IN VariantS OF SINGULARITIES OF POLYNOMIALS IN TWO COMPLEX VARIABLES AND THE NEWton DIAGRAMS

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Abstract. For any polynomial mapping $f : \mathbb{C}^2 \to \mathbb{C}$ with a finite number of critical points we consider the Milnor number $\mu(f)$, the jump of the Milnor numbers at infinity $\lambda(f)$, the number of branches at infinity $r_\infty(f)$ and the genus $\gamma(f)$ of the generic fiber $f^{-1}(t_{\text{gen}})$. The aim of this note is to estimate these invariants of $f$ in terms of the Newton diagram $\Delta_\infty(f)$.

1. Introduction. Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial with a finite number of critical points. We define the global Milnor number $\mu(f)$ by putting

$$\mu(f) := \sum_{P \in \mathbb{C}^2} \left( \frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y} \right)_P$$

where the symbol $(\cdot, \cdot)_P$ denotes the multiplicity of intersection at the point $P \in \mathbb{C}^2$. Note that $\mu(f) < +\infty$.

Let $C_t \subset \mathbb{P}^2(\mathbb{C})$ be the projective closure of the fiber $f^{-1}(t)$ where $t \in \mathbb{C}$. If $d = \deg f$ and $F(X, Y, Z)$ is the homogeneous form corresponding to $f = f(X, Y) = \sum_{\alpha + \beta \leq d} c_{\alpha \beta} X^{\alpha} Y^{\beta}$, then $C_t$ is given by the equation $F(X, Y, Z) - tZ^d = 0$. Let $L_\infty \subset \mathbb{P}^2(\mathbb{C})$ be the line at infinity given by $Z = 0$ and let $(C_t)_\infty = C_t \cap L_\infty$. Obviously, $(C_t)_\infty = (C_0)_\infty$. In the sequel we write $C = C_0$ and $C_\infty = (C_0)_\infty$. If $f^+(X, Y) = \sum_{\alpha + \beta = d} c_{\alpha \beta} X^{\alpha} Y^{\beta}$ is the leading part of the polynomial $f$, then

$$C_\infty = \{ (x : y : z) \in \mathbb{P}^2(\mathbb{C}) : z = 0 \text{ and } f^+(x, y) = 0 \} .$$

For every $P \in C_t$ we denote by $\mu_P^t = \mu_P^t(C_t)$ the Milnor number of the curve $C_t$ at the point $P$. There exist numbers $\mu_P^{\text{gen}} \geq 0$ ($P \in C_\infty$) such that
\( \mu_P^t \geq \mu_P^{\text{gen}} \) for all \( t \in \mathbb{C} \). Moreover, \( \mu_P^t = \mu_P^{\text{gen}} \) for almost all \( t \in \mathbb{C} \). This fact is due to Broughton \[1\] (see also \[4\] for a simple direct proof). Hence the set

\[ \Lambda(f) = \{ t \in \mathbb{C} : \mu_P^t > \mu_P^{\text{gen}} \text{ for some } P \in C_{\infty} \} \]

is finite and the numbers

\[
\lambda'(f) := \sum_{P \in C_{\infty}} (\mu_P^t - \mu_P^{\text{gen}}) \quad \text{and} \quad \lambda(f) := \sum_{t \in \mathbb{C}} \lambda'(f)
\]

are well defined. At any point \( P \in C \) we consider the number \( r_P(C) \) of branches of the curve \( C \) centered at \( P \). We define the number \( r_{\infty}(C) \) of branches at infinity of the curve \( C \) by putting

\[ r_{\infty}(C) := \sum_{P \in C_{\infty}} r_P(C) . \]

It is known (see \[4\]) that the function

\[ C \setminus \Lambda(f) \ni t \to r_{\infty}(C^t) \in \mathbb{N} \]

is constant. Let \( r_{\infty}(f) := r_{\infty}(C^t) \) for \( t \in \mathbb{C} \setminus \Lambda(f) \). We call \( r_{\infty}(f) \) the generic number of branches at infinity.

Let \( \text{supp } f = \{ (\alpha, \beta) \in \mathbb{N}^2 : c_{\alpha\beta} \neq 0 \} \). The Newton diagram at infinity \( \Delta_{\infty}(f) \) is the convex hull of \( \{ (0,0) \} \cup \text{supp } f \). For any \( f \) we define its global Newton number \( \mu(\Delta_{\infty}(f)) \) by putting

\[
\mu(\Delta_{\infty}(f)) := 2 \text{Area } \Delta_{\infty}(f) - A - B + 1
\]

where \( A = \max \{ \alpha \in \mathbb{N} : (\alpha,0) \in \Delta_{\infty}(f) \} \) and \( B = \max \{ \beta \in \mathbb{N} : (0,\beta) \in \Delta_{\infty}(f) \} \). The Newton polygon at infinity \( \partial \Delta_{\infty}(f) \) is the set of the faces of \( \Delta_{\infty}(f) \) not included in the coordinate axes. We define the number

\[ r(\Delta_{\infty}(f)) := \sum_{S \in \partial \Delta_{\infty}(f)} r(S) \]

where \( r(S) = \) (number of integer points lying on the segment \( S \)) \(- 1 \). Hence the integer points divide \( S \) into \( r(S) \) segments.

For any segment \( S \in \partial \Delta_{\infty}(f) \) we let \( \text{in}(f,S)(X,Y) \) be the sum of all monomials \( c_{\alpha\beta}X^\alpha Y^\beta \) such that \( (\alpha, \beta) \in S \). The polynomial \( f \) is nondegenerate on \( S \in \partial \Delta_{\infty}(f) \) if the system of equations

\[
\text{in}(f,S)(X,Y) = \frac{\partial}{\partial X} \text{in}(f,S)(X,Y) = \frac{\partial}{\partial Y} \text{in}(f,S)(X,Y) = 0
\]

has no solution in \( \mathbb{C}^* \times \mathbb{C}^* \) where \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \). Our main result is the following:

**Theorem 1.1.**

Let \( f : \mathbb{C}^2 \to \mathbb{C} \) be a polynomial such that \( \mu(f) < +\infty \). Suppose that the diagram \( \Delta_{\infty}(f) \) has a nonempty interior. Then
(1) \( \mu(\Delta_\infty(f)) - (\mu(f) + \lambda(f)) \geq r(\Delta_\infty(f)) - r_\infty(f) \geq 0, \)
(2) the equalities hold if \( f \) is nondegenerate on each segment \( S \in \partial\Delta_\infty(f) \) not included in a line passing through the origin.

We give the proof in Section 3. Our theorem implies the following estimation due to Cassou-Noguès:

**Corollary 1.2** ([2], Theorem 10).
Let \( f : \mathbb{C}^2 \to \mathbb{C} \) be a polynomial such that \( \mu(f) < +\infty \). Then

(1) \( \mu(f) + \lambda(f) \leq \mu(\Delta_\infty(f)) \),
(2) the equality holds if \( f \) is nondegenerate on each segment \( S \in \partial\Delta_\infty(f) \) not included in a line passing through the origin.

**Proof.** If \( \Delta_\infty(f) \) does not have interior points then \( \deg f \leq 2 \) (otherwise \( \mu(f) = \infty \)) and the result is easily seen. Therefore we can assume that \( \Delta_\infty(f) \) has a nonempty interior and (1.2) follows from (1.1).

To give another application of our result let us put \( \gamma(f) = \) the genus of the Riemann surface corresponding to the generic fiber \( f^{-1}(t_{\text{gen}}) \). Let \( \gamma(\Delta_\infty(f)) \) be the number of integer points lying inside \( \Delta_\infty(f) \).

**Corollary 1.3.** With the assumptions given above we have

(1) \( \gamma(f) \leq \gamma(\Delta_\infty(f)) \),
(2) the equality holds if \( f \) is nondegenerate on each segment \( S \in \partial\Delta_\infty(f) \) not included in a line passing through the origin.

**Proof.** We may assume that \( \Delta_\infty(f) \) has interior points. By Abhyankar-Sathaye’s formula (see [3], Formula 4.4) we have

\[ 2 \gamma(f) = \mu(f) + \lambda(f) - r_\infty(f) + 1. \]

On the other hand, by Pick’s formula we get

\[ 2 \gamma(\Delta_\infty(f)) = \mu(\Delta_\infty(f)) - r(\Delta_\infty(f)) + 1 \]
and we obtain 1.3 directly from the main result. \( \square \)

2. The Newton diagrams. Let \( f(X,Y) = \sum c_{\alpha\beta}X^\alpha Y^\beta \in \mathbb{C}[X,Y] \) be a nonzero polynomial of degree \( d \). We say that the polynomial \( f \) is quasi–convenient if \( c_{0\beta} \neq 0 \) and \( c_{\alpha0} \neq 0 \) for some integers \( \alpha, \beta \geq 0 \). If the above condition holds for some positive \( \alpha, \beta \), then \( f \) is called convenient polynomial.
Let \( \text{supp} f = \{(\alpha,\beta) \in \mathbb{N}^2 : c_{\alpha\beta} \neq 0\} \). We define

\[ \Delta(f) := \text{conv}(\text{supp} f) \text{ and } \Delta_\infty(f) := \text{conv}(\{(0,0)\} \cup \text{supp} f). \]

The polygons \( \Delta(f) \) and \( \Delta_\infty(f) \) are called respectively Newton diagram and Newton diagram at infinity of the polynomial \( f \). For every quasi–convenient polynomial we consider additionaly its Newton diagram at zero. This polygon is the closure of the set \( \Delta_\infty(f) \setminus \Delta(f) \). We denote it by \( \Delta_0(f) \). If \( a,b > 0 \)
are smallest integer numbers such that \((a, 0), (0, b) \in \text{supp} f\), then \(\Delta_0(f)\) is the polygon bounded by the segments joining the points \((0, 0)\) with \((a, 0)\) and \((0, 0)\) with \((0, b)\) and by the faces of the diagram \(\Delta(f)\) that separate it from the origin.

Obviously, \(\Delta_{\infty}(f) = \Delta_0(f) \cup \Delta(f)\). If \((0, 0) \in \text{supp} f\), then \(\Delta_{\infty}(f) = \Delta(f)\) and \(\Delta_0(f) = \emptyset\). Similarly as in the definition of \(\partial \Delta_{\infty}(f)\), we define for every quasi–convenient polynomial \(f\) its Newton polygon at zero \(\partial \Delta_0(f)\) as the set of the faces of \(\Delta_0(f)\) not included in the coordinate axes. By \(\partial \Delta(f)\) we denote the set of all faces of \(\Delta(f)\) and we call it Newton polygon of the polynomial \(f\). If \(f\) is quasi–convenient, then \(\partial \Delta_0(f), \partial \Delta_{\infty}(f) \subset \partial \Delta(f)\). But if \(f(0, 0) \neq 0\), then \(\partial \Delta_0(f) = \emptyset\) and \(\partial \Delta_{\infty}(f) = \partial \Delta(f)\).

Newton Diagrams in affine systems of coordinates. If \(U = (\vec{u}; \vec{e}_1, \vec{e}_2)\) is an affine system of coordinates of the real plane \(R^2\) (i.e. \(\vec{u}, \vec{e}_1, \vec{e}_2 \in R^2\) and \(\vec{e}_1, \vec{e}_2\) are linearly independent), then we define the support of the polynomial \(f(X, Y) \in C[X, Y]\) in the system \(U\): \(\text{supp}^U f := \{\vec{u} + \alpha \vec{e}_1 + \beta \vec{e}_2 : (\alpha, \beta) \in \text{supp} f\}\) and Newton diagram of the polynomial \(f(X, Y)\) in the system \(U\): \(\Delta^U(f) := \text{conv}(\text{supp}^U f)\). Similarly to the standard case we define \(\Delta_{\infty}^U(f) := \text{conv}(\{\vec{u}\} \cup \text{supp}^U f)\) and if \(f\) is quasi–convenient we put \(\Delta_0^U(f) := \text{closure of } (\Delta_{\infty}^U(f) \setminus \Delta^U(f))\). If \(f(0, 0) \neq 0\), then \((0, 0) \in \text{supp} f\), hence \(\vec{u} \in \text{supp}^U f\) and then \(\Delta_{\infty}^U(f) = \Delta^U(f)\) and \(\Delta_0^U(f) = \emptyset\). Analogously to the standard case we define the polygons \(\partial \Delta_0^U(f), \partial \Delta_0^U(f)\) and \(\partial \Delta_{\infty}^U(f)\) of the polynomial \(f\) in the system \(U\).

If \(U = (\vec{0}, \vec{v}, \vec{j})\) where \(\vec{v} = [1, 0], \vec{j} = [0, 1]\), then the notions introduced above correspond to the standard constructions presented above, i.e. \(\Delta^U(f) = \Delta(f), \Delta_{\infty}^U(f) = \Delta_{\infty}(f), \text{in}^U(f, S)(X, Y) = \text{in}(f, S)(X, Y), \) etc.

Let \(F(X, Y, Z) = Z^d f(X/Z, Y/Z)\), where \(d = \deg f > 0\), be a homogenization of a polynomial \(f(X, Y)\). The projective curve \(F(X, Y, Z) = 0\) is the projective closure of the affine curve \(f(X, Y) = 0\). It is natural to consider the affine curves \(F(1, Y, Z) = 0\) and \(F(X, 1, Z) = 0\). If \(f(0, 0, 0) \neq 0\), then \(F(X, Y, Z)\) is also the homogenization of \(F(1, Y, Z)\) and \(F(X, 1, Z)\). The notion of the Newton diagram in an affine system of coordinates is useful while
comparing the Newton diagrams of the polynomials \( f(X, Y) = F(X, Y, 1), F(X, 1, Z) \) and \( F(1, Y, Z) \).

**Lemma 2.1 (Main Lemma).**

Let \( U = (0; \vec{i}, \vec{j}) \), \( V = (d\vec{i}; \vec{j} - \vec{i}, -\vec{i}) \), \( W = (d\vec{j}; \vec{i} - \vec{j}, -\vec{j}) \). Then
\[
\supp^U F(X, Y, 1) = \supp^V F(1, Y, Z) = \supp^W F(X, 1, Z).
\]

**Proof.** We prove the first equality. Denote \( N = \supp f \). Hence if \( f(X, Y) = \sum_{(\alpha, \beta) \in N} c_{\alpha\beta} X^\alpha Y^\beta \), then \( F(X, Y, Z) = \sum_{(\alpha, \beta) \in N} c_{\alpha\beta} X^\alpha Y^\beta Z^{d-\alpha-\beta} \) and \( F(1, Y, Z) = \sum_{(d-\beta-\gamma, \beta) \in N} c_{d-\beta-\gamma, \beta} Y^\beta Z^\gamma \). We have
\[
\supp^V F(1, Y, Z) = \\
\{ \beta(\vec{j} - \vec{i}) + \gamma(\vec{-i}) + d\vec{i}: (\beta, \gamma) \in \supp F(1, Y, Z) \} =
\{ \beta(\vec{j} - \vec{i}) + \gamma(\vec{-i}) + d\vec{i}: \beta = d - \alpha - \beta \text{ and } (\alpha, \beta) \in N \} =
\{ (d - \beta - \gamma)\vec{i} + \beta \vec{j}: \gamma = d - \alpha - \beta \text{ and } (\alpha, \beta) \in N \} =
\{ \alpha \vec{i} + \beta \vec{j}: (\alpha, \beta) \in N \} = N = \supp^U F(X, Y, 1).
\]

In the same way we prove that \( \supp^U F(X, Y, 1) = \supp^W F(X, 1, Z) \).

Directly from the above lemma we get the following corollaries:

**Corollary 2.2.**

1. \( \Delta(f) = \Delta^U (F(X, Y, 1)) = \Delta^V (F(1, Y, Z)) = \Delta^W (F(X, 1, Z)) \).
2. \( \partial \Delta(f) = \partial \Delta^U (F(X, Y, 1)) = \partial \Delta^V (F(1, Y, Z)) = \partial \Delta^W (F(X, 1, Z)) \).
3. If \( f \) is a quasi–convenient polynomial, then the polynomials \( F(1, Y, Z) \) and \( F(X, 1, Z) \) are also quasi–convenient and the triangle with vertices at \( (0, 0) \), \( (0, \deg f) \), \( (\deg f, 0) \) is the union of the polygons \( \Delta_\infty(f), \Delta_I(f) \) and \( \Delta_{II}(f) \), whose interiors are disjoint, where
\[
\Delta_I(f) := \Delta^V (F(1, Y, Z)), \quad \Delta_{II}(f) := \Delta^W (F(X, 1, Z)).
\]

Suppose that the polynomial \( f(X, Y) \) is quasi–convenient. We denote
\[
\partial \Delta_I(f) := \partial \Delta^V (F(1, Y, Z)), \quad \partial \Delta_{II}(f) := \partial \Delta^W (F(X, 1, Z)).
\]

The leading part \( f^+(X, Y) \) is a homogeneous form and all points of its support lie on the line \( \alpha + \beta = \deg f \). Hence the diagram \( \Delta(f^+) \) is a segment or a point. Therefore the polygon \( \partial \Delta(f^+) \) is the empty set or a one-element set. The segment \( \Delta(f^+) \) is called the main segment of the polynomial \( f \).

**Corollary 2.3.** If \( f \) a is quasi–convenient polynomial, then
\[
\partial \Delta_\infty(f) = \partial \Delta_I(f) \cup \partial \Delta_{II}(f) \cup \partial \Delta(f^+).
\]
Remark 2.4. The description of the Newton diagram at infinity by means of local diagrams was given by numerous authors [2], [7], [8]. Our version of this description allows us to give a simple proof of the main result.

Nondegeneracy. A nonzero polynomial \( f \) is nondegenerate on \( S \in \partial \Delta(f) \) if the system of equations

\[
\text{in}(f, S)(X, Y) = \frac{\partial}{\partial X} \text{in}(f, S)(X, Y) = \frac{\partial}{\partial Y} \text{in}(f, S)(X, Y) = 0
\]

has no solution in \( \mathbb{C}^* \times \mathbb{C}^* \). We say that a quasi–convenient polynomial \( f = f(X, Y) \) is nondegenerate at zero (at infinity) if it is nondegenerate on each segment \( S \in \partial \Delta_0(f) (S \in \partial \Delta_{\infty}(f)) \). The introduced notions of nondegeneracy at zero and at infinity can be defined using the Newton diagram constructed at any affine system \( U = (\vec{u}; \vec{1}, \vec{e}_2). \) Instead of the notions \( \text{in}(f, S), \partial \Delta_0(f), \partial \Delta_{\infty}(f) \) we consider their counterparts \( \text{in}^U(f, S), \partial \Delta^U_0(f), \partial \Delta^U_{\infty}(f) \). The nondegeneracy at zero (at infinity) does not depend on the choice of the system \( U \) because the diagram \( \Delta^U(f) \) is the image of the diagram \( \Delta(f) \) by the affine transformation of the real plane:

\[
\mathbb{R}^2 \ni (\alpha, \beta) \rightarrow (\alpha, \beta)_U := \vec{u} + \alpha \vec{e}_1 + \beta \vec{e}_2 \in \mathbb{R}^2.
\]

Proposition 2.5. Let \( f(X, Y) \in \mathbb{C}[X, Y] \) be a quasi–convenient polynomial of degree \( d > 0 \) and let \( F(X, Y, Z) \) be its homogenization. Then \( f(X, Y) \) is nondegenerate at infinity if and only if

1. the polynomials \( F(1, Y, Z), F(X, 1, Z) \) are nondegenerate at zero and
2. the leading part \( f^+(X, Y) \) is a homogeneous form without multiple factors of the form \( \xi X - \eta Y \) where \( \xi \eta \neq 0 \).

Proof. Let \( U = (\vec{0}; \vec{i}, \vec{j}), V = (d\vec{i}; \vec{j} - \vec{i}, -\vec{i}_V), W = (d\vec{j}; \vec{i} - \vec{j}, -\vec{j}_W) \) and let \( S \in \partial \Delta(f) \). We may consider the nondegeneracy of the polynomials \( f(X, Y) = \)
$F(X, Y, 1, F(1, Y, Z)$ and $F(X, 1, Z)$ on $S$ respectively in the systems $U$, $V$ and $W$ (see Corollary 3.2 (2)). By direct calculation we have

(a) $\text{in}^V(F(1, Y, Z), S)(Y, Z) = Z^d \text{in}(f, S)(1/Z, Y/Z)$,
(b) $\text{in}^W(F(1, 1, Z), S)(X, Z) = Z^d \text{in}(f, S)(X/Z, 1/Z)$.

We show that the following conditions are equivalent:

(1) $f(X, Y)$ is nondegenerate on $S$,
(2) $F(1, Y, Z)$ is nondegenerate on $S$,
(3) $F(X, 1, Z)$ is nondegenerate on $S$.

We prove the equivalence $(1) \iff (2)$. The proof of $(1) \iff (3)$ runs analogously.

We denote $g(X, Y) = \text{in}(f, S)(X, Y)$ and $h(Y, Z) = \text{in}^V(F(1, Y, Z), S)(Y, Z)$.

We have to show that the system $\frac{\partial g}{\partial X}(X, Y) = g(X, Y) = 0$ has a solution in $\mathbb{C}^* \times \mathbb{C}^*$ if and only if the system $\frac{\partial h}{\partial Y}(Y, Z) = \frac{\partial h}{\partial Z}(Y, Z) = h(Y, Z) = 0$ has a solution in $\mathbb{C}^* \times \mathbb{C}^*$. From (a) we get

\[ h(Y, Z) = Z^d g\left(\frac{1}{Z}, \frac{Y}{Z}\right), \quad Z \frac{\partial h}{\partial Y}(Y, Z) = Z^d \frac{\partial g}{\partial Y}\left(\frac{1}{Z}, \frac{Y}{Z}\right) \text{ and} \]

\[ Z^2 \frac{\partial g}{\partial Z}(Y, Z) = Z^d \left(dZ g\left(\frac{1}{Z}, \frac{Y}{Z}\right) - \frac{\partial g}{\partial X}\left(\frac{1}{Z}, \frac{Y}{Z}\right) - Y \frac{\partial g}{\partial Y}\left(\frac{1}{Z}, \frac{Y}{Z}\right)\right). \]

These equalities imply the above equivalence.

By Corollary 2.3 we have

\[ \partial \Delta(f) = \partial \Delta_\infty(f) = \partial \Delta_I(f) \cup \partial \Delta_{II}(f) \cup \partial \Delta(f^+). \]

Note that $f$ is nondegenerate on each segment $S \in \partial \Delta_I(f)$ ($S \in \partial \Delta_{II}(f)$) if and only if the polynomial $F(1, Y, Z)$ ($F(X, 1, Z)$) is nondegenerate at zero.

Let $\phi = \phi(X, Y)$ be a homogeneous form of positive degree. It is easy to check that the system $\phi = \phi_\infty = \phi_0 = 0$ has no solution in $\mathbb{C}^* \times \mathbb{C}^*$ if and only if $\phi(X, Y) = X^m Y^n \phi_1(X, Y)$ for some $m, n \in \mathbb{N}$ where $\phi_1$ is a reduced homogeneous form such that $\phi_1(0, 0) \neq 0$. Hence the polynomial $f$ is nondegenerate on the main segment $S = \Delta(f^+)$ if and only if the homogeneous form $f^+(X, Y) = \text{in}(f, S)(X, Y)$ has only single factors of the form $\xi X - \eta Y$ where $\xi \eta \neq 0$. The above observations complete the proof of our proposition.

\[ 3. \text{The Milnor numbers and number of branches. Let } f(X, Y) \in \mathbb{C}[X, Y] \text{ be a convenient polynomial without constant term and let } \Delta_0(f) \text{ be its Newton diagram at zero. We define the numbers}\]

\[
\mu(\Delta_0(f)) = 2 \text{Area } \Delta_0(f) - \text{ord } f(X, 0) - \text{ord } f(0, Y) + 1;
\]

\[
r(\Delta_0(f)) := \sum_{S \in \partial \Delta_0(f)} r(S).
\]
We denote by \( r_0(f) \) the number of branches of the curve \( f(X,Y) = 0 \) at zero.

Let us recall the following:

**Theorem 3.1** ([9], Theorem 1.2).

If \( f(X,Y) \in \mathbb{C}[X,Y] \) is a convenient polynomial without constant term, then

1. \( \mu_0(f) - \mu(\Delta_0(f)) \geq r(\Delta_0(f)) - r_0(f) \geq 0 \),
2. the equality holds if \( f \) is nondegenerate at zero.

**Theorem 3.2** (Cassou-Noguès' formula, [2], Proposition 12).

Let \( c = \#C_\infty \). If \( \mu(f) < +\infty \), then

\[
\sum_{P \in C_{\infty}} \mu_{\text{gen}}(P) - c + \mu(f) + \lambda(f) - 1 = d(d-3).
\]

A proof of the above formula without using Eisenbud-Neumann diagrams is given in [3].

**Proof of the main result.** Without loss of generality we can assume that the polynomial \( f \) is quasi-convenient with the generic fiber \( f^{-1}(0) \). Otherwise we consider the polynomial \( f^t = f - t \), where \( t \in \mathbb{C} \setminus \Lambda(f) \) is such that \( f^t(0,0) \neq 0 \). Then

\( a) \ \mu(\Delta_{\infty}(f)) = \mu(\Delta_{\infty}(f^t)), \ r(\Delta_{\infty}(f)) = r(\Delta_{\infty}(f^t)), \)
\( b) \ \mu(f) = \mu(f^t), \ \lambda(f) = \lambda(f^t) \) and \( r_{\infty}(f) = r_{\infty}(f^t) \).

Moreover, if \( f \) satisfies the assumption of the second part of our theorem then we can choose \( t \in \mathbb{C} \setminus \Lambda(f) \) such that \( f^t \) is nondegenerate at infinity.

Therefore, it is enough to check our theorem for a quasi-convenient polynomial \( f \) such that \( 0 \not\in \Lambda(f) \). Moreover, in the proof of (2) we may assume that \( f \) is nondegenerate at infinity.

Let \( P_1 = (1:0:0), \ P_2 = (0:1:0) \in \mathbb{P}^2 \). We have the following cases:

(i) \( P_1 \in C_\infty, \ P_2 \in C_\infty \)
(ii) \( P_1 \in C_\infty, \ P_2 \notin C_\infty \)
(iii) \( P_1 \notin C_\infty, \ P_2 \in C_\infty \)
(iv) \( P_1 \notin C_\infty, \ P_2 \notin C_\infty \)

We give the proof in the case (i). In other cases the proof runs analogously. We prove the both parts of our theorem paralelly. In the case under consideration \( f^+(X,Y) = aX^pY^{d-p} + \cdots + bX^{d-q}Y^q \) where \( a, b \in \mathbb{C}^* \) and \( p, q \) are integers such that \( p, q > 0, \ p + q \leq d \). Hence \( c - 2 \leq d - p - q \). It is easily seen that \( c = d - p - q + 2 \) if and only if the polynomial \( f \) is nondegenerate on the main segment \( \Delta(f^+) \).

Let \( A = \deg f(X,0) \) and \( B = \deg f(0,Y) \). Let \( F(X,Y,Z) \) be the homogeneous form corresponding to the polynomial \( f(X,Y) \). Note that

\[
\text{Area} \Delta_0(F(1,Y,Z)) = \text{Area} \Delta_0^Y(F(1,Y,Z)).
\]
and

\[ \text{Area } \Delta_0(F(X, 1, Z)) = \text{Area } \Delta_0^W(F(X, 1, Z)) \]

where \( V = (d \vec{i}; \vec{j} - \vec{i}, -\vec{i}) \) and \( W = (d \vec{j}; \vec{i} - \vec{j}, -\vec{j}) \). Hence

\[ \mu(\Delta_0((F(1, Y, Z)))) = 2 \text{Area } \Delta_I(f) - (d - A) - q + 1 \]

and

\[ \mu(\Delta_0((F(1, Y, Z)))) = 2 \text{Area } \Delta_{II}(f) - (d - B) - p + 1. \]

Recall that \( \mu(\Delta_\infty(f)) = 2 \text{Area } \Delta_{\infty}(f) - A - B + 1 \). Therefore, by Corollary 2.2 (3) we get

\[
\mu(\Delta_0(F(1, Y, Z))) = 2 \text{Area } \Delta_I(f) - (d - A) - q + 1 \]

and

\[
\mu(\Delta_0(F(X, 1, Z))) = 2 \text{Area } \Delta_{II}(f) - (d - B) - p + 1. \]

Recall that \( \mu(\Delta_\infty(f)) = 2 \text{Area } \Delta_{\infty}(f) - A - B + 1 \). Therefore, by Corollary 2.2 (3) we get

\[
\mu(\Delta_0(F(1, Y, Z))) + \mu(\Delta_0(F(X, 1, Z))) + \mu(\Delta_\infty(f)) =
\]

\[
d(d - 3) + d - p - q + 3 \geq d(d - 3) + c + 1
\]

and the equality holds if and only if the polynomial \( f \) is nondegenerate on the main segment. In the case under consideration the polynomials \( F(1, Y, Z) \) and \( F(X, 1, Z) \) are convenient without constant term. By Theorem 3.1 we have

\[
\mu_{P_1} = \mu_{P_1} = \mu_0(F(1, Y, Z)) \geq \mu(\Delta_0(F(1, Y, Z)))
\]

and

\[
\mu_{P_2} = \mu_{P_2} = \mu_0(F(X, 1, Z)) \geq \mu(\Delta_0(F(X, 1, Z))).
\]

Applying Theorem 3.1 and the Main Lemma we get

\[
\mu_{P_1} = \mu_{P_1} = \mu_0(F(1, Y, Z)) \geq \mu(\Delta_0(F(1, Y, Z))) \]

and

\[
\mu_{P_2} = \mu_{P_2} = \mu_0(F(X, 1, Z)) \geq \mu(\Delta_0(F(X, 1, Z))).
\]

On the other hand we have \( \mu_{P} = \mu_0(F - 1) \) for each \( P \in C_\infty \). Thus

\[
\sum_{P \in C_\infty \setminus \{P_1, P_2\}} \mu_{P} \geq \sum_{P \in C_\infty \setminus \{P_1, P_2\}} (\text{ord } F - 1) = (d - p - q) - (c - 2) \geq
\]
\[ r(\Delta(f^+)) - (c - 2) \geq r(\Delta(f^+)) - \sum_{P \in C_\infty \setminus \{P_1, P_2\}} r_P(C). \]

It is clear that the equalities above we get if \( f \) is nondegenerate on the main segment. The above estimate and the inequality (***) give

\[ \mu(\Delta_\infty(f)) - (\mu(f) + \lambda(f)) \geq \sum_{S \in \partial \Delta_{\text{II}}(f)} r(S) + \sum_{S \in \partial \Delta_{\text{I}}(f)} r(S) + r(\Delta(f^+)) - \sum_{P \in C_\infty} r_P(C). \]

The diagram \( \Delta_\infty(f) \) has a nonempty interior. Hence by Corollary 2.3 we get

\[ \mu(\Delta_\infty(f)) - (\mu(f) + \lambda(f)) \geq r(\Delta_\infty(f)) - \sum_{P \in C_\infty} r_P(C). \]

This completes the proof of (1). To proof of (2) we use Proposition 2.5. □

References


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