ON LEMPERT FUNCTIONS IN $C^2$

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Abstract. We give a characterization of all cartesian products $D_1 \times D_2 \subset C^2$ for which the Lempert function and the injective Lempert function coincide. In particular, we show that there exist domains in $C^2$ for which they are different.

1. Introduction. The main result of this paper is very similar to the one presented in [2], which concerns equality between the Kobayashi-Royden and Hahn pseudometrics for product domains in $C^2$. The ideas and techniques used here are mostly the same; therefore, only essentially different parts are presented.

For a domain $D \subset C^n$, the Lempert function $L$ and the injective Lempert function $H$ are defined by the formulae:

\[ L_D(z_1, z_2) := \inf \{ p(\lambda_1, \lambda_2) : \exists f \in O(E, D) f(\lambda_1) = z_1, f(\lambda_2) = z_2 \}, \quad z_1, z_2 \in D, \]
\[ H_D(z_1, z_2) := \inf \{ p(\lambda_1, \lambda_2) : \exists f \in O(E, D) f(\lambda_1) = z_1, f(\lambda_2) = z_2, f \text{ is injective} \}, \quad z_1, z_2 \in D, \]

where $E$ denotes the unit disc and $p$ denotes the Poincaré distance (cf. [1]). Put $H_D(z, z) := 0$. Obviously, $L \leq H$. It is known that both functions are invariant under biholomorphic mappings, i.e., if $f : D \rightarrow \tilde{D}$ is biholomorphic, then

\[ H_D(z_1, z_2) = H_{\tilde{D}}(f(z_1), f(z_2)), \quad L_D(z_1, z_2) = L_{\tilde{D}}(f(z_1), f(z_2)), \quad z_1, z_2 \in D. \]

It is also known that $H_C \equiv L_C \equiv 0$ and that for a hyperbolic (in the sense of the uniformization theorem) domain $D \subset C$ and for any $z_1, z_2 \in D$, $z_1 \neq z_2$ we have $H_D(z_1, z_2) \equiv L_D(z_1, z_2)$ iff $D$ is simply connected. Using methods similar to [3], one can prove that $H_D \equiv L_D$ for any domain $D \subset C^n, n \geq 3$.

\[ \text{1Observe that for any } z_1, z_2 \in D, \ z_1 \neq z_2, \text{ there exists an injective holomorphic disc } f : E \rightarrow D \text{ such that } z_1, z_2 \in f(E). \text{ Indeed, first we take an injective } C^1 \text{-curve } \alpha : [0, 1] \rightarrow D \text{ with } \alpha(0) = z_1, \alpha(1) = z_2, \text{ and } \alpha'(t) \neq 0 \text{ for all } t \in [0, 1]. \text{ Next, we take a } C^1 \text{-approximation of } \alpha \text{ by a polynomial mapping } P \text{ with } P(0) = z_1 \text{ and } P(1) = z_2; \text{ } P \text{ has to be injective when close enough to } \alpha. \text{ Finally, we proceed as in Remark 3.1.1 in [1].} \]
Let $D_1, D_2 \subset \mathbb{C}$. The aim of this paper is to show that $H_{D_1 \times D_2} \equiv L_{D_1 \times D_2}$ iff at least one of $D_1, D_2$ is simply connected or biholomorphic to $\mathbb{C}_\star$. In particular, there are domains $D \subset \mathbb{C}^2$ for which $H_D \not\equiv L_D$.

2. The main result.

Theorem 1. Let $D_1, D_2 \subset \mathbb{C}$ be domains. Then:
1. If at least one of $D_1, D_2$ is simply connected, then $H_{D_1 \times D_2} \equiv L_{D_1 \times D_2}$.
2. If at least one of $D_1, D_2$ is biholomorphic to $\mathbb{C}_\star$, then $H_{D_1 \times D_2} \equiv L_{D_1 \times D_2}$.
3. Otherwise, $H_{D_1 \times D_2} \not\equiv L_{D_1 \times D_2}$.

Let $p_j : D_j^1 \rightarrow D_j$ be a holomorphic universal covering of $D_j$ ($D_j^1 \in \{\mathbb{C}, E\}$), $j = 1, 2$. Recall that if $D_j$ is simply connected, then $H_{D_j} \equiv L_{D_j}$. If $D_j$ is not simply connected and $D_j$ is not biholomorphic to $\mathbb{C}_\star$, then, by the uniformization theorem, $D_j^1 = E$ and $p_j$ is not injective.

Hence, Theorem 1 is an immediate consequence of the following three propositions (we keep the above notation).

Proposition 2. If $H_{D_1} \equiv L_{D_1}$, then $H_{D_1 \times D_2} \equiv L_{D_1 \times D_2}$ for any domain $D_2 \subset \mathbb{C}$.

Proposition 3. If $D_1$ is biholomorphic to $\mathbb{C}_\star$, then $H_{D_1 \times D_2} \equiv L_{D_1 \times D_2}$ for any domain $D_2 \subset \mathbb{C}$.

Proposition 4. If $D_j^* = E$ and $p_j$ is not injective, $j = 1, 2$, then $H_{D_1 \times D_2} \not\equiv L_{D_1 \times D_2}$.

Observe that for any domain $D \subset \mathbb{C}^n$ we have: $H_D \equiv L_D$ iff for any $f \in \mathcal{O}(E, D)$, $0 < \alpha < \vartheta < 1$ with $f(0) \neq f(\alpha)$, there exists an injective $g \in \mathcal{O}(E, D)$ such that $g(0) = f(0)$ and $g(\vartheta) = f(\alpha)$. 

Proof of Proposition 2. Let $f = (f_1, f_2) \in \mathcal{O}(E, D_1 \times D_2)$, $0 < \alpha < \vartheta < 1$, and $f(0) \neq f(\alpha)$.

First, consider the case where $f_1(0) \neq f_1(\alpha)$. By $(\ast)$, there exists an injective function $g_1 \in \mathcal{O}(E, D_1)$ such that $g_1(0) = f_1(0)$ and $g_1(\vartheta) = f_1(\alpha)$. Put $g(z) := (g_1(z), f_2(\frac{\alpha}{\vartheta} z))$.

Obviously, $g \in \mathcal{O}(E, D_1 \times D_2)$ and $g$ is injective. Moreover, $g(0) = f(0)$ and $g(\vartheta) = (g_1(\vartheta), f_2(\alpha)) = (f_1(\alpha), f_2(\alpha)) = f(\alpha)$.

Suppose now that $f_1(0) = f_1(\alpha)$. Take $0 < d < \text{dist}(f(0), \partial D_1)$ \footnote{dist($z_0, A$) := $\inf\{|z - z_0| : z \in A\}$, where $\| \cdot \|$ is the Euclidean norm; dist($z_0, \emptyset$) := $+ \infty$.} and put

$$h(z) := \frac{f_2(z \frac{\alpha}{\vartheta}) - f_2(0)}{f_2(\alpha) - f_2(0)}, \quad M := \max\{|h(z)| : z \in \overline{E}\},$$

$$g_1(z) := f_1(0) + \frac{d}{M + \frac{\alpha}{\vartheta}} (h(z) - \frac{z}{\vartheta}), \quad g(z) := \left( g_1(z), f_2 \left( \frac{\alpha}{\vartheta} z \right) \right), \quad z \in E.$$
Obviously, $g \in \mathcal{O}(E, \mathbb{C} \times D_2)$. Since $|g_1(z) - f_1(0)| < d$, we get $g_1(z) \in B(f_1(0), d) \subset D_1$ for $z \in E$. Hence $g \in \mathcal{O}(E, D_1 \times D_2)$. Take $z_1, z_2 \in E$ such that $g(z_1) = g(z_2)$. Then $h(z_1) = h(z_2)$, and consequently $z_1 = z_2$.

Finally $g(0) = (g_1(0), f_2(0)) = (f_1(0) + \frac{d}{M+\vartheta} h(0), f_2(0)) = f(0)$ and $g(\vartheta) = (g_1(\vartheta), f_2(\alpha)) = (f_1(0) + \frac{d}{M+\vartheta} (h(\vartheta) - 1), f_2(\alpha)) = (f_1(0), f_2(\alpha)) = f(\alpha)$. \hfill \(

\textbf{Proof of Proposition 3} \text{ \footnote{\begin{footnotesize} \textnormal{B}(z_0, r) := \{z \in \mathbb{C}^n : \|z - z_0\| < r\}.\end{footnotesize}}}

We may assume that $D_1 = \mathbb{C}_*$ and $D_2 \neq \mathbb{C}$. Using (*), let $f = (f_1, f_2) \in \mathcal{O}(E, \mathbb{C}_* \times D_2)$, $0 < \alpha < \vartheta < 1$, and $f(0) \neq f(\alpha)$. Applying an appropriate automorphism of $\mathbb{C}_*$, we may assume that $f_1(0) = 1$.

For the case where $f_2(0) = f_2(\alpha)$, we apply the above construction to the domains $\widetilde{D}_1 = f_2(0) + \text{dist}(f_2(0), \partial D_2)E$, $\widetilde{D}_2 = \mathbb{C}_*$ and mappings $\widetilde{f}_1 \equiv f_2(0)$, $\widetilde{f}_2 = f_1$.

Now, consider the case where $f_2(0) \neq f_2(\alpha)$ and $f_1(\alpha) = 1 + \vartheta$. We put

$$g_1(z) := 1 + z, \quad g(z) := (g_1(z), f_2\left(\frac{\alpha}{\vartheta}z\right)), \quad z \in E.$$ 

Obviously, $g \in \mathcal{O}(E, \mathbb{C}_* \times D_2)$ and $g$ is injective. We have $g(0) = (f_2(0)) = f(0)$ and $g(\vartheta) = (1 + \vartheta, f_2(\alpha)) = f(\alpha)$.

In all other cases, define a sequence $(d_k)$ such that we have

$$d_k = \frac{f_1(\alpha)}{1 + \vartheta}, \quad k \in \mathbb{N},$$

$$\text{Arg}(d_k) \to 0.$$ 

Observe that $d_k \to 1$. Let $M := \max\{|f_2(z)| : |z| \leq \frac{\alpha}{\vartheta}\}$. Take a $k \in \mathbb{N}$ such that $|c_k| > M$, where

$$c_k := \frac{f_2(\alpha) - d_k f_2(0)}{1 - d_k}.$$ 

Put

$$h(z) := \frac{f_2\left(\frac{\alpha}{\vartheta}z\right) - c_k}{f_2(0) - c_k},$$

$$g_1(z) := (1 + z)h^k(z), \quad g_2(z) := f_2\left(\frac{\alpha}{\vartheta}z\right), \quad g(z) := (g_1(z), g_2(z)), \quad z \in E.$$ 

Obviously, $g \in \mathcal{O}(E, \mathbb{C} \times D_2)$. Since $h(z) \neq 0$, we have $g_1(z) \neq 0, z \in E$. Hence $g \in \mathcal{O}(E, \mathbb{C}_* \times D_2)$. Take $z_1, z_2 \in E$ such that $g(z_1) = g(z_2)$. Then $h(z_1) = h(z_2)$, and consequently $z_1 = z_2$.

Finally, $g(0) = (h^k(0), f_2(0)) = f(0)$ and
Proof of Proposition 1. One can show (see [2]) that there exist \( \varphi_1, \varphi_2 \in \text{Aut}(E) \) and a point \( q = (q_1, q_2) \in E^2 \), \( q_1 \neq q_2 \), such that \( p_j(\varphi_j(q_1)) = p_j(\varphi_j(q_2)), j = 1, 2 \), and \( \det[(p_j \circ \varphi_j)'(q_k)]_{j,k=1,2} \neq 0 \). Put \( \tilde{p}_j := p_j \circ \varphi_j, j = 1, 2 \), and suppose that \( H_{D_1 \times D_2} \equiv L_{D_1 \times D_2} \). Put \( z = (z_1, z_2) := (\tilde{p}_1(0), \tilde{p}_2(0)) \) and \( w = (w_1, w_2) := (\tilde{p}_1(r), \tilde{p}_2(r)) \), where \( r \in (0, 1) \) is such that \( \tilde{p}_j : B(0, r) \to D_j \) is injective.

Let \( (1, 1/\sqrt{r}) \ni \alpha_n \searrow 1 \). Fix an \( n \in \mathbb{N} \). Since \( L_{D_1 \times D_2}(z, w) = p(0, r) \), there exists \( f_n \in \mathcal{O}(E, D_1 \times D_2) \) such that \( f_n(0) = z \) and \( f_n(\alpha_n r) = w \). By \((*)\), there exists an injective holomorphic mapping \( g_n = (g_{n,1}, g_{n,2}) : E \to D_1 \times D_2 \) such that \( g_n(0) = z \) and \( g_n(\alpha_n^2 r) = w \). Let \( \tilde{g}_{n,j} \) be the lifting with respect to \( \tilde{p}_j \) of \( g_{n,j} \) with \( \tilde{g}_{n,j}(0) = 0, j = 1, 2 \). Observe that \( \tilde{g}_{n,j}(\alpha_n^2 r) = r \) for \( n \) large enough, \( j = 1, 2 \).

By the Montel theorem, we may assume that the sequence \( (\tilde{g}_{n,j})_{n=1}^{\infty} \) is locally uniformly convergent, \( \tilde{g}_{0,j} := \lim_{n \to \infty} \tilde{g}_{n,j} \). We have \( \tilde{g}_{0,j}(0) = 0 \), \( \tilde{g}_{0,j}(r) = r \) and \( \tilde{g}_{0,j} : E \to E \). By the Schwarz lemma we have \( \tilde{g}_{0,j} = \text{id}_E \), \( j = 1, 2 \). From now on, we proceed as in [2].

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References


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