LINEAR FORMS ON MODULES OF PROJECTIVE DIMENSION ONE

by Udo Vetter

Let $R$ be a noetherian ring and $M$ an $R$–module which has a presentation

$$0 \to F \xrightarrow{\psi} G \to M \to 0$$

with finite free $R$–modules $F$ and $G$ of rank $m$ and $n$. In [2] we proved:

**Proposition 1.** Assume that $r = n - m > 1$ and that the first non-vanishing Fitting ideal of $M$ has grade $r + 1$. Then the following conditions are equivalent.

1. There is a $\chi \in M^* = \text{Hom}_R(M, R)$ such that the ideal $\text{Im} \chi$ has grade $r + 1$.
2. There exists a submodule $U$ of $M$ with the following properties:
   (i) rank $U = r - 1$;
   (ii) $U$ is reflexive, orientable, and $U_p$ is a free direct summand of $M_p$ for all primes $p$ of $R$ such that grade $p \leq r$.
3. $m = 1$ and $r$ is odd.

The equivalence $(1) \Leftrightarrow (2)$ can easily be proved directly (see Proof of Proposition 7 in [2]) while the equivalence $(1) \Leftrightarrow (3)$ results from a description of the homology of the Koszul complex associated to a linear form on $M$ (see Theorem 5 in [2]).

With the assumptions of Proposition [1] let $m = 1$ and $n \geq 4$ be even (which means that the rank $r$ of $M$ is odd). Fix a basis $e_1, \ldots, e_n$ of $G$ and let

$$\psi(1) = \sum_{i=1}^n (-1)^i x_i e_{n+1-i}.$$ 

The map $\varphi : \sum_{i=1}^n a_i e_i \mapsto \sum_{i=1}^n a_i x_i$ then obviously induces a linear form $\chi$ on $M$ such that grade $\text{Im} \chi = n$. The submodule $U = \text{Ker} \chi$ has properties (i) and (ii) of Proposition [1] and the (skewsymmetric) map $\rho : G \to G^*$ given by $\rho(e_i) = (-1)^i e_{n+1-i}^*$, $i = 1, \ldots, n$, induces an
isomorphism $\tilde{\rho} : U \to U^*$. (Here as in the following $e_1^*, \ldots, e_n^*$ denotes the basis of $G^*$ dual to $e_1, \ldots, e_n$.) So condition (2) in Proposition 1 may be replaced by

(2') There exists a submodule $U$ of $M$ with the following properties:

(i) $\text{rank } U = r - 1$;

(ii) $U$ is orientable, and $U_p$ is a free direct summand of $M_p$ for all primes $p$ of $R$ such that grade $p \leq r$;

(iii) $U$ is self-dual in a skew-symmetric way, i.e. there is an isomorphism $\rho : U \to U^*$ such that $\rho^* \circ h = -\rho$, $h : U \to U^{**}$ being the natural map.

The Koszul complex associated to $\varphi$ induces an exact sequence

\[ \bigwedge^3 G \xrightarrow{\tau} \bigwedge^2 G^* \xrightarrow{\varphi^*} \text{Ker } \varphi \to 0, \]

and there is a map $p$ from Ker $\varphi$ onto $U$ which has the kernel $\psi(1)$. So, in particular, $U$ is minimally generated by $\binom{n}{2} - 1$ elements. The aim of this note is to give an explicit construction of $U$ as a submodule of the free module $R^{(\binom{n}{2})-1}$. Since

\[ \text{Ker}(p \circ \sigma) = R \cdot \sum_{i=1}^{n/2} (-1)^{i-1} e_i \wedge e_{n+1-i} + \text{Ker } \sigma, \]

in view of (1) we obtain an exact sequence

\[ R \oplus \bigwedge^3 G \xrightarrow{\tilde{\tau}} \bigwedge^2 G^* \xrightarrow{p \circ \sigma} U \to 0, \]

where

\[ \tilde{\tau}(1,0) = \sum_{i=1}^{n/2} (-1)^{i-1} e_i \wedge e_{n+1-i} \quad \text{and} \quad \tilde{\tau}(0,y) = \tau(y) \]

for all $y \in \bigwedge^3 G$. Dualizing yields the exact sequence

\[ 0 \to U^* \to \bigwedge^2 G^* \xrightarrow{\tilde{\tau}^*} R \oplus \bigwedge^3 G^* \to 0, \]

where we used the natural isomorphims $\bigwedge^k G^* \cong (\bigwedge^k G)^*$. We shall explicitly represent $U^* = \text{Ker } \tilde{\tau}^*$ as a submodule of $\bigwedge^2 G^*$.

**Proposition 2.** The elements

\[ r_{ij} = \varphi \wedge ((-1)^j x_i e_n^* e_{n-j+1} + (-1)^{j+1} x_j e_n^* e_{n-i+1}), \]

$i, j = 1, \ldots, n$, generate $U^*$.

**Proof.** Since $\varphi \wedge \eta$ vanishes on $\text{Im } \tau$ for all $\eta \in G^*$, we have $r_{ij} \circ \tau = 0$. Moreover,

\[ r_{ij}(\tilde{\tau}(1,0)) = (x_j x_i (-1)^j e_n^* e_{n-j+1} + (-1)^{j+1} x_i x_j e_n^* e_{n-i+1})(\tilde{\tau}(1,0)) = 0. \]

So $r_{ij} \in U^*$ for all $i, j$. 
Let \( \alpha = \sum_{1 \leq k < l \leq n} a_{kl} e_k^* \wedge e_l^* \in U^* \). Then, in particular,

\[
(2) \quad a_{1n} - a_{2,n-1} + \ldots + (-1)^{\frac{n}{2}+1} a_{n,\frac{n}{2}+1} = 0.
\]

Since \( \alpha \circ \tau(e_k \wedge e_l \wedge e_m) = 0 \) for all \( k, l, m \), we have in addition, that \( a_{kl} \in Rx_k + Rx_l \) for all \( k, l \).

Next we claim that there is an element \( \beta = \sum_{1 \leq k < l \leq n} b_{kl} e_k^* \wedge e_l^* \in \sum_{i,j} R \cdot r_{ij} \), such that \( a_{kl} = b_{kl} \) for \( k + l = n + 1 \). To prove this let \( 1 \leq k < n/2 \), \( k + l = n + 1 \), and \( a_{st} = 0 \) if \( s < k, s + t = n + 1 \). We show that there is a \( \beta \) which satisfies \( b_{st} = 0 \) for \( s < k, s + t = n + 1 \), and \( b_{kl} = a_{kl} \). Because of (2) this will prove our claim. First we deduce

\[
a_{kl} \in (Rx_k + Rx_l) \cap (Rx_{k+1} + \ldots + Rx_{l-1})
\]

\[
= Rx_k x_{k+1} + \ldots + Rx_k x_{l-1} + Rx_l x_{k+1} + \ldots + Rx_l x_{l-1},
\]

since \( x_1, \ldots, x_n \) is a regular sequence in \( R \). Consider \( r_{kj}, r_{jl} \) for \( j = k + 1, \ldots, l - 1 \). Using the canonical isomorphism \( G \to G^{**} \), we get

\[
(e_s \wedge e_t)(r_{kj}) = \begin{cases} 0 & \text{if } 1 \leq s < k, \ s + t = n + 1, \\ \pm x_k x_j & \text{if } (s, t) = (k, l), \end{cases}
\]

and

\[
(e_s \wedge e_t)(r_{jl}) = \begin{cases} 0 & \text{if } 1 \leq s < k, \ s + t = n + 1, \\ \pm x_j x_l & \text{if } (s, t) = (k, l). \end{cases}
\]

So we can find an appropriate \( b \in \sum_{i,j} R \cdot r_{ij} \).

In proving the proposition, namely \( \alpha \in \sum_{i,j} R \cdot r_{ij} \), we may now assume that \( a_{kl} = 0 \) whenever \( k + l = n + 1 \). We then show that there is an element \( \gamma = \sum c_{kl} e_k^* \wedge e_l^* \in \sum_{i,j} R \cdot r_{ij} \) with \( c_{kl} = 0 \) for \( k + l = n + 1 \) and \( c_{1l} = a_{1l} \) for \( l = 2, \ldots, n/2 \). Since

\[
(3) \quad x_1 a_{1n} - x_i a_{1n} + x_n a_{1l} = 0 = x_1 a_{1,n-l+1} - x_{i} a_{1,n-l+1} + x_{n-l+1} a_{1l}
\]

(which follows from \( \alpha \circ \tau(e_1 \wedge e_l \wedge e_n) = 0 = \alpha \circ \tau(e_1 \wedge e_l \wedge e_{n-l-1}) \)) we obtain \( a_{1l} \in Rx_1 x_l \). Obviously

\[
(e_s \wedge e_t)(r_{l,n-l+1}) = \begin{cases} 0 & \text{if } s + t = n + 1 \text{ or } s = 1, t < l, \\ (-1)^{l+1} x_1 x_l & \text{if } s = 1, t = l. \end{cases}
\]

So there is an appropriate \( c \in \sum_{i,j} R \cdot r_{ij} \).

Finally suppose that \( a_{kl} = 0 \) for \( k + l = n + 1 \) and \( a_{1l} = 0 \) for \( l = 2, \ldots, n/2 \). Then, because of (3), \( a_{1j} = 0 \) for \( j = 2, \ldots, n \). Let \( 1 < i < j \leq n \). Since

\[
x_1 a_{ij} - x_i a_{1j} + x_j a_{1i} = 0,
\]

we get \( a_{ij} = 0 \). The proof is complete now. \( \square \)
Proposition 3. With the above notation, for \( i, j = 1, \ldots, n \) and \( 1 \leq k < l \leq n \)

\((*)\) \((e_k \wedge e_l)(r_{ij}) = -(e_i \wedge e_j)(r_{kl})\)

holds. Furthermore, \( r_{ii} = 0 \), \( r_{ij} = -r_{ji} \), and

\((**\)) \( r_{1n} - r_{2n-1} + \ldots + (-1)^{\frac{n}{2}+1}r_{\frac{n}{2}\frac{n+1}{2}} = 0.\)

Consequently, \( U^* \) is minimally generated by the elements \( r_{ij} \) for which \( i < j \) and \((i,j) \neq (1,n)\), and is represented by the skewsymmetric matrix

\[
(e_k \wedge e_l)(r_{ij}), \quad 1 \leq k < l \leq n, \quad 1 \leq i < j \leq n, \quad (k,l) \neq (1,n) \neq (i,j).
\]

**Proof.** Equation \((*)\) is obtained by a straightforward computation, and \((**)\) is a direct consequence of \((*)\) since for all \( k,l, k<l \):

\[
(e_k \wedge e_l)(r_{ij}) = (\sum_{i<j}i+j=n+1(-1)^{i+j}e_i \wedge e_j)(r_{kl}) = r_{kl}(\tau(1,0)) = 0.
\]

The remaining assertions follow from Proposition 2, the definition of the \( r_{ij} \), and equations \((*)\),\((**\)).

Remarks 4. Suppose that \( R = K[X_1, \ldots, X_n] \) is the polynomial ring in \( n \) indeterminates over a field \( K \). In case \( x_i = X_i \), the module \( U \) considered above seems to have already been studied in [4]; this is definitely true for \( n = 4 \). In this case it also coincides with the rank \( n-2 \) module \( M_n \) constructed in [5] which likewise satisfies conditions (i) and (ii) of Proposition 1. For \( n > 4 \), the two modules are definitely different: from [2] we know that \( \text{projdim} U = (\text{projdim} U^*) = n-2 \), while \( \text{projdim} M_n^* = 2 \) for all \( n \).

Besides the fact that \( M_n \) is defined for arbitrary \( n \geq 2 \), its dual has, in contrast to \( U \), the remarkable property to be “optimal” in view of the Evans-Griffith syzygy theorem (cf. [1], 9.5.6 for example) because \( M_n^* \) is an \((n-2)-\text{th}\) syzygy of rank \( n-2 \). A concrete description of \( M_n^* \) similar to that we gave for \( U \)...
in Proposition[3] seems to be more complicated. An attempt with SINGULAR
[3] for $n = 5, 6$ leads to the supposition that the entries of a representing matrix
are homogeneous of degree $n - 2$ if the characteristic of $K$ is $\neq 2$.

References

2. Bruns W., Vetter U., *The Koszul complex in projective dimension one*, In: Geometric and
   Combinatorial Aspects of Commutative Algebra, J. Herzog and G. Restuccia eds., Marcel
3. Greuel G.-M., Pfister G., Schönemann H., *Singular version 1.2*, University of Kaiser-
4. Trautmann G., *Darstellung von Vektorraumbündeln über $\mathbb{C}^n \setminus \{0\}$*, Arch. Math. 24 (1973),
   303–313.

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