for $W(2, 2)$, and otherwise as *typical*. Again, we refer to a highest weight $W(2, 2)$-module as (a)typical depending on its highest weight. So a Verma module over $W(2, 2)$ contains a nontrivial cosingular vector if and only if it is atypical.

**Proposition 2.3.** Let $h_W = \frac{1-p^2}{24} c_W$, $p \in \mathbb{Z}_{>0}$.

(i) Let $(h_p, h_W), r \in \mathbb{Z}_{>0}$ be an atypical weight and $k \in \mathbb{Z}$. Then $(h_p + kp, h_W)$ is *atypical* if and only if $k < \frac{r}{2}$.

\(^2\)This module is denoted by $L'$ in [15, 19]. We change notation to $\tilde{L}$ due to use of superscript $W(2, 2)$.
(ii) Atypical Verma module $W^{(2,2)}(h_{p,r},h_W)$ contains exactly $\left\lfloor \frac{r+1}{2} \right\rfloor$ cosingular vectors. The weights of these vectors are $h_{p,r} + (r-i)p = h_{p,r+2i}, i = 0, \ldots, \left\lfloor \frac{r+1}{2} \right\rfloor$.

**Proof.** (i) Directly from Theorem 2.1 since $h_{p,r} + kp = h_{p,r-2k}$.
(ii) Follows from (i) since $W^{(2,2)}(h_{p,r},h_W)$ contains a chain of submodules which are isomorphic to $W^{(2,2)}(h_{p,r} + ip, h_W), i \in \mathbb{Z}_{>0}$.

**Remark 2.4.** Standard PBW basis for $W^{(2,2)}(c_L,c_W,h,h_W)$ consists of vectors

$$Wt(-m_s) \cdots W(-m_1)L(-n_t) \cdots L(-n_1)v_{h,h_W}$$

such that $m_s \geq \cdots \geq m_1 \geq 1, n_t \geq \cdots \geq n_1 \geq 1$. The only nonzero component of $u_{rp}$ belonging to $\mathbb{C}[L(-1), L(-2), \ldots]v$ is $L(-p)^rv_{h,h_W}$ [19].

Define $P_2(n) = \sum_{i=0}^{n} P(n-i)P(i)$ where $P$ is a partition function with $P(0) = 1$. We have the following character formulas [19]

\[
\text{char } W^{(2,2)}(c_L,c_W,h,h_W) = q^h \sum_{n \geq 0} P_2(n) q^n = q^h \prod_{k \geq 1} (1 - q^k)^{-2},
\]

for all $h, h_W \in \mathbb{C}$. If $h_W = \frac{1-p^2}{24}c_W$, then

\[
\text{char } W^{(2,2)}(c_L,c_W,h,h_W) = q^h (1 - q^p) \sum_{n \geq 0} P_2(n) q^n = q^h (1 - q^p) \prod_{k \geq 1} (1 - q^k)^{-2}.
\]

If $(h, h_W)$ is typical for $W(2,2)$, then this is the character of an irreducible highest weight module. Finally, the character of atypical irreducible module is

\[
\text{char } W^{(2,2)}(c_L,c_W,h_{p,r},h_W) = q^{h_{p,r}} (1 - q^p) (1 - q^{2p}) \sum_{n \geq 0} P_2(n) q^n
\]

\[
= q^{h_{p,r}} (1 - q^p) (1 - q^{2p}) \prod_{k \geq 1} (1 - q^k)^{-2}.
\]

The twisted Heisenberg–Virasoro algebra $\mathcal{H}$ is the universal central extension of the Lie algebra of differential operators on a circle of order at most one. It is the infinite-dimensional complex Lie algebra with a basis

$$\{L(n), I(n) : n \in \mathbb{Z} \} \cup \{C_L, C_{LI}, C_I\}$$

and commutation relations

\[
[L(n), L(m)] = (n - m)L(n + m) + \delta_{n,-m} \frac{n^3 - n}{12} C_L,
\]

\[
[L(n), I(m)] = -mI(n + m) - \delta_{n,-m}(n^2 + n) C_{LI},
\]

\[
[I(n), I(m)] = n\delta_{n,-m} C_I,
\]

\[
[\mathcal{H}, C_L] = [\mathcal{H}, C_{LI}] = [\mathcal{H}, C_I] = 0.
\]

The Lie algebra $\mathcal{H}$ admits the following triangular decomposition

\[
\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^0 \oplus \mathcal{H}^+,
\]

\[
\mathcal{H}^\pm = \text{span}_\mathbb{C}\{I(\pm n), L(\pm n) | n \in \mathbb{Z}_{>0}\}, \quad \mathcal{H}^0 = \text{span}_\mathbb{C}\{I(0), L(0), C_L, C_{LI}, C_I\}.
\]

Although they seem to be two similar extensions of the Virasoro algebra, representation theories of $W(2,2)$ and $\mathcal{H}$ are different. The main reason for that lies in the fact that $I(0)$ is
a central element, while $W(0)$ is not. However, applying free field realization, we shall see that highest weight modules over the two algebras are related.

Denote by $V^H(c_L, c_I, c_{L,I}, h, h_I)$ the Verma module and by $v_{h,I}$ its highest weight vector. $C_L$, $C_I$, $C_{L,I}$, $L(0)$ and $I(0)$ act on $v_{h,I}$ by scalars $c_L$, $c_I$, $c_{L,I}$, $h$ and $h_I$, respectively. Then $(c_L, c_I, c_{L,I})$ is called a central charge, and $(h, h_I)$ a highest weight. In this paper we consider central charges $(c_L, 0, c_{L,I})$ such that $c_{L,I} \neq 0$.

**Theorem 2.5** ([8]). Let $c_{L,I} \neq 0$. Verma module $V^H(c_L, 0, c_{L,I}, h, h_I)$ is reducible if and only if $h_I = (1 \pm p)c_{L,I}$ for some $p \in \mathbb{Z}_{\geq 0}$. In that case, there is a singular vector $v_p^\pm$ of weight $p$, which generates a maximal submodule in $V^H(c_L, 0, c_{L,I}, h, h_I)$ isomorphic to $V^H(c_L, 0, c_{L,I}, h + p, h_I)$.

**Remark 2.6.** In case $h_I = (1 + p)c_{L,I}$ an explicit formula for a singular vector $v_p^+$ is obtained using Schur polynomials in $I(−1), \ldots, I(−p)$. See [4] for details. Suppose that $xv_p^+ \in V^H(c_L, 0, c_{L,I}, h, h_I)$ lies in a maximal submodule. Then $x$ does not have a nontrivial additive component belonging to $\mathbb{C}[L(−1), L(−2), \ldots]$ [8].

There is an infinite chain of Verma submodules generated by singular vectors $v_p^\pm$, $k \in \mathbb{Z}_{>0}$, with all the subquotients being irreducible. Note that there is no mention of $L^H$ since there are no cosingular vectors in $V^H$.

The following character formulas were obtained in [8]:

$$
\text{char } V^H(c_L, 0, c_{L,I}, h, h_I) = q^h \sum_{n \geq 0} P_2(n) q^n = q^h \prod_{k \geq 1} (1 - q^k)^{-2},
$$

$$
\text{char } L^H(c_L, 0, c_{L,I}, h, h_I) = q^h \sum_{n \geq 0} P_2(n) q^n = q^h \prod_{k \geq 1} (1 - q^k)^{-2}.
$$

**Remark 2.7.** Throughout the rest of the paper we work with highest weight modules over the Lie algebras $W(2, 2)$ and $\mathcal{H}$ so we always denote algebra in superscript. In order to avoid too cumbersome notation, we omit central charges. Therefore, we write $V^H(h, h_I)$ for Verma module over $\mathcal{H}$, $V^{W(2, 2)}(h, h_W)$ for Verma module over $W(2, 2)$ and so on. We always assume that $c_W$ and $c_{L,I}$ are nonzero. Moreover, if we work with several modules over both algebras, $c_L$ is equal for all modules.

We shall write $(x)_{W(2, 2)}$ for a cyclic submodule $U(W(2, 2))x$ and $(x)_{\mathcal{H}}$ for $U(\mathcal{H})x$. Finally, $\cong_{W(2, 2)}$ denotes an isomorphism of $W(2, 2)$-modules.

## 3 Irreducible highest weight modules

In this section we present main results of the paper which completely describe the structure of (irreducible) highest weight modules for $\mathcal{H}$ as $W(2, 2)$-modules. The main tool is the homomorphism between $W(2, 2)$ and the Heisenberg–Virasoro vertex algebras from [4].

$L^{W(2, 2)}(c_L, c_W, 0, 0)$ is a simple universal vertex algebra associated to Lie algebra $W(2, 2)$ (cf. [19, 20]) which we denote by $L^{W(2, 2)}(c_L, c_W)$. It is generated by fields

$$
L(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \quad W(z) = Y(W, z) = \sum_{n \in \mathbb{Z}} W(n) z^{-n-2}
$$

where $\omega = L(−2)1$ and $W = W(−2)1$. Each highest weight $W(2, 2)$-module is also a module over a vertex operator algebra $L^{W(2, 2)}(c_L, c_W)$.

Likewise (see [7]) $L^H(c_L, 0, c_{L,I}, 0, 0)$ is a simple Heisenberg–Virasoro vertex operator algebra, which we denote by $L^H(c_L, c_{L,I})$. This algebra is generated by the fields

$$
L(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \quad I(z) = Y(I, z) = \sum_{n \in \mathbb{Z}} I(n) z^{-n-1}
$$
where \( \omega = L(-2)1 \) and \( I = I(-1)1 \). Moreover, highest weight \( \mathcal{H} \)-modules are modules over a vertex operator algebra \( L^H(c_L,c_L,1) \).

It was shown in [4] that there is a monomorphism of vertex operator algebras

\[
\Psi: L^{W(2,2)}(c_L,c_W) \rightarrow L^H(c_L,c_L,1),
\]

\[
\omega \mapsto L(-2)1, \\
W \mapsto (I(-1)^2 + 2c_L,I(-2))1,
\]

where \( c_W = -24c_L^2 \). By means of \( \Psi \), each highest weight module over \( \mathcal{H} \) becomes an \( L^{W(2,2)}(c_L,c_W) \)-module and therefore a module over \( W(2,2) \). In particular, \( \Psi \) induces a non-trivial \( W(2,2) \)-homomorphism (which we shall denote by the same letter)

\[
\Psi: V^{W(2,2)}(c_L,c_W,h,h_W) \rightarrow V^H(c_L,0,c_L,1,h,h_I),
\]

where \( c_W = -24c_L^2 \) and \( h_W = h_I(h_I - 2c_L,1) \). \( \Psi \) maps the highest weight vector \( v_{h,h_W} \) to the highest weight vector \( v_{h,h_I} \) and the action of \( W(-n) \) on \( V^H(c_L,0,c_L,1,h,h_I) \) is given by

\[
W(-n) \equiv 2c_L,I(n-1)I(-n) + \sum_{i \in \mathbb{Z}} I(-i)I(-n+i),
\]

\[
W(-n) \equiv 2c_L,I \left( n - 1 + \frac{h_I}{c_L,1} \right) I(-n) + \sum_{i \neq 0,n} I(-i)I(-n+i).
\]

Note that \( h_W = \frac{1-p^2}{24} c_W \) if and only if \( h_I = (1 \pm p)c_L,1 \), so either both of these Verma modules are irreducible, or they are reducible with singular vectors at equal levels. Moreover, \( (h,h_W) \in AT_{W(2,2)}(c_L,c_W) \) if and only if \( (h,h_I) \in AT_{H}(c_L,c_L,1) \).

Throughout the rest of this section we assume that \( c_W = -24c_L^2 \).

**Lemma 3.1** ([4, Lemma 7.2]). Suppose that \( h_I \neq (1 - p)c_L,1 \) for all \( p \in \mathbb{Z}_{>0} \). Then \( \Psi \) is an isomorphism of \( W(2,2) \)-modules. In particular, if \( h_I \neq (1 \pm p)c_L,1 \) for \( p \in \mathbb{Z}_{>0} \), then

\[
L^H(h,h_I) \cong_{W(2,2)} L^{W(2,2)}(h,h_W),
\]

where \( h_W = h_I(h_I - 2c_L,1) \).

**Lemma 3.2.** Suppose that \( x \in V^H(h,h_I) \) is \( \mathcal{H} \)-singular. Then \( x \) is \( W(2,2) \)-singular as well. In particular, if \( x = \Psi(y) \) is an \( \mathcal{H} \)-singular vector, then \( y \) is a (co)singular vector in \( V^{W(2,2)}(h,h_W) \).

**Proof.** Follows directly from (3.2) since

\[
W(n)x = -2c_L,I(n+1)I(n)x + \sum_{i \in \mathbb{Z}} I(-i)I(n+i)x.
\]

If \( I(k)x = 0 \) for all \( k \in \mathbb{Z}_{>0} \), then \( W(n)x = 0 \) for all \( n \in \mathbb{Z}_{>0} \). If \( x = \Psi(y) \), then \( W(2,2)y \in \text{Ker} \Psi \) so \( y \) is cosingular (or singular if \( \text{Ker} \Psi = 0 \)).

**Theorem 3.3.** Let \( p \in \mathbb{Z}_{>0} \).

(i) If \( (h,(1+p)c_L,1) \) is typical for \( \mathcal{H} \) (equivalently if \( (h,\frac{1-p^2}{24} c_W) \) is typical for \( W(2,2) \)) then

\[
L^H(h,(1+p)c_L,1) \cong_{W(2,2)} L^{W(2,2)} \left( h,\frac{1-p^2}{24} c_W \right).
\]

(ii)
(ii) If \((h_{p,r}, (1 + p)c_{L,I}) \in \mathcal{AT}_H(c_L, c_{L,I})\) (equivalently if \((h_{p,r}, \frac{1 - p^2}{24}c_W) \in \mathcal{AT}_{W(2,2)}(c_L, c_W)\)) then

\[
L^H(h_{p,r}, (1 + p)c_{L,I}) \cong_{W(2,2)} \tilde{L}^{W(2,2)}\left(h_{p,r}, \frac{1 - p^2}{24}c_W\right)
\]

and the short sequence of \(W(2,2)\)-modules

\[
0 \to L^{W(2,2)}\left(h_{p,r} + rp, \frac{1 - p^2}{24}c_W\right) \to L^H(h_{p,r}, (1 + p)c_{L,I}) \to \tilde{L}^{W(2,2)}(h_{p,r}, \frac{1 - p^2}{24}c_W) \to 0
\]

is exact.

**Proof.** By Lemma 3.1, \(\Psi\) is an isomorphism of Verma modules and thus by Lemma 3.2 it maps a \(W(2,2)\)-singular vector \(u'_p\) to an \(H\)-singular vector \(v'_p\). If \(h \neq h_{p,r}\), both of these vectors generate maximal submodules in respective Verma modules so (3.3) follows.

Now suppose that \(h = h_{p,r}\). We need to show that a cosingular vector \(u_{rp}\) is not mapped to a maximal submodule of \(V_H(h_{p,r}, h_I)\). But \(u_{rp}\) has \(L(-p)v\) as an additive component (see Remark 2.4), and by construction (3.1), \(\Psi(u_{rp})\) also must have this additive component. However, \(\Psi(u_{rp})\) can not lie in a maximal \(H\)-submodule of \(V_H(h, h_I)\) (see Remark 2.6). This means that isomorphism \(\Psi\) of Verma modules induces a \(W(2,2)\)-isomorphism of \(\tilde{L}^{W(2,2)}(h, h_W)\) and \(L^H(h, h_I)\) for all \(h \in \mathbb{C}\). Exactness of (3.4) is just an application of (2.1).

**Remark 3.4.** Note that the image \(\Psi(u_{rp})\) of a \(W(2,2)\)-cosingular vector is neither \(H\)-singular, nor \(H\)-cosingular in \(V_H(h_{p,r}, (1 + p)c_{L,I})\). For example, \(L(-1)v_{0,0}\) in \(V_H(0, 2c_{L,I})\) is \(W(2,2)\)-cosingular, but not \(H\)-singular since \(I(1)L(-1)v_{0,0} = 2c_{L,I}v_{0,0}\).

If \(h_I = (1 - p)c_{L,I}\), then \(\Psi\) is not an isomorphism. We shall present a \(W(2,2)\)-structure of Verma module later. In order to examine irreducible \(W(2,2)\)-modules we apply the properties of contragredient modules.

Let us recall the definition of contragredient module (see [12]). Assume that \((M, Y_M)\) is a graded module over a vertex operator algebra \(V\) such that \(M = \oplus_{n=0}^{\infty} M(n)\), \(\dim M(n) < \infty\) and suppose that there is \(\gamma \in \mathbb{C}\) such that \(L(0)M(n) \equiv (\gamma + n)Id\). The contragredient module \((M^*, Y_{M^*})\) is defined as follows. For every \(n \in \mathbb{Z}_{>0}\) let \(M(n)^* = \oplus_{n=0}^{\infty} M(n)^*\) be a restricted dual of \(M\). Consider the natural pairing \(\langle \cdot, \cdot \rangle : M^* \otimes M \to \mathbb{C}\). Define the linear map \(Y_{M^*} : V \to \text{End} M^*[z, z^{-1}]\) such that

\[
\langle Y_{M^*}(v, z)m', m\rangle = \langle m', Y_M(z^{L(1)}(z^{-2})^{L(0)}v, z^{-1})m\rangle
\]

for each \(v \in V, m \in M, m' \in M^*\). Then \((M^*, Y_{M^*})\) is a \(V\)-module.

In particular, choosing \(v = \omega = L_{-2}1\) in (3.5) one gets

\[
\langle L(n)m', m\rangle = \langle m', L(-n)m\rangle.
\]

Simple calculation with \(I \in \mathcal{L}^H(c_L, c_{L,I})\) and \(W \in \mathcal{L}^{W(2,2)}(c_L, c_W)\) shows that

\[
\langle I(n)m', m\rangle = \langle m', -I(-n)m + \delta_{n,0}2c_{L,I}\rangle, \quad \langle W(n)m', m\rangle = \langle m', W(-n)m\rangle.
\]

Therefore we get the following result (the first and third relations were given in [4]):
Lemma 3.5.

\[ L^H(h, h_1)^* \cong L^H(h, -h_1 + 2c_{L,I}), \quad L^{W(2,2)}(h, h_W)^* \cong L^{W(2,2)}(h, h_W). \]

In particular,

\[ L^H(h, (1 \pm p)c_{L,I})^* \cong L^H(h, (1 \mp p)c_{L,I}). \]

Directly from Theorem 3.3 and Lemma 3.5 follows

Corollary 3.6. Let \( p \in \mathbb{Z}_{>0}. \)

(i) If \((h, (1 - p)c_{L,I})\) is typical for \( H \) (equivalently if \((h, \frac{1-p^2}{24}c_W)\) is typical for \( W(2,2) \)) then

\[ L^H(h, (1 - p)c_{L,I}) \cong_{W(2,2)} L_{W(2,2)}(h, \frac{1-p^2}{24}c_W). \]

(ii) If \((h_{p,r}, (1 - p)c_{L,I}) \in \mathcal{AT}_H(c_{L,I},c_{L,I})\) (equivalently if \((h_{p,r}, \frac{1-p^2}{24}c_W) \in \mathcal{AT}_{W(2,2)}(c_{L,I},c_W)\) then

\[ L^H(h_{p,r}, (1 - p)c_{L,I}) \cong_{W(2,2)} L_{W(2,2)}(h_{p,r}, \frac{1-p^2}{24}c_W)^* \]

and the short sequence of \( W(2,2) \)-modules

\[ 0 \to L_{W(2,2)}(h_{p,r}, \frac{1-p^2}{24}c_W) \to L^H(h_{p,r}, (1 - p)c_{L,I}) \]
\[ \to L_{W(2,2)}(h_{p,r} + rp, \frac{1-p^2}{24}c_W) \to 0 \]

is exact.

From Lemma 3.1, Theorem 3.3 and Corollary 3.6 follow assertions of Theorem 1.1.

Finally, we show that Verma module over \( H \) is an infinite direct sum of irreducible \( W(2,2) \)-modules. Recall that \( V^H(h, (1 - p)c_{L,I}) \) has a series of singular vectors \( v_{-ip} \), \( i \in \mathbb{Z}_{\geq 0} \) (for \( i = 0 \), we set \( v_{-0} = v_{h,h_I} \)) which generate a descending chain of Verma submodules over \( H \):

\[
\begin{align*}
\langle v_{-ip} \rangle_H &= V^H(h, h_I) \\
\langle v_{-i(p+1)} \rangle_H &\cong V^H(h + i, h_I) \\
\langle v_{-i(p+2)} \rangle_H &\cong V^H(h + (i + 1)p, h_I) \\
\langle v_{-i(p+3)} \rangle_H &\cong V^H(h + (i + 2)p, h_I) \\
\langle v_{-i(p+4)} \rangle_H &\cong V^H(h + (i + 3)p, h_I) \\
&\vdots
\end{align*}
\]

Therefore one may identify \( V^H(h + ip, h_I) \) with a submodule of \( V^H(h, h_I) \) and a singular vector \( v_{-ip} \in V^H(h, h_I) \) with the highest weight vector \( v_{h+ip,h_I} \in V^H(h + ip, h_I) \). We will prove that in a typical case each of those vectors generates an irreducible \( W(2,2) \)-submodule.
Theorem 3.7. Let \( p \in \mathbb{Z}_{\geq 0} \). Suppose that \((h, (1 - p)c_{L,I}) \notin \mathcal{AT}_H(c_{L}, c_{L,I})\). Then we have the following isomorphism of \(W(2,2)\)-modules

\[
V^H(h, (1 - p)c_{L,I}) \cong W(2,2) \bigoplus_{i \geq 0} L^{W(2,2)} \left( h + ip, \frac{1 - p^2}{24} c_W \right).
\]

Proof. First we notice that the vertex algebra homorphism \( \Psi: L^{W(2,2)}(c_W, c_L) \to L^H(c_W, c_L) \), for every \( i \in \mathbb{Z}_{\geq 0} \) induces the following non-trivial homomorphism of \(W(2,2)\)-modules:

\[
\Psi^{(i)}: V^{W(2,2)} \left( h + ip, \frac{1 - p^2}{24} c_W \right) \to \langle v^-_{ip}\rangle_{W(2,2)} \subset V^H(h + ip, h_I),
\]

which maps the highest weight vector of \( V^{W(2,2)}(h + ip, \frac{1 - p^2}{24} c_W) \) to \( v^-_{ip} \). Since \((h, \frac{1 - p^2}{24} c_W)\) is typical it follows from Proposition 2.3(i) that \((h + ip, \frac{1 - p^2}{24} c_W)\) are typical for all \( i \in \mathbb{Z}_{\geq 0} \) as well.

Let \( h_W = \frac{1 - p^2}{24} c_W \). Consider the homomorphism \( \Psi^{(i)}: V^{W(2,2)}(h + ip, h_W) \to V^H(h + ip, h_I) \) above. Applying (3.2), we get

\[
\Psi^{(i)}(W(-p)v_{h+ip,h_I}) = \sum_{i=1}^{p-1} I(-i) I(i - p) v_{h+ip,h_I},
\]

so \( I(-p)v_{h+ip,h_I} \notin \text{Im} \Psi^{(i)} \). Since the Verma modules \( V^{W(2,2)}(h + ip, h_W) \) and \( V^H(h + ip, h_I) \) have equal characters, it follows that \( \ker \Psi^{(i)} \) contains a singular vector in \( V^{W(2,2)}(h + ip, h_W) \) of conformal weight \( h + (i+1)p \). Since the weight \((h + ip, h_W)\) is typical, the maximal submodule in \( V^{W(2,2)}(h + ip, h_W) \) is generated by this singular vector so we conclude that \( \ker \Psi^{(i)} \) is the maximal submodule in \( V^{W(2,2)}(h + ip, h_W) \). Therefore

\[
\text{Im} \Psi^{(i)} = \langle v_{h+ip,h_I}\rangle_{W(2,2)} \cong L^{W(2,2)}(h + ip, h_W).
\]

In this way we get a series of \((2,2)\)-monomorphisms

\[
L^{W(2,2)}(h + ip, h_W) \hookrightarrow V^H(h, (1 - p)c_{L,I}), \quad i \in \mathbb{Z}_{\geq 0}
\]

mapping \( v_{h+ip,h_W} \) to a singular vector \( v^-_{ip} \). Let \( v^-_{jp} \) be an \( H \)-singular vector in \( V^H(h + ip, (1 - p)c_{L,I}) \) of weight \( h + jp \), for \( j > i \). By Lemma 3.2, \( v^-_{jp} \) is singular for \( W(2,2) \) and therefore \( v^-_{jp} \notin \langle v_{h+ip,h_I}\rangle_{W(2,2)} \) for \( j > i \). We conclude that the images of morphisms (3.6) have trivial pairwise intersections (since these images are non-isomorphic irreducible modules), so their sum is direct. The assertion follows from the observation that the character of this sum is

\[
\sum_{i=0}^{\infty} q^{h+ip} (1 - q^p) \prod_{k \geq 1} (1 - q^k)^{-2} = q^{h} \prod_{k \geq 1} (1 - q^k)^{-2} = \text{char} V^H(h, (1 - p)c_{L,I}).
\]

From the previous theorem follows

\[
\begin{align*}
V^H(h, h_I) & \bigoplus \langle v_{h,h_I}\rangle_{W(2,2)} = L^{W(2,2)}(h, h_I) \\
& \bigoplus \langle v^-_p\rangle_{W(2,2)} \cong L^{W(2,2)}(h + p, h_I) \\
& \bigoplus \langle v^-_h\rangle_{W(2,2)} \cong L^{W(2,2)}(h, h_I)
\end{align*}
\]
Remark 3.8. It is interesting to notice that our Theorem 3.7 shows that \( V^H(h, h_I) \) can be considered as a \( W(2, 2) \)-analogue of certain Feigin-Fuchs modules for the Virasoro algebra which are also direct sums of infinitely many irreducible modules (cf. [11], [2, Theorem 5.1]).

In atypical case however, these irreducible \( W(2, 2) \)-submodules intertwine as follows. Consider \( V^H(h_{p,r}, h_I) \) where \( (h_{p,r}, h_I) \in \mathcal{AT}_H(c_L, c_{L,I}) \). Then \( \Psi \) maps a cosingular vector \( u_{rp} \in V^{W(2,2)}(h_{p,r}, h_W) \) to a singular vector \( v_{-rp}^- \). In other words we have

\[
\langle v_{-ip}^- \rangle_{W(2,2)} \subseteq \langle u_{hp}, h_I \rangle_{W(2,2)}.
\]

Using the same argument in view of Proposition 2.3 we see that

\[
\langle v_{-(r-1)p}^- \rangle_{W(2,2)} \subseteq \langle v_{ip}^- \rangle_{W(2,2)}, \quad i = 1, \ldots, \left\lfloor \frac{r-1}{2} \right\rfloor.
\]

In this case, \( I(-p)^{r-i} v_{hp}, h_I \), are \( W(2, 2) \)-subsingular vectors in \( V^H(h_{p,r}, h_I) \).

Example 3.9. Consider \( p = 1 \) case. Singular vector in \( V^H(h, 0) \) is \( u'_1 = (L(-1) + \frac{h}{c_{L,I}} I(-1)) v_{0,0} \), and \( u'_1 \) generates a copy of \( V^H(h + 1, 0) \).

\( r = 1 \): \( \Psi: V^{W(2,2)}(0, 0) \rightarrow V^H(0, 0) \) maps a singular vector \( u'_1 = W(-1)v_{0,0} \) to \( 0 \) and a cosingular vector \( u_1 = L(-1)v_{0,0} \) to a \( H \)-singular vector \( v^-_1 = L(-1)v_{0,0} \). We get the short exact sequence of \( W(2, 2) \)-modules

\[
0 \rightarrow L^{W(2,2)}(0, 0) \rightarrow L^H(0, 0) \rightarrow L^{W(2,2)}(1, 0) \rightarrow 0,
\]

which is an expansion of (3.1) considered as a homomorphism of \( W(2, 2) \)-modules. The rightmost module is generated by a projective image of \( I(-1)v_{0,0} \). Therefore, \( L^H(c_L, c_{L,I}) \) is generated over \( W(2, 2) \) by \( v_{0,0} \) and \( I(-1)v_{0,0} \).

\( r \in \mathbb{Z}_{>0} \). In general, a cosingular vector \( u_{rp} \in V^{W(2,2)}(\frac{1-r}{2}, 0) \) maps to a singular vector \( v^-_r \in V^H(\frac{1-r}{2}, 0) \) of weight \( \frac{1+r}{2} \).

\[
v^-_r = \prod_{i=0}^{r-1} \left( L(-1) + \frac{1-r+2i}{2c_{L,I}} I(-1) \right) v_{1-r,0}.
\]

4 Screening operators and \( W(2, 2) \)-algebra

We think that the vertex algebra \( L^{W(2,2)}(c_L, c_W) \) is a very interesting example of non-rational vertex algebra, which admits similar fusion ring of representations as some \( W \)-algebras appearing in LCFT (cf. [1, 2, 10, 13]). Since \( W \)-algebras appearing in LCFT are realized as kernels of screening operators acting on certain modules for Heisenberg vertex algebras, it is natural to ask if \( L^{W(2,2)}(c_L, c_W) \) admits similar realization. In [4] we embedded the \( W(2,2) \)-algebra as a subalgebra of the Heisenberg–Virasoro vertex algebra. In this section we shall construct a screening operator \( S_1 \) such that the kernel of this operator is exactly \( L^{W(2,2)}(c_L, c_W) \).
Let us first construct a non-semisimple extension of the vertex algebra $L^H(c_L, c_{L, I})$. Recall that the Lie algebra $H$ admits the triangular decomposition (2.2). Let $E = \text{span}_C\{v^0, v^1\}$ be 2-dimensional $H^{\geq 0} = H^0 \oplus H^+$-module such that $H^+$ acts trivially on $E$ and

$$L(0)v^i = v^i, \quad i = 0, 1, \quad I(0)v^1 = v^0, \quad I(0)v^0 = 0,$$

$$C_Lv^i = c_Lv^i, \quad C_{L, I}v^i = c_{L, I}v^i, \quad C_Iv^i = 0, \quad i = 1, 2.$$

Consider now induced $H$-module

$$\tilde{E} = U(H) \otimes _{U(H^{\geq 0})} E.$$

By construction, $\tilde{E}$ is a non-split self-extension of the Verma module $V^H(1, 0)$:

$$0 \to V^H(1, 0) \to \tilde{E} \to V^H(1, 0) \to 0.$$

Moreover, $\tilde{E}$ is a restricted module for $H$ and therefore it is a module over vertex operator algebra $L^H(c_L, c_{L, I})$. Since

$$\tilde{E} \cong E \otimes U(H^-)$$

as a vector space, the operator $L(0)$ defines a $\mathbb{Z}_{\geq 0}$-gradation on $\tilde{E}$.

Note that $$(L(-1) + I(-1)/c_{L, I})v_0$$ is a singular vector in $\tilde{E}$ and it generates the proper submodule. Finally we define the quotient module

$$U = \frac{\tilde{E}}{U(H)(L(-1) + I(-1)/c_{L, I})v_0}.$$

**Proposition 4.1.** $U$ is a $\mathbb{Z}_{\geq 0}$-graded module for the vertex operator algebra $L^H(c_L, c_{L, I})$:

$$U = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} U(m), \quad L(0)|U(m) \equiv (m + 1)\text{Id}.$$

The lowest component $U(0) \cong E$. Moreover, $U$ is a non-split extension of the Verma module $V^H(1, 0)$ by the simple highest weight module $L^H(1, 0)$:

$$0 \to L^H(1, 0) \to U \to V^H(1, 0) \to 0.$$

**Proof.** By construction $U$ is a graded quotient of a $\mathbb{Z}_{\geq 0}$-graded $L^H(c_L, c_{L, I})$-module $\tilde{E}$. The lowest component is $U(0) \cong E$. Submodule $U(H)v^0$ is isomorphic to $L^H(1, 0)$, and the projective image of $v^1$ generates the Verma module $V^H(1, 0)$ since $I(0)v^1 = v^0$. For the same reason, this exact sequence does not split.

Now we consider $L^H(c_L, c_{L, I})$-module

$$\mathcal{V}^\text{ext} := L^H(c_L, c_{L, I}) \oplus U.$$

By using [16, Theorem 4.8.1] (see also [3, 17]) we have that $\mathcal{V}^\text{ext}$ has the structure of a vertex operator algebra with vertex operator map $Y_{\text{ext}}$ defined as follows:

$$Y_{\text{ext}}(a_1 + w_1, z)(a_2 + w_2) = Y(a_1, z)(a_2 + w_2) + e^{xL(-1)}Y(a_2, -z)w_1,$$

where $a_1, a_2 \in L^H(c_L, c_{L, I})$, $w_1, w_2 \in U$.

Take now $v^i \in E \subset U$, $i = 0, 1$ as above and define

$$S_i(z) = Y_{\text{ext}}(v^i, z) = \sum_{n \in \mathbb{Z}} S_i(n)z^{-n-1}.$$

By construction

$$S_1(z) \in \text{End} \left( L^H(c_L, c_{L, I}), L^H(1, 0) \right)(z).$$
Proposition 4.2. For all \( n, m \in \mathbb{Z} \) we have:

\[
[L(n), S_i(m)] = -m S_i(n + m), \quad i = 0, 1,
\]

\[
[W(n), S_0(m)] = 0, \quad [W(n), S_1(m)] = 2mc_{L,I}S_0(n + m).
\]

In particular, \( S_0(0) \) and \( S_1(0) \) are screening operators. Moreover,

\[
S_1 = S_1(0) : L^H(c_L, c_{L,I}) \to L^H(1, 0)
\]

is nontrivial and \( S_1(0)I(-1)1 = -v_0 \).

Proof. Since \( L(k)v^i = \delta_{k,0}v^i \) for \( k \geq 0 \), commutator formula gives that

\[
[L(n), S_i(m)] = -m S_i(n + m).
\]

Next we calculate \([W(n), S_1(m)]\). We have

\[
W(-1)v^1 = 2I(-1)v^0 = -2c_{L,I}L(-1)v^0,
\]

\[
W(0)v^1 = -2c_{L,I}v^0, \quad W(n)v^1 = 0, \quad n \geq 0.
\]

This implies that

\[
[W(n), S_1(m)] = 2c_{L,I}mS_0(n + m).
\]

Since \( W(n)v^0 = 0 \) for \( n \geq -1 \) we get

\[
[W(n), S_0(m)] = 0.
\]

Therefore we have proved that \( S_i(0), i = 0, 1 \) are screening operators. Next we have

\[
S_1(0)I(-1)1 = \text{Res}_z Y_{\text{ext}}(v^1, z)I(-1) = \text{Res}_z e^{zL(-1)}Y(I(-1)1, -z)v^1 = -v_0.
\]

The proof follows.

Theorem 4.3. \( S_1 \) is a derivation of the vertex algebra \( \mathcal{V}^{\text{ext}} \) and we have

\[
\text{Ker}_{L^H(c_L, c_{L,I})}S_1 \cong L^{W(2,2)}(c_L, c_W).
\]

Proof. By construction \( S_1 = \text{Res}_z Y_{\text{ext}}(v^1, z) \), so \( S_1 \) is a derivation so \( \overline{W} = \text{Ker}_{L^H(c_L, c_{L,I})}S_1 \) is a vertex subalgebra of \( L^H(c_L, c_{L,I}) \). Since

\[
S_1L(-2)1 = S_1W(-2)1 = 0
\]

we have that \( L^{W(2,2)}(c_L, c_W) \subset \overline{W} \). Since \( S_1I(-1)1 \neq 0 \), we have that \( I(-1)1 \) does not belong to \( \overline{W} \). By using the fact that \( L^H(c_L, c_{L,I}) \) is as \( W(2,2) \)-module generated by singular vector \( 1 \) and cosingular vector \( I(-1)1 \) (see Example 3.9) we get that \( \overline{W} = L^{W(2,2)}(c_L, c_W) \). The proof follows.

Remark 4.4. Of course, every \( \mathcal{V}^{\text{ext}} \)-module becomes a \( W(2,2) \)-module with screening operator \( S_1 \). Similar statement holds for intertwining operators. Constructions of such modules and intertwining operators require different techniques which we will present in our forthcoming paper [5].

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