The stability of sextic functional equation in fuzzy modular spaces

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Abstract

By using the fixed point technique, we prove the stability of sextic functional equations. Our results are proved in the framework of fuzzy modular spaces (briefly, $F,M$-space) lower semi continuous (briefly, l.s.c.) and $\beta$-homogeneous are necessary for this work.

Keywords: Stability, Sextic mapping, Fuzzy modular space

1. Introduction

In 1940 during a conference at Wisconsin University, S.M. Ulam [17] presented the following question concerning stability of group homomorphisms:

Let $G_1$ be a group and let $G_2$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $g : G_1 \to G_2$ with $d(f(x), g(x)) < \epsilon$ for all $x \in G_1$?

When the homomorphisms are stable? It is situation from the question such that we are interested. That is, if a mapping is almost a homomorphism, then there exists an exact homomorphism must be close.

In following year, Hyers [6] was the first to give an affirmative answer to Ulam’s question for the case where $G_1$ and $G_2$ are Banach spaces. After that, a generalized version of the theorem of Hyers for approximately linear mappings was given by Th.M. Rassias [13]. Later, the stability problems of various functional equation have been extensively investigated by a number of authors.

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One of the interesting functional equations studied is the system of additive-quadratic-cubic functional equations [5]:

\[
\begin{align*}
&f(ax_1 + bx_2, y, z) + f(ax_1 - bx_2, y, z) = 2 af(x_1, y, z), \\
&f(x, ay_1 + by_2, z) + f(x, ay_1 - by_2, z) = 2a^2 f(x, y_1, z) + 2b^2 f(x, y_2, z), \\
&f(x, y, az_1 + bz_2) + f(x, y, az_1 - bz_2) = \pm \sqrt{a^2 + b^2} f(x, y, z_1 + z_2) + f(x, y, z_1 - z_2)) + 2(a^2 - b^2) f(x, y, z_1)
\end{align*}
\]

where \(a, b \in \mathbb{Z} \setminus \{0\}\) with \(a \neq \pm 1, \pm b\).

The function \(f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) defined by \(f(x, y, z) = xyz^2 z^3\) is a solution of the system (1.1). In particular, letting \(y = z = x\), we get a sextic function \(h : \mathbb{R} \rightarrow \mathbb{R}\) in one variable given by \(h(x) := f(x, x, x) = cx^6\).

The concept of modular spaces was introduced by Nakano [11]. Soon after, the notation of modular spaces was redefined and generalized by Musielak and Orlicz [10]. In 2007, Nonnezi [12] presented probabilistic modular spaces related the theory of modular spaces.

After that, Shen and Chen [15] following the idea of probabilistic modular spaces and the definition of fuzzy metric spaces based on George and Veeramani’s sense [4], applied fuzzy concept to the classical notions of modular and modular spaces, and in 2013, Shen and Chen [15] presented the concept of a fuzzy modular space. After that, Kuman [8, 9], Wongkum and et al [19] studied fixed points and some properties in modular or fuzzy modular spaces.

In this paper, we investigate the generalized Ulam-Hyers-Rassies (briefly, UHR) stability of a sextic functional equations from linear spaces into \(\mathbb{F}\)-modules, by using some ideas of [3, 19].

2. Preliminaries

In this section, conventionally, we write throughout the paper \(\mathbb{R}, \mathbb{C},\) and \(\mathbb{N}\) to denote respectively the set of all reals, complexes, and nonnegative integers.

Moreover, we recall some basic definitions and properties of a fuzzy modular space.

Definition 2.1. (Vasuki [18]). A fuzzy set \(A\) in \(X\) is a function with domain \(X\) and value in \([0, 1]\).

Definition 2.2. (Arul Selvaraj and Sivakumar [2]). A triangular norm (briefly, \(t\)-norm) is a function \(\ast : [0, 1] \times [0, 1] \rightarrow [0, 1]\) satisfying, for each \(a, b, c, d \in [0, 1]\), the following conditions:

1. \(a \ast 1 = a;\)
2. \(a \ast b \leq c \ast d\) whenever \(a \leq c, b \leq d;\)
3. \(a \ast b = b \ast a;\) and \((a \ast b) \ast c = a \ast (b \ast c).\)

Definition 2.3. Let \(X\) be a vector space over a field \(\mathbb{K}\) (\(\mathbb{R}\) or \(\mathbb{C}\)). A generalized functional \(\rho : X \rightarrow [0, \infty]\) is called a modular if for arbitrary \(x, y \in X\),

\(\rho(x) = 0\) if and only if \(x = 0,\)

\(\rho(\alpha x) = \rho(x)\) for every scalar \(\alpha\) with \(|\alpha| = 1,\)
(m3) $\rho(z) \leq \rho(x) + \rho(y)$, whenever $z$ is a convex combination of $x$ and $y$. The corresponding modular space, denoted by $X_\rho$, is then defined by

$$X_\rho := \{ x \in X : \rho(\lambda x) \rightarrow 0 \text{ as } \lambda \rightarrow 0 \}.$$

**Remark 2.4.** Note that for a fixed $x \in X_\rho$, the valuation $\gamma \in \mathbb{R} \mapsto \rho(\gamma x)$ is increasing.

Unlike a norm, a modular needs not be continuous or convex in general. However, it is often occur that some weaker form of them are assumed.

**Remark 2.5.** In case a modular $\rho$ is convex, one has $\rho(x) \leq \delta \rho(\frac{x}{\delta})$ for all $x \in X_\rho$, provided that $0 < \delta \leq 1$.

**Definition 2.6.** Let $X_\rho$ be a modular space and $\{x_n\}$ be a sequence in $X_\rho$. Then,

(i) $\{x_n\}$ is $\rho$-convergent to a point $x \in X_\rho$ and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$.

(ii) $\{x_n\}$ is called $\rho$-Cauchy if for all $\epsilon > 0$, we have $\rho(x_n - x_m) < \epsilon$ for sufficiently large $m, n \in \mathbb{N}$.

(iii) A subset $K \subset X_\rho$ is called $\rho$-complete if any $\rho$-Cauchy sequence is $\rho$-convergent.

Another unnatural behavior one usually encounter is that the convergence of a sequence $\{x_n\}$ to $x$ does not imply that $\{e x_n\}$ converges to $e x$, where $e$ is chosen from the corresponding scalar field. Thus, many mathematicians imposed some additional conditions for a modular to meet in order to make the multiples of $\{x_n\}$ converge naturally. Such preferences are referred to mostly under the term related to the $\Delta_2$ conditions.

A modular $\rho$ is said to satisfy the $\Delta_2$ condition if there exists $\kappa \geq 2$ such that $\rho(2x) \leq \kappa \rho(x)$ for all $x \in X_\rho$. Some authors varied the notion so that only $\kappa > 0$ is required and called it the $\Delta_2$-type condition. In fact, one may see that these two notions coincide. There are still a number of equivalent notions related to the $\Delta_2$ conditions.

**Remark 2.7.** We have to be very careful about the convergence behaviors on multiples and sums of $\rho$-convergent sequences. In general, we suppose that $(x_{n1}), (x_{n2}), \cdots, (x_{ nk})$, for some $k \in \mathbb{N}$, are sequences in $X_\rho$, in which they $\rho$-converge to the points $x^1, x^2, \cdots, x^k \in X_\rho$, respectively. Then, the averaged sequence $\left\{ \frac{1}{k} \sum_{i=1}^{k} x_{ni} \right\}$ $\rho$-converges to $\frac{1}{k} \sum_{i=1}^{k} x^i$.

In [7], Khamsi proved a series of fixed point theorems in modular spaces where the modulars do not satisfy $\Delta_2$ conditions. His results exploit one unifying hypothesis in which the boundedness of an orbit is assumed.

**Definition 2.8.** Given a modular space $X_\rho$, a nonempty subset $C \subset X_\rho$, and a mapping $T : C \rightarrow C$, the orbit of $T$ around a point $x \in X_\rho$ is the set

$$\mathcal{O}(x) := \{ x, Tx, T^2x, \cdots \}.$$

The quantity $\delta_\rho(x) := \sup \{ \rho(u - v) : u, v \in \mathcal{O}(x) \}$ is then associated and is called the *orbital diameter* of $T$ at $x$. In particular, if $\delta_\rho(x) < \infty$, we say that $T$ has a bounded orbit at $x$. 
Lemma 2.9 (If). Let $X_\rho$ be a modular space whose the induced modular is l.s.c. and $C \subset X_\rho$ be a $\rho$-complete subset. If $T : C \to C$ is a $\rho$-contraction, i.e., there is a constant $k \in [0, 1)$ such that

$$\rho(Tx - Ty) \leq k \rho(x - y), \quad \forall x, y \in X_\rho,$$

and $T$ has a bounded orbit at a point $x_0 \in X_\rho$, then the sequence $\{T^n x_0\}$ is $\mu$-convergent to a point $w \in C$.

Definition 2.10. (Shen and Chen [15]). Let $V$ be a real or complex vector space with a zero $\theta$, $*$ a continuous triangular norm, and $\mu$ a fuzzy set on the product $V \times \mathbb{R}^+$. Suppose that the following properties hold for $x, y \in V$ and $s, t > 0$:

(FM1) $\mu(x, t) > 0$;

(FM2) $\mu(x, t) = 1$ for all $t > 0$ if and only if $x = \theta$;

(FM3) $\mu(x, t) = \mu(-x, t)$;

(FM4) $\mu(z, s + t) \geq \mu(x, s) \ast \mu(y, t)$ whenever $z$ is the convex combination between $x$ and $y$;

(FM5) the mapping $t \mapsto \mu(x, t)$ is continuous at each fixed $x \in V$.

Then, we write $(V, \mu, *)$ to represent the space with the pre-defined properties. In particular, we call $\mu$ a fuzzy modular and the triple $(V, \mu, *)$ a fuzzy modular space (briefly, $\mathcal{FM}$-space).

It is worth noting that every fuzzy modular is non-decreasing with respect to $t > 0$.

Example Let $X$ be a real or complex vector space and $\rho$ be a modular on $X$. Take the $t$-norm $a \ast b = \min\{a, b\}$. For every $t \in (0, \infty)$, define $\mu(x, t) = \frac{t}{t + \rho(x)}$ for all $x \in X$. Then $(X, \mu, *)$ is a $\mathcal{FM}$-space.

Remark Note that the above conclusion still holds even if the $t$-norm is replaced by $a \ast b = a \cdot b$ and $a \ast b = \max\{a + b - 1, 0\}$, respectively.

Definition 2.11. Let $(X, \mu)$ be a $\mathcal{FM}$-space, $\{x_n\}$ be a sequence in $X$ and $x \in X$.

1. The sequence $\{x_n\}$ with $x_n \in (X, \mu)$ is said to be $\mu$ - convergent to $x$ (write $x_n \xrightarrow{\mu} x$) if, for any $t > 0$ and $\lambda \in (0, 1)$, there exists a positive integer $n_0$ such that

$$\mu(x_n - x, t) > 1 - \lambda$$

for all $n \geq n_0$.

2. The sequence $\{x_n\}$ with $x_n \in (X, \mu)$ is called a $\mu$ - Cauchy sequence if, for any $t > 0$ and $\lambda \in (0, 1)$, there exists a positive integer $n_0$ such that

$$\mu(x_n - x_m, t) > 1 - \lambda$$

for all $n, m \geq n_0$. 
3. Every $\mu$-convergent sequence in $\mathcal{FM}$-space is $\mu$-Cauchy sequence. If each $\mu$-Cauchy sequence is $\mu$-convergent sequence in a $\mathcal{FM}$-space $(X, \mu)$, then $(X, \mu)$ is called a $\mu$-complete $\mathcal{FM}$-space.

Shen and Chen [15] also studied the topological properties of a fuzzy modular space with a special property that for every $x \in V$ and a non-zero real $\lambda$, the equality

$$
\mu(\lambda x, t) = \mu\left(x, \frac{t}{|\lambda|^2}\right)
$$

holds for some fixed $\beta \in (0, 1)$. If the fuzzy modular $\mu$ has this property, we shall say that it is $\beta$-homogeneous.

The $\mu$-ball in $(V, \mu, \ast)$ is the set of the form

$$
B(x, r, t) := \{y \in V : \mu(x - y, t) > 1 - r\},
$$

where $r \in (0, 1)$ and $t > 0$.

Now, suppose that $\mu$ is $\beta$-homogeneous for some $\beta \in (0, 1)$. According to Shen and Chen [15], the family $\mathcal{B}$ of all $\mu$-balls forms a base for a first-countable Hausdorff topology, written as $\mathcal{T}_\mu$. With the notion of the $\mu$-balls, it is easy to see that a sequence $(x_n)$ in $V$ $\mu$-converges (i.e., it converges in the topology $\mathcal{T}_\mu$) to its $\mu$-limit $x \in V$ if and only if $\mu(x_n - x, t) \to 1$ as $n \to \infty$ for all $t > 0$. Note here that the $\mu$-limit is unique if it does exist after all. It is then natural to say that $(x_n)$ is $\mu$-Cauchy if for any given $\varepsilon \in (0, 1)$ and $t > 0$, there exists $N \in \mathbb{N}$ with $\mu(x_m - x_n, t) > 1 - \varepsilon$ whenever $m, n > N$.

From here, let us turn to a typical example of a triangular norm which is defined by $(a \ast b) = \min\{a, b\}$. This triangular norm has a very special property that if $\ast'$ be an arbitrary triangular norm, then $(a \ast' b) \leq (a \ast b)$ for all $a, b \in [0, 1]$. With this property, it is suitable to call this $\ast$ a strongest triangular norm. As is claimed by Shen and Chen [15], if $V$ is a real vector space equipped with a $\beta$-homogeneous fuzzy modular $\mu$ and a strongest triangular norm $\ast$, then a $\mu$-convergent sequence is $\mu$-Cauchy. The authors also mentioned that if $\ast$ is not the strongest one, such implementation is not always true.

We say that $\mathcal{FM}$-space $(X, \mu, \ast)$ satisfy the lower semi continuous if, for any sequence $x_n$ of $X$ and $\mu$-converging to a point $x \in X$,

$$
\mu(x, t) \leq \liminf_{n \to \infty} \mu(x_n, t)
$$

for all $t > 0$.

Theorem 2.12. [7] Let $X_\rho$ be a modular space satisfying L.s.c. property. Let $C$ be a $\rho$-complete nonempty subset of $X_\rho$ and $T : C \to C$ be a quasi-contraction, that is, there exists $K < 1$ such that

$$
\rho(T(x) - T(y)) \leq K \max\{\rho(x - y), \rho(x - T(x)), \rho(y - T(y)), \rho(x - T(y)), \rho(y - T(x))\}.
$$

Let $X \in \mathcal{C}$ such that

$$
\delta_\rho(x) := \sup\{\rho(T^n(x) - T^m(x)) : m, n \in \mathbb{N}\} < \infty.
$$

Then $(T^n(x))$ $\rho$-converges to a point $w \in C$. Moreover, if $\rho(w - T(w)) < \infty$ and $\rho(x - T(w)) < \infty$, then the $\rho$-limit of $T^n(x)$ is a fixed point of $T$. Furthermore, if $w^*$ is any fixed point of $T$ in $C$ such that $\rho(w - w^*) < \infty$, then one has $w = w^*$.
3. Generalized UHR stability of sextic mappings

This section, assume that \( \mu \) is a fuzzy modular on \( V \) with the LSC. (in the fuzzy modular sense) and \((V, \mu, *)\) is a \( \mu \)-complete \( \beta \)-homogeneous \( F,M \)-space with \( \beta \in (0, 1] \) and \( * \) define by minimum t-norm. In this section, we establish the conditional UHR stability of sextic functional equations in a \( F,M \)-space.

**Theorem 3.1.** Let \( E \) be a linear space and \((V, \mu, *)\) be a \( \mu \)-complete \( \beta \)-homogeneous \( F,M \)-space and \( p \in \{-1, 1\} \) be fixed. Suppose that \( f : E \times E \times E \to (V, \mu, *) \) satisfies the condition \( f(x, 0, z) = 0 \) and the inequalities of the form:

\[
\begin{align*}
\mu(f(ax + bx, y, z) + f(ax - bx, y, z) - 2af(x, y, z), t) & \geq \tau(x, y, z, t), \\
\mu(f(ax_1 + by_1, z) + f(ax_1 - by_1, z) - 2abf(x_1, y_1, z) & - 2b^2f(x_1, y_1, z), t) \\
\geq \varsigma(x_1, y_1, z, t), \\
\mu(f(x, y, axz + bz) + f(x, y, axz - bz) - 2abf(x, y, z_1 + z_2) & - f(x, y, z_1 - z_2) - 2a^2b^2f(x, y, z_1), t) \\
\geq \upsilon(x, y, z_1, z_2, t),
\end{align*}
\]

where \( \tau, \varsigma, \upsilon : E^4 \to \Delta \), and \( \Delta \) is the set of all non-decreasing function, are given function such that:

\[
\begin{align*}
limit_{n \to \infty} \tau(a^n x_1, a^n x_2, a^n y, a^n z, a^{6\beta p_m}t) &= 1, \\
limit_{n \to \infty} \varsigma(a^n x, a^n y_1, a^n y_2, a^n z, a^{6\beta p_m}t) &= 1, \\
limit_{n \to \infty} \upsilon(a^n x, a^n y, a^n z_1, a^n z_2, a^{6\beta p_m}t) &= 1,
\end{align*}
\]

for all \( x, x_1, y, y_1, z_1, z_2 \in E, i = 1, 2 \). Assume that

\[
\Phi(x, y, z, t) = \upsilon(a^{\frac{n+1}{2}} x, a^{\frac{n+1}{2}} y, a^{\frac{n+1}{2}} z, 0, a^{(6-3p)\beta t/2^{\beta t+2}} \times \varsigma(a^{\frac{n+1}{2}} x, a^{\frac{n+1}{2}} y, 0, a^{\frac{n+1}{2}} z, a^{(6-3p)\beta t/2^{\beta t+2}}) \\
+ \tau(a^{\frac{n+1}{2}} x, 0, a^{\frac{n+1}{2}} y, a^{\frac{n+1}{2}} z, a^{(4-3p)\beta t/2^{\beta t+2}}) \times (3.4)
\]

has the property:

\[
\Phi(a^p x, a^p y, a^p z, a^{6\beta p}tL) \geq \Phi(x, y, z, t) \quad (3.5)
\]

for all \( x, y, z \in E \) with a constant \( 0 < L < \frac{1}{2\upsilon} \). Then there exists a unique sextic function \( s : E \times E \times E \to (V, \mu, *) \) satisfying the system (1.1) such that

\[
\mu(s(x, y, z) - f(x, y, z), \frac{2^\beta}{1 - 2^\beta L}t) \geq \Phi(x, y, z, t). \quad (3.6)
\]

**Proof.** Let \( x_1 = 2x \) and \( x_2 = 0 \) and replacing \( y, z \) by \( 2y, 2z \) in (3.1), respectively, we get

\[
\mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), t) \geq \tau(2x, 0, 2y, 2z, t) \quad (3.7)
\]
for all \(x, y, z \in E\).

Let \(y_1 = 2y\) and \(y_2 = 0\) and replacing \(x, z\) by \(2ax, 2z\) in (3.2), respectively, we have

\[
\mu(2f(2ax, 2ay, 2z) - 2a^2f(2ax, 2y, 2z), t) \geq \varsigma(2ax, 2y, 0, 2z, t) \tag{3.8}
\]

for all \(x, y, z \in E\).

Let \(z_1 = 2z\) and \(z_2 = 0\) and replacing \(x, y\) by \(2ax, 2ay\) in (3.3), respectively, we obtain

\[
\mu(2f(2ax, 2ay, 2az) - 2a^3f(2ax, 2ay, 2z), t) \geq \nu(2ax, 2ay, 2z, 0, t) \tag{3.9}
\]

for all \(x, y, z \in E\). Since \(\mu\) is \(\beta\)-homogeneous, it follows from (3.8), (3.9) and consider:

\[
\mu(2f(2ax, 2ay, 2z) - 2a^2f(2ax, 2y, 2z) + 2f(2ax, 2ay, 2az) - 2a^3f(2ax, 2ay, 2z), t).
\]

But, we know that

\[
\begin{align*}
\mu(2f(2ax, 2ay, 2az) - 2a^3f(2ax, 2ay, 2z), t) \\
\geq \mu\left(\frac{1}{a^2}(2f(2ax, 2ay, 2az) - 2a^3f(2ax, 2ay, 2z)), t\right).
\end{align*}
\]

Hence, we get

\[
\begin{align*}
\mu(2f(2ax, 2ay, 2z) - 2a^2f(2ax, 2y, 2z) \\
+ 2f(2ax, 2ay, 2az) - 2a^3f(2ax, 2ay, 2z), t) \\
\geq \mu(2f(2ax, 2ay, 2z) - 2a^2f(2ax, 2y, 2z) \\
+ 2a^3f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z), t) \\
= \mu(2a^3f(2ax, 2ay, 2az) - 2a^2f(2ax, 2y, 2z), t) \\
\geq \mu(a^{-3}f(2ax, 2ay, 2az) - a^2f(2ax, 2y, 2z), t) \\
= \mu\left(\frac{1}{2}a^{-3}f(2ax, 2ay, 2az) - \frac{1}{2}a^2f(2ax, 2y, 2z), t\right) \\
= \mu(2a^{-3}f(2ax, 2ay, 2az) - 2a^2f(2ax, 2y, 2z), 2^5t) \\
= \mu(2a^{-3}f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z) + 2f(2ax, 2ay, 2z) \\
- 2a^2f(2ax, 2ay, 2z), 2^5t) \\
= \mu(2a^{-3}f(2ax, 2ay, 2az) - f(2ax, 2ay, 2z) + f(2ax, 2ay, 2z) \\
- a^2f(2ax, 2ay, 2z), 2^5t) \\
= \mu((a^{-3}f(2ax, 2ay, 2az) - f(2ax, 2ay, 2z)) + (f(2ax, 2ay, 2z) \\
- a^2f(2ax, 2ay, 2z)), t) \\
= \mu(\frac{1}{2}(2a^{-3}f(2ax, 2ay, 2az) - f(2ax, 2ay, 2z)) + \frac{1}{2}(2f(2ax, 2ay, 2z) \\
- 2a^2f(2ax, 2ay, 2z)), t/2 + t/2) \\
\geq \mu(2a^{-3}f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z), t/2) \\
* \mu(2f(2ax, 2ay, 2z) - 2a^2f(2ax, 2y, 2z), t/2) \\
= \mu(2f(2ax, 2ay, 2az) - 2a^3f(2ax, 2ay, 2z), a^{3/2}t/2) \\
* \mu(2f(2ax, 2ay, 2z) - 2a^2f(2ax, 2y, 2z), t/2) \\
\geq \nu(2ax, 2ay, 2z, 0, a^{3t/2}) * \varsigma(2ax, 2y, 0, 2z, t/2)
\end{align*}
\]
and hence
\[
\mu(2a^{-3}f(2ax, 2ay, 2az) - 2a^2 f(2ax, 2y, 2z), t) \\
\geq \mu\left(\frac{1}{a^2}(2a^{-3}f(2ax, 2ay, 2az) - 2a^2 f(2ax, 2y, 2z)), t\right) \\
= \mu(2a^{-5}f(2ax, 2ay, 2az) - 2f(2ax, 2y, 2z), t) \\
= \mu\left((2a^{-5})\frac{a^2}{a^2}f(2ax, 2ay, 2az) - 2\frac{a^2}{a^2} f(2ax, 2y, 2z), t\right) \\
= \mu\left(\frac{1}{a^2}(2a^{-3}f(2ax, 2ay, 2az) - 2a^2 f(2ax, 2y, 2z)), t\right) \\
= \mu(2a^{-3}f(2ax, 2ay, 2az) - 2a^2 f(2ax, 2y, 2z), a^{2\beta}t) \\
= \mu(2a^{-3}f(2ax, 2ay, 2az) - 2f(2ax, 2y, 2z) + 2f(2ax, 2ay, 2z) \\
- 2a^2 f(2ax, 2y, 2z), a^{2\beta}t) \\
= \mu(2a^{-3}f(2ax, 2ay, 2az) - f(2ax, 2ay, 2z) + f(2ax, 2ay, 2z) \\
- a^2 f(2ax, 2y, 2z), a^{2\beta}t) \\
= \mu(a^{-3}f(2ax, 2ay, 2az) - f(2ax, 2ay, 2z) + f(2ax, 2ay, 2z) \\
- a^2 f(2ax, 2y, 2z), a^{2\beta}t/2^{\beta+1}) \\
= \mu\left(\frac{1}{2}(2a^{-3}f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z)) + \frac{1}{2}(2f(2ax, 2ay, 2z) \\
- 2a^2 f(2ax, 2y, 2z)) + a^{2\beta}t/2^{\beta+1} + a^{2\beta}t/2^{\beta+1}\right) \\
\geq \mu(2a^{-3}f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z), a^{2\beta}t/2^{\beta+1}) \\
* \mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z), a^{2\beta}t/2^{\beta+1}) \\
= \mu((2a^{-3})\frac{a^3}{a^3}f(2ax, 2ay, 2az) - 2\frac{a^3}{a^3} f(2ax, 2ay, 2z), a^{2\beta}t/2^{\beta+1}) \\
* \mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z), a^{2\beta}t/2^{\beta+1}) \\
= \mu(2f(2ax, 2ay, 2az) - 2a^3 f(2ax, 2ay, 2z), a^{2\beta}t/2^{\beta+1}) \\
* \mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z), a^{2\beta}t/2^{\beta+1}) \\
\geq \nu(2ax, 2ay, 2z, 0, a^{2\beta}t/2^{\beta+1})*bl(2ax, 2y, 0, 2z, a^{2\beta}t/2^{\beta+1})
for all \(x, y, z \in E\). By (3.7) and the last inequality, we get

\[
\begin{aligned}
\mu(a^{-5}f(2ax, 2ay, 2az) - af(2x, 2y, 2z), t) \\
= \mu(a^{-5}f(2ax, 2ay, 2az) - f(2ax, 2y, 2z) \\
+ f(2ax, 2y, 2z) - af(2x, 2y, 2z), t) \\
= \mu(\frac{1}{2}(2a^{-5}f(2ax, 2ay, 2az) - 2f(2ax, 2y, 2z)) \\
+ \frac{1}{2}(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), t/2 + t/2) \\
\geq \mu(2a^{-5}f(2ax, 2ay, 2az) - 2f(2ax, 2y, 2z), t/2) \\
\geq \mu(af(2x, 2y, 2z) - 2af(2x, 2y, 2z), t/2) \\
\geq \mu(a^{-5}f(2ax, 2ay, 2az) - f(2ax, 2y, 2z), t/2 + t/2) \\
\geq \mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), t/2) \\
\geq \mu(2a^{-5}f(2ax, 2ay, 2az) - 2a^{-2}f(2ax, 2ay, 2az)) \\
+ \frac{1}{2}(2a^{-2}f(2ax, 2ay, 2az) - 2f(2ax, 2y, 2z), t/2 + t/2) \\
\geq \mu(2a^{-5}f(2ax, 2ay, 2az) - 2a^{-2}f(2ax, 2ay, 2az), t/2^{3+2}) \\
+ \mu(2a^{-2}f(2ax, 2ay, 2az) - 2f(2ax, 2y, 2z), t/2^{3+2}) \\
+ \mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), t/2) \\
\geq \mu(2f(2ax, 2y, 2z) - 2a^{5/2}f(2ax, 2ay, 2az), t/2^{3+2}) \\
\geq \mu(2f(2ax, 2ay, 2z) - 2a^{2}f(2ax, 2y, 2z), t/2^{3+2}) \\
\geq \mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), t/2) \\
\geq \mu(2(2ax, 2ay, 2z, 0, a^{5/2}t/2^{3+2}) + \phi(2ax, 2ay, 2z, 0, a^{2}t/2^{3+2}) \\
\geq \phi(2x, 0, 2y, 2z, t/2).
\end{aligned}
\]
for all $x, y, z \in E$. Therefore, we get

\[
\begin{align*}
\mu(a^{-6} f(2ax, 2ay, 2az) - f(2x, 2y, 2z), t) \\
= \mu\left(\left(a^{-6}\right) \frac{a}{a} f(2ax, 2ay, 2az) - \frac{a}{a} f(2x, 2y, 2z), t\right) \\
= \mu\left(\frac{1}{a} (a^{-5} f(2ax, 2ay, 2az) - af(2x, 2y, 2z)), t\right) \\
= \mu(a^{-5} f(2ax, 2ay, 2az) - af(2x, 2y, 2z), a^2 t) \\
= \mu(a^{-5} f(2ax, 2ay, 2az) - f(2ax, 2y, 2z) \\
+ f(2ax, 2y, 2z) - af(2x, 2y, 2z), a^3 t) \\
= \mu\left(\frac{1}{2} (2a^{-5} f(2ax, 2ay, 2az) - 2f(2ax, 2y, 2z)) \\
+ \frac{1}{2} (2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z)), a^3 t/2 + a^2 t/2\right) \\
\geq \mu(2a^{-5} f(2ax, 2ay, 2az) - 2f(2ax, 2y, 2z), a^3 t/2) \\
* \mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), a^2 t/2) \\
= \mu(a^{-5} f(2ax, 2ay, 2az) - f(2ax, 2y, 2z), a^3 t/2^2 + 1) \\
* \mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), a^3 t/2) \\
= \mu\left(\frac{1}{2} (2a^{-5} f(2ax, 2ay, 2az) - 2a^{-2} f(2ax, 2ay, 2az)) \\
+ \frac{1}{2} (2a^{-2} f(2ax, 2ay, 2az) - 2f(2ax, 2y, 2z)), a^3 t/2^2 + 2 + a^2 t/2^3 + 2\right) \\
* \mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), a^2 t/2) \\
\geq \mu(2a^{-2} f(2ax, 2ay, 2az) - 2a^{-2} f(2ax, 2ay, 2az), a^3 t/2^3 + 2) \\
* \mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), a^2 t/2) \\
= \mu(2f(2ax, 2ay, 2az) - 2a^3 f(2ax, 2ay, 2az), a^6 t/2^3 + 2) \\
* \mu(2f(2ax, 2ay, 2az) - 2a^2 f(2ax, 2y, 2z), a^3 t/2^3 + 2) \\
* \mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), a^3 t/2) \\
\geq v(2ax, 2ay, 2z, 0, a^6 t/2^3 + 2) \ast \zeta(2ax, 2y, 0, 2z, a^3 t/2^3 + 2) \\
* \tau(2x, 0, 2y, 2z, a^3 t/2).
\end{align*}
\]
Replacing $x, y$ and $z$ by $\frac{2x}{3}, \frac{x}{2}$ and $\frac{x}{3}$ in the last inequality, respectively, we get

\[
\mu \left( \frac{f(ax, ay, az)}{a^6} - f(x, y, z), t \right) \geq \mu \left( \frac{1}{2} \left( 2f(ax, ay, az) - \frac{2f}{a} (ax, y, z) \right) + \frac{1}{2} \left( \frac{2f}{a} (ax, y, z) - 2f(x, y, z) \right), t/2 + t/2 \right) 
\]

\[
\geq \mu \left( \frac{2f(ax, ay, az)}{a^6} - \frac{2}{a^3} f(ax, ay, z) - 2f(x, y, z), t/2 \right) 
\]

\[
= \mu \left( \frac{1}{2} \left( 2 \cdot 2f(ax, ay, az) - 2 \cdot 2f(ax, ay, z) \right) + \frac{1}{2} \left( 2 \cdot 2f(ax, ay, z) - 2f(x, y, z) \right), t/2 + t/2 \cdot 2 \right) 
\]

\[
\geq \mu \left( \frac{2}{a} f(ax, ay, z) - 2f(x, y, z), t/2 \right) 
\]

\[
= \mu \left( \frac{2f(ax, ay, az)}{a^6} - \frac{2}{a^3} f(ax, ay, z), t/2 \cdot 2 \right) 
\]

\[
\geq \mu \left( 2f(ax, ay, az) - 2a^3 f(ax, ay, z), a^{6\beta/2} t/2^{\beta+2} \right)
\]

\[
+ \mu(2f(ax, ay, z) - 2a^3 f(ax, y, z), a^{6\beta/2} t/2^{\beta+2})
\]

\[
+ \mu(2f(ax, y, z) - 2a^3 f(x, y, z), a^{6\beta/2} t/2^{\beta+2})
\]

\[
+ \mu(2f(x, y, z) - 2a^3 f(x, y, z), a^{6\beta/2} t/2^{\beta+2})
\]

\[
\geq \nu(ax, ay, z, 0, a^{6\beta/2} t/2^{\beta+2}) \cdot \psi(ax, y, 0, z, a^{6\beta/2} t/2^{\beta+2})
\]

\[
+ \tau(x, 0, y, z, a^{6\beta/2} t/2^{\beta+2})
\]
for all $x, y, z \in E$. Replacing $x, y, z$ by $a^{-1}x, a^{-1}y, a^{-1}z$ in (3.10), we get

\[
\begin{align*}
\mu \left( \frac{f(x, y, z)}{a^6} - f(a^{-1}x, a^{-1}y, a^{-1}z), t \right) \\
\geq \mu \left( \frac{1}{a^6} \left( f(x, y, z) - f(a^{-1}x, a^{-1}y, a^{-1}z) \right), t \right) \\
= \mu \left( \frac{f(x, y, z)}{a^6} - f(a^{-1}x, a^{-1}y, a^{-1}z), a^6t \right) \\
= \mu \left( \frac{1}{2} \left( \frac{2f}{a^6}(x, y, z) - \frac{2f}{a}(x, a^{-1}y, a^{-1}z) \right) \\
+ \frac{1}{2} \left( \frac{2f}{a^3}(x, a^{-1}y, a^{-1}z) - 2f(a^{-1}x, a^{-1}y, a^{-1}z) \right), a^{6t/2} + a^{6t/2} \right) \\
\geq \mu \left( \frac{1}{a^6} \left( f(x, y, z) - \frac{2f}{a}(x, a^{-1}y, a^{-1}z) \right), a^{6t/2+1} \right) \\
\times \mu \left( \frac{f(x, a^{-1}y, a^{-1}z) - 2f(a^{-1}x, a^{-1}y, a^{-1}z), a^{6t/2}}{a^6} \right) \\
= \mu \left( \frac{1}{2} \left( \frac{2f}{a^6}(x, y, z) - \frac{2f}{a}(x, y, a^{-1}y) \right) \\
+ \frac{1}{2} \left( \frac{2f}{a^3}(x, y, a^{-1}y) - \frac{2f}{a}(x, a^{-1}y, a^{-1}z) \right), a^{6t/2+2} + a^{6t/2+2} \right) \\
\times \mu \left( \frac{2f(x, a^{-1}y, a^{-1}z) - 2f(a^{-1}x, a^{-1}y, a^{-1}z), a^{7t/2}}{a^6} \right) \\
\geq \mu \left( \frac{1}{a^6} \left( 2f(x, y, z) - 2a^3f(x, y, a^{-1}z) \right), a^{6t/2+2} \right) \\
\times \mu \left( \frac{1}{a^3} \left( 2f(x, y, a^{-1}z) - 2a^2f(x, a^{-1}y, a^{-1}z) \right), a^{6t/2+2} \right) \\
\times \mu \left( \frac{2f(x, a^{-1}y, a^{-1}z) - 2a^3f(x, a^{-1}y, a^{-1}z), a^{7t/2}}{a^6} \right) \\
\geq \mu \left( \frac{2f(x, y, z) - 2a^3f(x, y, a^{-1}z), a^{12t/2+2}}{a^6} \right) \\
\times \mu \left( \frac{2f(x, y, a^{-1}z) - 2a^2f(x, a^{-1}y, a^{-1}z), a^{9t/2+2}}{a^6} \right) \\
\times \mu \left( \frac{2f(x, a^{-1}y, a^{-1}z) - 2a^3f(x, a^{-1}y, a^{-1}z), a^{7t/2}}{a^6} \right) \\
\geq v(a^{-1}x, y, a^{-1}z, 0, a^{12t/2+2}) + \zeta(x, a^{-1}y, 0, a^{-1}z, a^{9t/2+2}) \\
\times \tau(a^{-1}x, 0, a^{-1}y, a^{-1}z, a^{7t/2})
\end{align*}
\]

but, we know that

\[
\begin{align*}
\mu \left( \frac{1}{a^6} \left( f(a^{-1}x, a^{-1}y, a^{-1}z) - f(x, y, z) \right), t \right) \\
\geq \mu \left( \frac{1}{a^6} \left( f(a^{-1}x, a^{-1}y, a^{-1}z) - f(x, y, z) \right), t \right)
\end{align*}
\]
therefore
\[
\mu\left(\frac{f(a^{-1}x, a^{-1}y, a^{-1}z)}{a^{-6}} - f(x, y, z), t\right) \\
\geq \nu(a^{-1}x, y, a^{-1}z, 0, a^{12\beta}\tau/a^{6\beta+2}) + \tau(a^{-1}x, 0, a^{-1}y, a^{-1}z, a^{7\beta}t/2)
\]
and so
\[
\mu\left(\frac{f(a^p x, a^p y, a^p z)}{a^{6p}} - f(x, y, z), t\right) \geq \Phi(x, y, z, t). \tag{3.11}
\]

Now, we consider the set
\[\mathcal{D} = \{ h : E \times E \times E \to V : h(x, 0, z) = 0 \text{ for all } x, z \in E \}\]
and introduce the modular \(\rho\) on \(\mathcal{D}\) as follows:
\[\rho(h) = \inf\{ c > 0 : \mu(h(x, y, z), ct) \geq \Phi(x, y, z, t) \}\].

We know that \(\rho\) is even from \(\rho(-h) = \rho(h)\) and \(\rho(0) = 0\). If \(\rho(h) = 0\), then, for each \(c > 0\),
\[\rho(h(x, y, z), ct) \geq \Phi(x, y, z, t)\]
for all \(t > 0\) and \(x, y \in E\). Now, if \(c = ct\) is fixed and \(t \to +\infty\), then
\[\rho(h(x, y, z), c) = 0\], which implies that \(h = 0\). It is sufficient to show that \(\rho\)
satisfies the following condition:
\[\rho(\alpha g + \beta h) \leq \rho(g) + \rho(h)\]
if \(\alpha + \beta = 1\) for all \(\alpha, \beta \geq 0\). Let \(\epsilon > 0\) be given. Then there exist \(c_1 > 0\) and \(c_2 > 0\) such that
\[c_1 \leq \rho(g) + \epsilon, \quad \mu(g(x, y, z), c_1 t) \geq \Phi(x, y, z, t)\]
and
\[c_2 \leq \rho(h) + \epsilon, \quad \mu(h(x, y, z), c_2 t) \geq \Phi(x, y, z, t)\].

If \(\alpha + \beta = 1\) for all \(\alpha, \beta \geq 0\), then we get
\[
\mu(\alpha g(x, y, z) + \beta h(x, y, z), c_1 t + c_2 t) \geq \mu(g(x, y, z), c_1 t) \ast \mu(h(x, y, z), c_2 t) \\
\geq \Phi(x, y, z, t)
\]
and
\[\rho(\alpha g + \beta h) \leq c_1 + c_2 \leq \rho(g) + \rho(h) + 2\epsilon\]
thus
\[\rho(\alpha g + \beta h) \leq \rho(g) + \rho(h)\].

Now, we show that \(\rho\) has the \(\Delta_2\)-condition, where \(\kappa = 2^3\). For all \(\epsilon > 0\), there exists \(c > 0\) such that
\[c \leq \rho(h) + \epsilon, \quad \mu(h(x, y, z), ct) \geq \Phi(x, y, z, t)\].
Since \((V, \mu, \ast)\) is a \(\beta\)-homogeneous \(F,M\)-space, we get
\[
\mu(2\rho(x,y,z), 2^n c t) = \mu(h(x,y,z), ct) \geq \Phi(x,y,z, t),
\]
where \(\rho(2^n) \leq 2^n c \leq 2^n \rho(h) + 2^n \epsilon\). Thus \(\mu\) satisfies the \(\Delta_2\)-condition with \(\kappa = 2^n\).

Moreover, \(\rho\) satisfies the \(\text{Lsc}_\ast\) (in the modular sense). Indeed, if the sequence \(\{h_n\}\) in \(D\) is \(\rho\)-convergent to \(h\), then we can easily see that \(h_n(x,y,z)\) is \(\rho\)-convergent to \(h(x,y,z)\) for all \(x,y,z \in E\).

Let \(\rho := \lim \inf_{n \to \infty} \rho(h_n) < \infty\) and \(\rho(h) > \rho\). Then, we have
\[
\mu(h(x,y,z), pt) < \Phi(x,y,z, t)
\]
for all \(t > 0\). Since \(\mu\) satisfies the \(\text{Lsc}_\ast\) (in the fuzzy modular sense), we have
\[
\lim \sup_{n \to \infty} \mu(h_n(x,y,z), pt) \leq \mu(h(x,y,z), pt) < \Phi(x,y,z, t).
\]
From the last inequality, we know that there exists a positive integer \(n_0 \in \mathbb{N}\) such that
\[
\mu(h_n(x,y,z), pt) < \Phi(x,y,z, t)
\]
and so \(\rho(h_n) > \rho\) for all \(n \geq n_0\). Thus \(\lim \inf \rho(h_n) > \rho\) where \(n \to \infty\), which is a contradiction. Therefore, \(\rho\) satisfies the \(\text{Lsc}_\ast\).

If \(\delta > 0\) and \(\lambda \in (0,1)\) are given, it follows from \(\Phi(x,y,z) \in \Delta\) that there exists \(t_0 > 0\) such that \(\Phi(x,y,z, t_0) > 1 - \lambda\). Let \(\{h_n\}\) be a \(\rho\)-Cauchy sequence in \(D_\rho\) and let \(\epsilon < \frac{1}{\rho h_n - h_m}\) be given. Then there exists a positive integer \(n_0 \in \mathbb{N}\) such that \(\rho(h_n - h_m) \leq \epsilon\) for all \(n,m \geq n_0\).

Now, by considering the definition of the modular \(\rho\), we see that
\[
\mu(h_n(x,y,z) - h_m(x,y,z), \delta) \geq \mu(h_n(x,y,z) - h_m(x,y,z), \epsilon t_0) \geq \Phi(x,y,z, t_0) \geq 1 - \lambda
\]
for all \(x,y,z \in E\) and \(n,m \geq n_0\).

If \(x,y\), and \(z\) are arbitrary given points of \(E\), then (3.12) implies that \(\{h_n(x,y,z)\}\) is a \(\rho\)-Cauchy sequence in \((V, \mu, \ast)\). Since \(V\) is \(\mu\)-complete, it follows that \(\{h_n(x,y,z)\}\) is \(\mu\)-convergent in \((V, \mu, \ast)\) for all \(x,y,z \in E\). Thus, we can define
\[
h(x,y,z) = \lim_{n \to \infty} h_n(x,y,z),
\]
where a function \(h : E \times E \times E \to (V, \mu, \ast)\) for all \(x,y,z \in E\). Moreover, \(\mu\) has the \(\text{Lsc}_\ast\). Then, we have
\[
\rho(h_n - h) \leq \epsilon
\]
for all \(n \geq n_0\). Thus \(\{h_n\}\) is a \(\rho\)-convergent sequence in \(D_\rho\). Therefore, \(D_\rho\) is \(\rho\)-complete. Now, we consider the function \(T : D_\rho \to D_\rho\) defined by
\[
T h(x,y,z) := a^{-\rho h(a^p x, a^p y, a^p z)}
\]
for all \( h \in \mathcal{D}_\rho \). Let \( g, h \in \mathcal{D}_\rho \) and \( c \in [0, \infty) \) be an arbitrary constant with \( \rho(g - h) \leq c \). From the definition of \( \rho \), we have

\[
\mu(g(x, y, z) - h(x, y, z), ct) \geq \Phi(x, y, z, t)
\]

for all \( x, y, z \in E \). By the assumption and the last inequality, we get

\[
\mu(Tg(x, y, z) - Th(x, y, z), Lct) = \mu(a^{-6p}g(a^p x, a^p y, a^p z) - a^{-6p}h(a^p x, a^p y, a^p z), Lct)
\]

\[
= \mu(g(a^p x, a^p y, a^p z) - h(a^p x, a^p y, a^p z), a^{6p}Lct)
\]

\[
\geq \Phi(a^p x, a^p y, a^p z, a^{6p}Lt)
\]

\[
\geq \Phi(x, y, z, t)
\]

for all \( x, y, z \in E \) and so \( \rho(Tg - Th) \leq L\rho(g - h) \) for all \( g, h \in \mathcal{D}_\rho \), that is, \( T \) is a \( \rho \)-contraction.

Now, we show that the \( \rho \)-strict mapping \( T \) satisfies the conditions of Theorem (2.12). Observe that

\[
\mu(a^{-6p} f(a^{2p} x, a^{2p} y, a^{2p} z) - f(a^p x, a^p y, a^p z), t) \geq \Phi(a^p x, a^p y, a^p z, t)
\]

and so

\[
\mu(a^{-2(6p)} f(a^{2p} x, a^{2p} y, a^{2p} z) - a^{-6p} f(a^p x, a^p y, a^p z), Lt)
\]

\[
= \mu(a^{-6p} f(a^{2p} x, a^{2p} y, a^{2p} z) - f(a^p x, a^p y, a^p z), a^{6p}Lt)
\]

\[
\geq \Phi(a^p x, a^p y, a^p z, a^{6p}Lt)
\]

\[
\geq \Phi(x, y, z, t)
\]

Thus, we get

\[
\mu\left(\frac{f(a^{2p} x, a^{2p} y, a^{2p} z)}{a^{2(6p)}} - f(x, y, z), 2^\beta(Lt + t)\right)
\]

\[
\geq \mu\left(\frac{f(a^{2p} x, a^{2p} y, a^{2p} z)}{a^{2(6p)}} - f(a^p x, a^p y, a^p z), Lt\right)
\]

\[
\ast\mu\left(\frac{f(a^{2p} x, a^{2p} y, a^{2p} z)}{a^{6p}} - f(x, y, z), t\right)
\]

\[
\geq \Phi(x, y, z)(t)
\]

for all \( x, y, z \in E \). By replacing \( x, y, z \) by \( a^p x, a^p y \) and \( a^p z \) in (3.13), respectively, we get

\[
\mu(a^{-2(6p)} f(a^{3p} x, a^{3p} y, a^{3p} z) - f(a^p x, a^p y, a^p z), a^{6p}2^\beta(L^2 t + Lt))
\]

\[
\geq \Phi(a^p x, a^p y, a^p z, a^{6p}Lt)
\]

\[
\geq \Phi(x, y, z, t)
\]

and so

\[
\mu(a^{-3(6p)} f(a^{3p} x, a^{3p} y, a^{3p} z) - a^{-6p} f(a^p x, a^p y, a^p z), 2^\beta(L^2 t + Lt)) \geq \Phi(x, y, z, t).
\]
Therefore, we get
\[
\begin{align*}
\mu\left( \frac{f(a^{3p}x, a^{3p}y, a^{3p}z)}{a_n^{6np}} - f(x, y, z), 2^\beta (L^2(t + L) + t) \right) \\
\geq \mu\left( \frac{f(a^{3p}x, a^{3p}y, a^{3p}z)}{a_n^{6np}} - \frac{f(a^{np}x, a^{np}y, a^{np}z)}{a_n^{6np}}, 2^\beta (L^2(t + L)) \right) \\
\cdot \mu\left( \frac{f(a^{np}x, a^{np}y, a^{np}z)}{a_n^{6np}} - f(x, y, z), t \right) \\
\geq \Phi(x, y, z, t)
\end{align*}
\]
for all \( x, y, z \in E \). By induction, we can easily see that
\[
\mu\left( \frac{f(a^{np}x, a^{np}y, a^{np}z)}{a_n^{6np}} - f(x, y, z), \left\{ (2^\beta L)^{n-1} + 2^\beta \sum_{i=1}^{n-1} (2^\beta L)^{i-1} \right\} t \right) \geq \Phi(x, y, z, t)
\]
for all \( x, y, z \in E \) and so
\[
\rho(T^n f - f) \leq (2^\beta L)^{n-1} + 2^\beta \sum_{i=1}^{n-1} (2^\beta L)^{i-1} \leq 2^\beta \sum_{i=1}^{n} (2^\beta L)^{i-1} \leq \frac{2^\beta}{1 - 2^\beta L} L^{n}, \quad (3.14)
\]
Next, we confirm that \( \delta_\rho(f) = \sup \{ \rho(T^n(f) - T^m(f)) : n, m \in \mathbb{N} \} < \infty \). Since \( \rho \) satisfies the \( \Delta_2 \)-condition with \( \kappa = 2^\beta \), it follows from the inequality (3.14) that
\[
\rho(T^n f - T^m f) \leq \frac{1}{2} \rho(2T^n f - 2f) + \frac{1}{2} \rho(2T^m f - 2f) \\
\leq \frac{\kappa}{2} \rho(T^n f - f) + \frac{\kappa}{2} \rho(T^m f - f) \\
\leq \frac{2^\beta}{1 - 2^\beta L}
\]
for all \( n, m \in \mathbb{N} \). By the definition of \( \delta_\rho(f) \), we have \( \delta_\rho(f) < \infty \). Thus Theorem (2.12) shows that \( \{T^n(f)\} \) is \( \rho \)-convergent to a point \( s \in D_\rho \). Since \( \rho \) has the lsc., the inequality (3.14) gives \( \rho(T(s) - f) < \infty \).

If we replace \( m \) by \( n+1 \) in the inequality (3.15), then we obtain
\[
\rho(T^{n+1} f - T^n f) \leq \frac{2^\beta}{1 - 2^\beta L}.
\]
Therefore, we get \( \rho(T(s) - s) \leq \frac{2^\beta}{1 - 2^\beta L} < \infty \). Therefore, it follows from Theorem (2.1) that \( \rho \)-limit of \( \{T^n(f)\}, s \in D_\rho \), is a fixed point of the mapping \( T \).

If we replace \( x_1, x_2, y \) and \( z \) by \( a^{np}x_1, a^{np}x_2, a^{np}y \) and \( a^{np}z \) in the inequality (3.1), respectively, then we obtain
\[
\begin{align*}
\mu\left( \frac{f(a^{np}(ax_1 + bx_2), a^{np}y, a^{np}z)}{a_n^{6np}} + \frac{f(a^{np}(ax_1 - bx_2), a^{np}y, a^{np}z)}{a_n^{6np}} \right) \\
- 2a_f \frac{f(a^{np}(ax_1, a^{np}y, a^{np}z))}{a_n^{6np}}, t \right) \\
= \mu\left( f(a^{np}(ax_1 + bx_2), a^{np}y, a^{np}z) + f(a^{np}(ax_1 - bx_2), a^{np}y, a^{np}z) \\
- 2a_f \frac{f(a^{np}(ax_1, a^{np}y, a^{np}z))}{a_n^{6np}}, t \right) \\
\geq \tau(a^{np}x_1, a^{np}x_2, a^{np}y, a^{np}z, a_n^{6np} t).
\end{align*}
\]
Similarly, by replacing \( x, y_1, y_2 \) and \( z \) by \( a^{np}x, a^{np}y_1, a^{np}y_2 \) and \( a^{np}z \) in the inequality (3.2), respectively, we get
\[
\mu \left( \frac{f(a^{np}x, a^{np}(ay_1 + by_2), a^{np}z)}{a^{6np}} + \frac{f(a^{np}x, a^{np}(ay_1 - by_2), a^{np}z)}{a^{6np}} - 2a^2 f(a^{np}x, a^{np}y_1, a^{np}z) - 2b^2 f(a^{np}x, a^{np}y_2, a^{np}z) + \frac{a^{b\mu}}{a^{6np}} \right) \\
\geq \zeta(a^{np}x, a^{np}y_1, a^{np}y_2, a^{np}z), a^{6b\mu + t})
\]
and, also by replacing \( x, y, z_1 \) and \( z_2 \) by \( a^{np}x, a^{np}y, a^{np}z_1 \) and \( a^{np}z_2 \) in the inequality (3.3), respectively, we get
\[
\mu \left( \frac{f(a^{np}x, a^{np}y, a^{np}(az_1 + bz_2))}{a^{6np}} + \frac{f(a^{np}x, a^{np}y, a^{np}(az_1 - bz_2))}{a^{6np}} - ab_2 f(a^{np}x, a^{np}y, a^{np}(z_1 + z_2)) + \frac{a^{b\mu}}{a^{6np}} \right) \\
\geq v(a^{np}x, a^{np}y, a^{np}z_1, a^{np}z_2), a^{6b\mu + t})
\]
for all \( x, x_i, y, y_i, z, z_i \in E, i = 1, 2 \). Taking \( n \to \infty \) in the inequalities (3.16), (3.17) and (3.18), we deduce that \( s \) satisfies the system (1.1), that is, \( s \) is sextic. It follows from the inequality (3.14) that
\[
\rho(s - f) \leq \frac{2^b}{1 - 2^b L}.
\]
Hence (3.5) holds. If \( s^* \) is another fixed point of \( T \), then we get
\[
\rho(s - s^*) \leq \frac{1}{2^b}(2\rho(T(s) - s) + \frac{1}{2^b}(2\rho(T(s^*) - s)) \\
\leq \frac{b}{2^b}(\rho(T(s) - f) + \frac{b}{2^b}(\rho(T(s^*) - f) \\
\leq \frac{2^b}{1 - 2^b L} < \infty.
\]
Since \( T \) is \( \rho \)-contraction, we get
\[
\rho(s - s^*) = \rho(T(s) - T(s^*)) \\
\leq L\rho(s - s^*),
\]
which implies that \( \rho(s - s^*) = 0 \) or \( s = s^* \). Since \( \rho(s - s^*) < \infty \), which proves the uniqueness of \( s \). This completes the proof.

Concluding remarks

Our results guarantee the generalized UHR stability of sextic mappings, whose codomain is equipped with a \( \beta \)-homogeneous and l.s.c. modular.
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