ON THE IRREDUCIBILITY OF FIBRES OF COMPLEX
POLYNOMIAL MAPPINGS

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Dedicated to Professor Tadeusz Winiarski
on the occasion of his 60th birthday

Abstract. We prove that for any open polynomial mapping $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$, $m \geq 3$, there is a linear change of coordinates $\alpha : \mathbb{C}^m \rightarrow \mathbb{C}^m$ such that for each component $f_i$ of $\alpha \circ f$, every fibre of $f_i$ is irreducible. This is a generalization of the Kaliman result to the multidimensional case.

Introduction. Let $X, Y$ be irreducible algebraic varieties over $\mathbb{C}$ in the sense of Weil-Serre (algebraic spaces in terms of [5], VII, §17) and $f : X \rightarrow Y$ be a regular dominating mapping. We call $f$ primitive if the generic fibre of $f$ is irreducible and we call $f$ totally primitive if each fibre of $f$ is irreducible. In the case that $f : \mathbb{C}^n \rightarrow \mathbb{C}$ is a polynomial, the primitivity of $f$ is equivalent to the indecomposability or non-compositeness of $f$ considered by Schinzel [7] and Stein [9] (the equivalence was proved by Płoski in [6]).

Kaliman in [4] proved that in the two-dimensional jacobian conjecture for a polynomial mapping $(p, q) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, $Jac(p, q) \equiv 1$, one can assume that one component, say $p$, is totally primitive. He showed it by finding for $(p, q)$ a polynomial automorphism $f = (f_1, f_2) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ such that $p_1 := f_1(p, q)$ is totally primitive. In this paper we prove that in the multi-dimensional case the result is stronger. Namely, for any open polynomial mapping $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$, $m \geq 3$, there is a linear change of coordinates $\alpha : \mathbb{C}^m \rightarrow \mathbb{C}^m$ such that each component of $\alpha \circ f$ is totally primitive (Theorem 3). The crucial role

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in the proof plays a version of the Bertini Theorem (Proposition 3, cf. [3], Theorem 6.6). The last result follows from a criterion for the primitivity of the composition of a regular mapping with a projection (Theorem 2). Namely, we prove that for any regular dominating mapping \( f : X \to \mathbb{C}^m \), and a projection \( \pi : \mathbb{C}^m \to \mathbb{C}^{m-1} \) such that \( \pi \) is proper on the set of bad values of \( f \), the composition \( \pi \circ f \) is a primitive mapping.

In Section 1 we collect general facts on the primitivity of regular mappings (cf. [8], II.6, [2], 2.13.14, [1], III.4.3). Section 2 is devoted to the criterion for primitivity of the composition of a regular mapping with a projection. In the last Section 3 we prove the Kaliman Theorem in the multidimensional case.

1. Primitive mappings on algebraic varieties. Let \( f : X \to Y \) be a regular dominating mapping between irreducible algebraic varieties \( X, Y \) over \( \mathbb{C} \). The mapping \( f \) is called primitive if there exists a proper algebraic subset \( \tilde{X} \subset X \) such that for each \( \xi \in Y \setminus \tilde{Y} \), the fibre \( f^{-1}(\xi) \) is an irreducible algebraic subvariety of \( X \), and the mapping \( f \) is called totally primitive if each fibre \( f^{-1}(\xi), \xi \in Y \), is irreducible.

In the sequel we write “for the generic \( \xi \in Y \)” instead of “there exists a proper algebraic subvariety \( \tilde{Y} \subset Y \) such that for each \( \xi \in Y \setminus \tilde{Y} \).”

From the definition of primitive mappings we immediately obtain

**Proposition 1.** Let \( X, Y \) be irreducible algebraic varieties. If \( \tilde{X} \subset X \) is a proper algebraic subvariety of \( X \), then a dominating mapping \( f : X \to Y \) is primitive if and only if the mapping \( f|_{X \setminus \tilde{X}} : X \setminus \tilde{X} \to Y \) is primitive.

**Proof.** Assume that \( f \) is primitive. Since \( f|_{X \setminus \tilde{X}} : X \setminus \tilde{X} \to Y \) is dominating, then, for the generic \( \xi \in Y \), \( f^{-1}(\xi) \) is irreducible in \( X \) and \( f^{-1}(\xi) \cap (X \setminus X) \neq \emptyset \). Then \( f^{-1}(\xi) \cap (X \setminus X) \) is irreducible in \( X \setminus X \). This implies that \( f|_{X \setminus \tilde{X}} \) is primitive.

Let us assume that \( f|_{X \setminus \tilde{X}} \) is primitive. Since the generic fibre of \( f \) has dimension \( \dim X - \dim Y \) and the generic fibre of \( f|_{X} \) has dimension at most \( \dim X - \dim Y - 1 \), then for the generic fibre of \( f \), its all irreducible components intersect \( X \setminus \tilde{X} \). In consequence, by the assumption, \( f \) is a primitive mapping.

**Proposition 2.** Let \( X, Y \) be irreducible algebraic varieties and \( f : X \to Y \) be a dominating regular mapping. Then the mapping \( f \) is primitive if and only if there exist nonempty Zariski open subsets \( X_0 \subset X, Y_0 \subset Y \), biregular to affine varieties (i.e. to irreducible algebraic subsets of some \( \mathbb{C}^m \)) such that \( f(X_0) \subset Y_0 \) and the mapping \( f|_{X_0} : X_0 \to Y_0 \) is primitive.

**Proof.** By Proposition[1] we may assume that \( X \) is an affine variety. Let \( Y_0 \neq \emptyset \) be a nonempty Zariski open subset of \( Y \) and biregular to an affine
variety. Let $X_1 = f^{-1}(Y_0)$. Then $X_1$ is a Zariski open subset of $X$. Hence it is an algebraic variety. So, there exists a nonempty Zariski open subset $X_0 \neq \emptyset$ of $X_1$, biregular to an affine variety. In consequence, $f(X_0) \subset Y_0$ and, by Proposition $1$, $f$ is primitive if and only if $f|_{X_0} : X_0 \to Y_0$ is primitive. This ends the proof.

From the proposition it follows that the investigation of the primitivity of regular mappings can be reduced to affine varieties. We shall use this proposition in the proof of general algebraic criterions for the primitivity. We start with definitions.

For an irreducible algebraic variety $X$, by $\mathbb{C}(X)$ we denote the field of rational functions on $X$.

Let $f : X \to Y$ be a dominating regular mapping between irreducible algebraic varieties $X, Y$. Let $f^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ be the homomorphism induced by $f$. Since $f$ is dominating, then $f^*$ is an embedding of $\mathbb{C}(Y)$ into $\mathbb{C}(X)$. Of course, the extension $\mathbb{C}(X)$ of $f^*(\mathbb{C}(Y))$ is finite generated, i.e. $\mathbb{C}(X) = f^*(\mathbb{C}(Y))(\varphi_1, ..., \varphi_m)$, for some $\varphi_1, ..., \varphi_m \in \mathbb{C}(X)$. Thus $\mathbb{C}(X)$ is isomorphic to the quotient field of the ring $f^*(\mathbb{C}(Y))[T_1, ..., T_m]/I_X$, where $T_1, ..., T_m$ are algebraically independent variables and $I_X$ is the ideal of relations between $\varphi_1, ..., \varphi_m$, i.e. the kernel of the natural homomorphism $f^*(\mathbb{C}(Y))[T_1, ..., T_m] \to f^*(\mathbb{C}(Y))[\varphi_1, ..., \varphi_m]$.

The algebraic variety $X$ is called irreducible over $Y$ if $I_X^e \subset f^*(\mathbb{C}(Y))[T_1, ..., T_m]$ is a prime ideal, where $f^*(\mathbb{C}(Y))$ is an algebraic closure of $f^*(\mathbb{C}(Y))$ and $I_X^e$ denotes the ideal in $f^*(\mathbb{C}(Y))[T_1, ..., T_m]$ generated by $I_X$.

The following theorem gives the known equivalent conditions for the primitivity of $f$.

**Theorem 1.** Let $f : X \to Y$ be a regular dominating mapping between irreducible algebraic varieties $X, Y$. The following conditions are equivalent:

(i) $f$ is a primitive mapping,

(ii) $f^*(\mathbb{C}(Y))$ is algebraically closed in $\mathbb{C}(X)$,

(iii) $X$ is irreducible over $Y$.

**Proof.** (i)$\Rightarrow$(ii). By Proposition $2$, one can assume that $X$ and $Y$ are affine varieties. Take any $\varphi \in \mathbb{C}(X)$ algebraic over $f^*(\mathbb{C}(Y))$. Since $f$ is primitive, it is easy to show that $\varphi$ is constant on the generic fibres of $f$. Hence, the minimal polynomial in $f^*(\mathbb{C}(Y))[T]$ for $\varphi$ is of degree 1. Thus $\varphi \in f^*(\mathbb{C}(Y))$. Thus we have (ii).

The implication (ii)$\Rightarrow$(iii) follows from $[11]$, Ch. VII, §11, Theorem 39.

One can find a proof of the implication (iii)$\Rightarrow$(i) in $[8]$, Ch. 2, §6, Theorem 1.
From the above Theorem we immediately obtain

**Corollary 1.** Let $X, Y, Z$ be irreducible algebraic varieties. If $f : X \to Y$ and $g : Y \to Z$ are primitive mappings, then $g \circ f : X \to Z$ is primitive, too.

**2. Composition of a regular mappings with a projection.** In this section we give a theorem on primitivity of the composition of a regular mapping with a projection (Theorem [2]). Let us start with definitions.

Let $X$ be an irreducible algebraic variety. By $X^*$ we denote the set of singular points of $X$. Let $f : X \to \mathbb{C}^m$ be a regular dominating mapping. We say that $\xi \in \mathbb{C}^m$ is a typical value of $f$ if there exists a neighbourhood $\Delta \subset \mathbb{C}^m$ of $\xi$ such that $f|_{f^{-1}(\Delta)} : f^{-1}(\Delta) \to \Delta$ is a trivial topological bundle. We call the remaining points of $\mathbb{C}^m$ bifurcation points of $f$ and denote by $B_f$ the set of such points. Let $A_f = \{\xi \in \mathbb{C}^m :$ some irreducible component of $f^{-1}(\xi)$ is contained in $X^*\}, C_f$ – the set of critical values of $f|_{X \setminus X^*} : X \setminus X^* \to \mathbb{C}^m$. We denote by $E_f$ the Zariski closure of the set $A_f \cup B_f \cup C_f$ and call it the set of bad values of $f$.

**Lemma 1.** If $X$ is an irreducible algebraic variety and $f : X \to \mathbb{C}^m$ is a dominating regular mapping, then

(a) $E_f$ is a proper algebraic subset of $\mathbb{C}^m$,
(b) there exists a positive integer $d_f$ such that for any $\xi \in \mathbb{C}^m \setminus E_f$, $f^{-1}(\xi)$ is the union of $d_f$ irreducible components of dimension $\dim X - m$,
(c) for any $\xi \in \mathbb{C}^m \setminus E_f$ there exists a connected neighbourhood $\Delta \subset \mathbb{C}^m \setminus E_f$ of $\xi$ and open connected sets $U_1, ..., U_{d_f} \subset X \setminus X^*$ such that for any $j$, $f|_{U_j} : U_j \to \Delta$ is a trivial holomorphic bundle and for any $\tilde{\xi} \in \Delta$, $f^{-1}(\tilde{\xi}) \cap U_j$, $j = 1, ..., d_f$, are subsets of different components of $f^{-1}(\tilde{\xi})$.

**Proof.** (a) From [10], Corollary 5.1 we conclude that $B_f$ is contained in a proper algebraic subset of $\mathbb{C}^m$.

By the Sard Lemma, $C_f$ is also contained in a proper algebraic subset of $\mathbb{C}^m$.

From the definition of $B_f$, for any $\xi \in \mathbb{C}^m \setminus B_f$, each irreducible component of $f^{-1}(\xi)$ has dimension $\dim X - m$. Let $Y \subset \mathbb{C}^m$ be the Zariski closure of $f(X^*)$. If $Y \neq \mathbb{C}^m$, then $A_f \subset Y$ is contained in a proper algebraic subset of $\mathbb{C}^m$. If $Y = \mathbb{C}^m$, then the mapping $\tilde{f} = f|_{X^*} : X^* \to \mathbb{C}^m$ is dominating. Then there exists a nonempty Zariski open subset $U \subset \mathbb{C}^m$ such that for all $\xi \in U$, each irreducible component of $\tilde{f}^{-1}(\xi)$ has dimension $\dim X - m - 1$, and so each irreducible component of $f^{-1}(\xi)$ intersects $X \setminus X^*$. Thus $A_f \subset B_f \cup (\mathbb{C}^m \setminus U)$ is contained in a proper algebraic subset of $\mathbb{C}^m$. Summing up we have (a).

(b) follows from the definition of $A_f$ and $B_f$. 

(c) follows from (b), the definition of \( E_f \) and the Implicit Function Theorem.

**Theorem 2.** Let \( X \) be an irreducible algebraic variety and \( f : X \to \mathbb{C}^m \), \( m \geq 2 \), be a regular dominating mapping. If \( \pi : \mathbb{C}^m \ni (\xi, \xi') \mapsto \xi \in \mathbb{C}^{m-1} \) is the canonical projection and \( \pi|_{E_f} : E_f \to \mathbb{C}^{m-1} \) is proper, then \( \tilde{f} = \pi \circ f : X \to \mathbb{C}^{m-1} \) is a primitive mapping.

**Proof.** Let \( X_1 = X \setminus (f^{-1}(E_f) \cup X^*) \). From Lemma 1(a), \( f^{-1}(E_f) \) is a proper algebraic subset of \( X \), and so, by Proposition 1 it suffices to prove that \( g = f|_{X_1} : X_1 \to \mathbb{C}^{m-1} \) is a primitive mapping. We shall use Theorem 1 implication (ii) \( \Rightarrow \) (i).

Let \( \varphi \in \mathbb{C}(X_1) \) be algebraic over \( g^*(\mathbb{C}(\lambda)) \), \( \lambda = (\lambda_1, ..., \lambda_{m-1}) \). Then there exists an irreducible polynomial

\[ p = u^s + a_1(\lambda)u^{s-1} + \ldots + a_s(\lambda) \in \mathbb{C}(\lambda)[u], \]

where \( a_j \in \mathbb{C}(\lambda) \), such that

\[ p(g, u) \in g^*(\mathbb{C}(\lambda))[u] \]

is the minimal polynomial of \( \varphi \). Obviously

\[ (g, \varphi)(X_1) \text{ is a dense constructible subset of } \]

\[ \Gamma = \{(\xi, t) \in \mathbb{C}^{m-1} \times \mathbb{C} : p(\xi, t) = 0\}. \]

To finish the proof we will show that \( s = 1 \). By the Monodromy Theorem it suffices to prove that for any \( \xi \in \mathbb{C}^{m-1} \) there exist a neighbourhood \( \Delta \subset \mathbb{C}^{m-1} \) of \( \xi \) and \( s \) different meromorphic functions \( \psi_1, ..., \psi_s \) on \( \Delta \) such that

\[ p(\xi, \psi_j(\xi)) = 0 \quad \text{on } \Delta, \quad j = 1, ..., s. \]

Take any \( \xi_0 \in \mathbb{C}^{m-1} \). Since \( \pi|_{E_f} \) is proper, then there exists \( \xi'_0 \in \mathbb{C} \) such that \( (\xi_0, \xi'_0) \in \mathbb{C}^m \setminus E_f \). Let \( d_f \) be the number defined in Lemma 1(b) for the mapping \( f \). Since \( \varphi \) is constant on each irreducible component of the generic fibre \( g^{-1}(\xi) \), then \( \varphi \) is constant on each irreducible component of the generic fibre \( f^{-1}(\xi, \xi') \cap X_1 \), and so, by (1), \( s \leq d_f \). By Lemma 1(c), there exist a neighbourhood \( \Delta_0 = \Delta \times D \subset \mathbb{C}^m \setminus E_f \) of \( (\xi_0, \xi'_0) \), where \( \Delta \subset \mathbb{C}^{m-1}, D \subset \mathbb{C} \), and open sets \( U_j \subset X_1, j = 1, ..., d_f \) (in the natural topology) such that for any \( j, f|_{U_j} : U_j \to \Delta_0 \) is a trivial holomorphic bundle, and for any \( (\xi, \xi') \in \Delta_0 \), \( f^{-1}(\xi, \xi') \cap U_j \) are subsets of different components of \( f^{-1}(\xi, \xi') \). In consequence there exist meromorphic functions \( \psi_1, ..., \psi_{d_f} \) on \( \Delta_0 \) such that

\[ \varphi|_{U_j} = \psi_j \circ f|_{U_j}, \quad j = 1, ..., d_f \]

and so,

\[ \psi_j(\xi, \xi') + a_1(\xi)\psi_j^{-1}(\xi, \xi') + \ldots + a_s(\xi) = 0 \quad \text{for } (\xi, \xi') \in \Delta_0, \quad j = 1, ..., d_f. \]
Hence we see that \( \psi_j \) does not depend on \( \xi' \), and so,

\[
\psi_j^s(\xi) + a_1(\xi)\psi_j^{s-1}(\xi) + \ldots + a_s(\xi) = 0 \quad \text{on } \Delta, \quad j = 1, \ldots, d_f.
\]

Moreover

\[
\varphi|U_j = \psi_j \circ g|U_j, \quad j = 1, \ldots, d_f.
\]

Note that for any \( \xi \in \Delta \), each irreducible component of \( g^{-1}(\xi) \) intersects \( U = U_1 \cup \ldots \cup U_{d_f} \). Indeed, for fixed \( \xi \in \Delta \), the set \( g^{-1}(\xi) \) is non-empty. Let \( V_0 \) be an irreducible component of \( g^{-1}(\xi) \). Obviously \( \dim V_0 \geq 1 \). Since \( g^{-1}(\xi) = f^{-1}(\{\xi\} \times \mathbb{C}) \cap X \) then from the definition of \( X \) and the Remmert Open Mapping Theorem there follows that \( f|V_0 : V_0 \to \{\xi\} \times \mathbb{C} \) is open and hence dominating mapping. In consequence, there exists \( \xi' \in \mathbb{C} \) such that \( (\xi, \xi') \in f(V_0) \cap \Delta_0 \). From the choice of \( U_j \) there follows that \( f^{-1}(\xi, \xi') \cap U_j \), \( j = 1, \ldots, d_f \) are subsets of different components of \( f^{-1}(\xi, \xi') \). But at least one of them is contained in \( V_0 \). This gives the announced observation. From (3) and the above observation we have

\[
\bigcup_{j=1}^{d_f} \{(\xi, \psi_j(\xi)) : \xi \in \Delta\} = \bigcup_{j=1}^{d_f} (g, \varphi)(U_j) = (g, \varphi)(g^{-1}(\Delta)).
\]

Since, by (1), \( (g, \varphi)(g^{-1}(\Delta)) \) is a dense subset of \( \{(\xi, t) \in \Delta \times \mathbb{C} ; p(\xi, t) = 0\} \), then we see that there exist \( s \) different functions \( \psi_{j_1}, \ldots, \psi_{j_s} \in \{\psi_1, \ldots, \psi_{d_f}\} \) satisfying (2).

This ends the proof. \( \square \)

3. Irreducibility of components of polynomial mappings. In this section we prove a theorem on irreducibility of components of polynomial mappings (Theorem 3). It is a generalization of the Kaliman result [4] from the two-dimensional case of locally diffeomorphic mappings to the multidimensional case of open mappings.

Let us start with a proposition and two lemmas. This proposition is a version of the Bertini Theorem (cf. [3], Theorem 6.6).

For positive integers \( k, m \) we shall denote by \( M^{(k,m)} \) the set of all complex matrices \( \alpha = [\alpha_{i,j}]_{j=1,\ldots,k \atop i=1,\ldots,m} \) with \( k \) rows and \( m \) columns. Since \( M^{(k,m)} \) can be identified with \( \mathbb{C}^{km} \), then \( M^{(k,m)} \) is an algebraic variety. For any \( \alpha \in M^{(k,m)} \), we shall also denote by \( \alpha \) the linear mapping of \( \mathbb{C}^m \) into \( \mathbb{C}^k \) defined by \( \alpha \). If \( k = m \) and \( \alpha \) is invertible, then \( \alpha \) is called a linear change of coordinates in \( \mathbb{C}^k \).

**Proposition 3.** Let \( X \) be an irreducible algebraic variety over \( \mathbb{C} \) and \( f = (f_1, \ldots, f_m) : X \to \mathbb{C}^m \), \( m \geq 2 \), be a regular mapping. Let \( 0 < k < \dim f(X) \).
Then the mapping \( \tau : X \times \mathbb{M}^{(k,m)} \to \mathbb{C}^k \times \mathbb{M}^{(k,m)} \) of the form

\[
\tau(z, \alpha) = (\alpha(f(z)), \alpha)
\]

is primitive.

**Proof.** By Corollary 1, it suffices to prove this proposition in the case \( k = \dim f(X) - 1 \). Let \( Y \subset \mathbb{C}^m \) be the Zariski closure of \( f(X) \). It is known that there exists a dense subset \( W \subset \mathbb{M}^{(k+1,m)} \) such that for any \( \beta \in W \), \( \beta|_Y : Y \to \mathbb{C}^{k+1} \) is proper. Moreover, for any \( \beta \in W \) there exists a dense subset \( U_\beta \subset \mathbb{M}^{(k,k+1)} \) such that for any \( \eta \in U_\beta \), \( \eta : \mathbb{C}^{k+1} \to \mathbb{C}^k \) restricted to \( E_{\beta \circ f} \) is proper. Hence

\[
U = \{ \alpha \in \mathbb{M}^{(k,m)} : \exists \beta \in W, \exists \eta \in U_\beta, \alpha = \eta \circ \beta \}
\]

is dense in \( \mathbb{M}^{(k,m)} \). By Theorem 2, for any \( \alpha \in U \) the mapping \( \alpha \circ f \) is primitive. Thus

\[
\{(\xi, \alpha) \in \mathbb{C}^k \times \mathbb{M}^{(k,m)} : \tau^{-1}(\xi, \alpha) \text{ is irreducible} \}
\]

is dense in \( \mathbb{C}^k \times \mathbb{M}^{(k,m)} \). Hence, by Lemma 1(b), for the generic \( (\xi, \alpha) \in \mathbb{C}^k \times \mathbb{M}^{(k,m)} \), the fibre \( \tau^{-1}(\xi, \alpha) \) is irreducible. Thus the mapping \( \tau \) is primitive. \( \square \)

In the proof of Theorem 3 we will need a lemma on a family of algebraic sets.

We say that an algebraic set \( V \subset \mathbb{C}^k \) is **in general position** if for any \( s \in \{1, \ldots, k\} \) the projection

\[
V \ni (\xi_1, \ldots, \xi_k) \mapsto (\xi_1, \ldots, \xi_{s-1}, \xi_{s+1}, \ldots, \xi_k) \in \mathbb{C}^{k-1}
\]

is proper.

Take any invertible \( \eta \in \mathbb{M}^{(k,k)} \). Put \( L_\eta : \mathbb{C}^k \times \mathbb{M}^{(k,m)} \to \mathbb{C}^k \times \mathbb{M}^{(k,m)} \),

\[
L_\eta(\xi, \alpha) = (\eta(\xi), \eta \alpha),
\]

where \( \eta \alpha \) denotes the multiplication of matrices. Obviously \( L_\eta \) is a linear automorphism of \( \mathbb{C}^k \times \mathbb{M}^{(k,m)} \).

**Lemma 2.** Let \( V \subset \mathbb{C}^k \times \mathbb{M}^{(k,m)} \) be an algebraic set. If for any invertible \( \eta \in \mathbb{M}^{(k,k)} \) there is

\[
L_\eta(V) = V,
\]

then for the generic \( \alpha \in \mathbb{M}^{(k,m)} \),

\[
V_\alpha = \{ \xi \in \mathbb{C}^k : (\xi, \alpha) \in V \}
\]

is in general position.
Thus $\eta$ and from (5),

\[ s \text{ common points at infinity for } s = 1, \ldots, k. \]

Moreover, there exists an invertible $H \in \mathcal{M}(k,m)$, $P_s \notin (V_\alpha)$ for $s = 1, \ldots, k$, where $(V_\alpha) \subset \mathbb{P}^k$ denotes the closure of $V_\alpha$.

Let $V$ be the closure of $V$ in $\mathbb{P}(\mathbb{C}) \times \mathcal{M}(k,m)$. Take an invertible $\eta \in \mathcal{M}(k,k)$. Let $\tilde{\eta} : \mathbb{P}^k \to \mathbb{P}^k$ be the canonical extension of $\eta$, i.e.

\[ \tilde{\eta}(z_0 : \ldots : z_k) = (z_0 : \eta(z_1, \ldots, z_k)), \]

and $\tilde{L}_\eta : \mathbb{P}^k \times \mathcal{M}(k,m) \to \mathbb{P}^k \times \mathcal{M}(k,m)$ be the automorphism of $\mathbb{P}^k \times \mathcal{M}(k,m)$ generated by $L_\eta$, i.e.

\[ \tilde{L}_\eta(P, \alpha) = (\tilde{\eta}(P), \eta\alpha). \]

By the assumption, $\tilde{L}_\eta(V) = V$. Since $V \neq \mathbb{C}^k \times \mathcal{M}(k,m)$, then

\[ \tag{4} V_\infty = V \cap (H_\infty \times \mathcal{M}(k,m)) \not\subset (H_\infty \times \mathcal{M}(k,m)). \]

Moreover,

\[ \tag{5} \tilde{L}_\eta(V_\infty) = V_\infty. \]

Let $V_{\infty,\alpha} = V_\infty \cap (H_\infty \times \{\alpha\})$. Observe that

\[ W_s = \{\alpha \in \mathcal{M}(k,m) : (P_s, \alpha) \in V_{\infty,\alpha}\}, \]

are proper algebraic subsets of $\mathcal{M}(k,m)$. Indeed, take $s \in \{1, \ldots, k\}$. Obviously $W_s$ is an algebraic set. By (4), there exists $\alpha^0 \in \mathcal{M}(k,m)$ such that $V_{\infty,\alpha^0} \neq H_\infty \times \{\alpha^0\}$. Thus there exists a point $Q \in H_\infty$ such that $(Q, \alpha^0) \not\in V_{\infty,\alpha^0}$. Moreover, there exists an invertible $\eta \in \mathcal{M}(k,k)$ such that $\tilde{\eta}(Q) = P_s$. Hence and from (5),

\[ (P_s, \eta\alpha^0) = \tilde{L}_\eta(Q, \alpha^0) \notin \tilde{L}_\eta(V_{\infty,\alpha^0}) = V_{\infty,\eta\alpha^0}. \]

Thus $\eta\alpha^0 \notin W_s$, i.e. $W_s$ is a proper algebraic subset of $\mathcal{M}(k,m)$.

Since, for any $\alpha \in \mathcal{M}(k,m)$, $(\overline{V_\alpha}) \cap H_\infty \times \{\alpha\} \subset V_{\infty,\alpha}$, then from the above there follows that for $\alpha \in \mathcal{M}(k,m) \setminus (W_1 \cup \ldots \cup W_k)$, $P_s \notin (V_\alpha)$ for $s = 1, \ldots, k$. This ends the proof.

**Lemma 3.** Let $f : \mathbb{C}^n \to \mathbb{C}^m$ be an open polynomial mapping. If $f$ is totally primitive, then the all components of $f$ are totally primitive, too.

**Proof.** Let $f = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$. For the simplicity of notations we prove that $f_1$ is totally primitive. Fix $t \in \mathbb{C}$. Since $\Gamma_t = f_1^{-1}(t) = f^{-1}(\{t\} \times \mathbb{C}^{m-1})$, then, by the Remmert Open Mapping Theorem, $f|_{\Gamma_t} : \Gamma_t \to \{t\} \times \mathbb{C}^{m-1}$ is an open mapping. So, it is dominating on each irreducible component of $\Gamma_t$. 
By the assumptions, for each $\xi \in \mathbb{C}^{m-1}$, $f^{-1}(t, \xi)$ is irreducible. Hence, for the generic $\xi \in \mathbb{C}^{m-1}$, $f^{-1}(t, \xi)$ is contained in each irreducible component of $\Gamma_t$. This implies that the intersection of the components of $\Gamma_t$ has dimension $\dim \Gamma_t$. Thus $\Gamma_t$ is irreducible.

\[\text{Theorem 3. Let } f : \mathbb{C}^n \to \mathbb{C}^m, \ m \geq 3, \text{ be an open polynomial mapping. For the generic linear change of coordinates } \alpha : \mathbb{C}^n \to \mathbb{C}^m, \text{ all components of } \alpha \circ f \text{ are totally primitive.}\]

\[\text{Proof. Define mappings } \kappa : \mathbb{C}^n \times \mathbb{M}^{(m,m)} \to \mathbb{C}^m \times \mathbb{M}^{(m,m)}, \ \tau : \mathbb{C}^n \times \mathbb{M}^{(m-1,m)} \to \mathbb{C}^{m-1} \times \mathbb{M}^{(m-1,m)}, \text{ by}\]

\[\kappa(z, \alpha) = (\alpha(f(z)), \alpha), \quad \tau(z, \beta) = (\beta(f(z)), \beta),\]

and projections $\pi_s : \mathbb{C}^m \to \mathbb{C}^{m-1}$, $\Pi_s : \mathbb{M}^{(m,m)} \to \mathbb{M}^{(m-1,m)}$, $s = 1, \ldots, m,$

\[\pi_s(\xi_1, \ldots, \xi_m) = (\xi_1, \ldots, \xi_s-1, \xi_{s+1}, \ldots, \xi_m),\]

\[\Pi_s([\alpha_{ij}]_{i,j=1,\ldots,s-1,s+1,\ldots,m}) = [\alpha_{ij}]_{i=1,\ldots,s-1,j=1,\ldots,m}.\]

Denote by $\text{id}$ the identity mapping on $\mathbb{C}^n$. Then the diagram

\[\begin{array}{ccc}
\mathbb{C}^n \times \mathbb{M}^{(m,m)} & \xrightarrow{\kappa} & \mathbb{C}^m \times \mathbb{M}^{(m,m)} \\
(id, \Pi_s) \downarrow & & \downarrow (\pi_s, \Pi_s) \\
\mathbb{C}^n \times \mathbb{M}^{(m-1,m)} & \xrightarrow{\tau} & \mathbb{C}^{m-1} \times \mathbb{M}^{(m-1,m)}
\end{array}\]

is commutative for $s = 1, \ldots, m$. By the assumption, $\dim f(\mathbb{C}^n) = m$. Thus, by Proposition 3, $\tau$ is a primitive mapping. Let $V \subset \mathbb{C}^{m-1} \times \mathbb{M}^{(m-1,m)}$ be the minimal algebraic set, such that for any $(\xi, \beta) \in (\mathbb{C}^{m-1} \times \mathbb{M}^{(m-1,m)}) \setminus V$, the fibre $\tau^{-1}(\xi, \beta)$ is irreducible. Observe that $V$ satisfies the assumptions of Lemma 2. Indeed, by primitivity of $\tau$, $V$ is a proper algebraic subset of $\mathbb{C}^{m-1} \times \mathbb{M}^{(m-1,m)}$. Take any invertible $\eta \in \mathbb{M}^{(m-1,m-1)}$. Let us observe that

\[L_\eta(V) = V.\]

Indeed, take any $(\xi, \beta) \in (\mathbb{C}^{m-1} \times \mathbb{M}^{(m-1,m)}) \setminus L_\eta(V)$. Then $(\xi, \beta) = L_\eta(\xi^1, \beta^1)$ and $(\xi^1, \beta^1) \notin V$. Therefore,

\[\tau^{-1}(\xi, \beta) = \{z \in \mathbb{C}^n : \beta(f(z)) = \xi\} \times \{\beta\}\]

\[= \{z \in \mathbb{C}^n : \eta \circ \beta^1(f(z)) = \eta(\xi^1)\} \times \{\beta\}\]

\[= \{z \in \mathbb{C}^n : \beta^1(f(z)) = \xi^1\} \times \{\beta\}\]

is irreducible, because

\[\tau^{-1}(\xi^1, \beta^1) = \{z \in \mathbb{C}^n : \beta^1(f(z)) = \xi^1\} \times \{\beta^1\}\]

is irreducible. Hence, by definition of $V$, $V \subset L_\eta(V)$. Since $L_\eta$ is an automorphism, then $L_\eta(V) = V$. So, by Lemma 2 there exists a Zariski open subset
Let $U \subset \mathcal{M}^{(m-1,m)}$ such that for each $\beta \in U$, the set $V_\beta = \{ \xi \in \mathbb{C}^{m-1} : (\xi, \beta) \in V \}$ is in general position in $\mathbb{C}^{m-1}$.

From the above, the set $W = \bigcap_{s=1}^{m} \Pi_s^{-1}(U)$ is a nonempty Zariski open subset of $\mathcal{M}^{(m,m)}$. Thus the set $D$ of all invertible $\alpha \in W$ is also a nonempty Zariski open subset of $\mathcal{M}^{(m,m)}$. Fix any $\alpha \in D$ and let
\[ \alpha \circ f = (\tilde{f}_1, \ldots, \tilde{f}_m). \]

Take any $i \in \{1, \ldots, m\}$. Since $m \geq 3$, then there exist $j_1, j_2 \in \{1, \ldots, m\}$, $j_1 \neq j_2$ such that $i \in \{1, \ldots, m\} \setminus \{j_1, j_2\}$. Without loss of generality we may assume that $i \in \{1, \ldots, m-2\}$. To prove that $f_i$ is totally primitive, by Lemma 3, it suffices to show that the mapping
\[ (\tilde{f}_1, \ldots, \tilde{f}_{m-2}) : \mathbb{C}^n \to \mathbb{C}^{m-2} \]
is totally primitive. Take $\xi \in \mathbb{C}^{m-2}$. Then
\[ I_\xi = (\tilde{f}_1, \ldots, \tilde{f}_{m-2})^{-1}(\xi) = (\tilde{f}_1, \ldots, \tilde{f}_{m-1})^{-1}(\{\xi\} \times \mathbb{C}). \]

By the Remmert Open Mapping Theorem,
\[ (\tilde{f}_1, \ldots, \tilde{f}_{m-1})|_{I_\xi} : I_\xi \to \{\xi\} \times \mathbb{C} \]
is an open mapping. Thus, it is dominating on each irreducible component of $I_\xi$. By the choice of $\alpha$, $\beta = \Pi_m(\alpha) \in U$. By the definition of $U$, $V_\beta$ is in general position, thus, for the generic $t \in \mathbb{C}$, $(\xi, t, \beta) \not\in V$. Moreover, by (6), for any $z \in \mathbb{C}^n$,
\[ ((\tilde{f}_1, \ldots, \tilde{f}_{m-1})(z), \beta) = (\pi_m, \Pi_m) \circ \kappa(z, \alpha) = \tau(z, \beta). \]

Thus, for the generic $t \in \mathbb{C}$,
\[ (\tilde{f}_1, \ldots, \tilde{f}_{m-1})^{-1}(\xi, t) \times \{\beta\} = \tau^{-1}(\xi, t, \beta) \]
is irreducible. Hence, for the generic $t \in \mathbb{C}$, the fibre $(\tilde{f}_1, \ldots, \tilde{f}_{m-1})^{-1}(\xi, t)$ is irreducible, and, consequently, is contained in each irreducible component of $I_\xi$. This implies that the intersection of the components of $I_\xi$ has dimension $\dim I_\xi$. Thus $I_\xi$ is irreducible. This proves Theorem 3.

**Remarks.** 1. The above theorem does not hold for $m = 2$. A simple example is the mapping: $f : \mathbb{C}^2 \to \mathbb{C}^2$, $f(x_1, x_2) = (x_1^2, x_2^2)$.

2. The assumption of openness of $f$ in the theorem is essential and cannot be replaced by the weaker one that $f$ is dominating. An example: $f : \mathbb{C}^n \to \mathbb{C}^n$, $f(x_1, \ldots, x_n) = (x_1 \ldots x_n, x_2 \ldots x_n, \ldots, x_{n-1}x_n, x_n)$.

3. From the proof it follows that for any $1 \leq k \leq m - 2$ and $\xi \in \mathbb{C}^k$ the fibre $(\tilde{f}_{i_1}, \ldots, \tilde{f}_{i_k})^{-1}(\xi)$, $1 \leq i_1 < \ldots < i_k \leq m$, is irreducible. It is not true for $k = m - 1$ as shown by the example: $f : \mathbb{C}^m \to \mathbb{C}^m$, $f(x_1, \ldots, x_m) = (x_1^2, \ldots, x_m^2)$.

4. Theorem 3 really is a generalization of the Kaliman Theorem, because every polynomial mapping with non-zero constant jacobian is open.
Corollary 2. Let \( f : \mathbb{C}^n \to \mathbb{C}^m, m \geq 3 \), be an open polynomial mapping. For the generic linear change of coordinates \( \alpha : \mathbb{C}^m \to \mathbb{C}^m \), where \( \alpha \in M(m,m) \), and for any component \( \tilde{f_j} \) of the mapping \( \alpha \circ f \),

\[ \tilde{f_j} - t \]

is an irreducible polynomial in \( \mathbb{C}[x_1, \ldots, x_n] \) for any \( t \in \mathbb{C} \).

Proof. By Theorem 3 for the generic \( \alpha \in M(m,m) \), any component of \( \alpha \circ f \) is totally primitive. Fix such \( \alpha \). Suppose to the contrary that there exist a component \( \tilde{f_j} \) of \( \alpha \circ f \) and \( t_0 \in \mathbb{C} \) such that the polynomial \( \tilde{f_j} - t_0 \) is reducible. Since \( \tilde{f_j}^{-1}(t_0) \) is an irreducible algebraic set and \( \tilde{f} - t_0 \) is a reducible polynomial, then there exist a polynomial \( g \in \mathbb{C}[x_1, \ldots, x_n] \) and \( k > 1 \) such that \( \tilde{f_j} - t_0 = g^k \). Thus, for any \( t \neq 0 \), \( \tilde{f_j} - t_0 - t = (g - \varepsilon_1) \cdots (g - \varepsilon_k) \), where \( \varepsilon_i, i = 1, \ldots, k \) are all \( k \)-th roots of \( t \). Thus \( \tilde{f_j}^{-1}(t_0 + t) \) is a reducible algebraic set for \( t \neq 0 \). This is impossible.

Corollary 3. Let \( f \in \mathbb{C}[x_1, \ldots, x_n], n \geq 3 \). If \( f \) is monic with respect to \( x_1 \), then for the generic \( (\alpha, \beta) \in \mathbb{C}^2 \),

\[ f + \alpha x_2 + \beta x_3 - t \]

is an irreducible polynomial for any \( t \in \mathbb{C} \).

Proof. If \( \deg f = 0 \), then the assertion is obvious. Let \( \deg f > 0 \). Since \( f \) is monic with respect to \( x_1 \), then for any \( (t_1, t_2, t_3) \in \mathbb{C}^3 \), the set

\[ \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : f(z) = t_1, z_2 = t_2, z_3 = t_3 \} \]

has dimension \( n - 3 \). Thus, by the Remmert Open Mapping Theorem, \( (f, x_2, x_3) : \mathbb{C}^n \to \mathbb{C}^3 \) is an open mapping. Hence, by Theorem 3 the assertion follows.

Analogously to the above we obtain

Corollary 4. Let \( f \in \mathbb{C}[x_1, \ldots, x_n], n \geq 3 \) be a nonconstant polynomial. If \( f \) is monic with respect to \( x_1 \), and \( g \in \mathbb{C}[x_2], h \in \mathbb{C}[x_3] \) are nonconstant polynomials, then for the generic \( (\alpha, \beta) \in \mathbb{C}^2 \),

\[ f + \alpha g + \beta h - t \]

is an irreducible polynomial for any \( t \in \mathbb{C} \).

References


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