CONVERGENCE IN CAPACITY OF THE PLURICOMPLEX GREEN FUNCTION

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Abstract. In this paper we prove that if Ω is a bounded hyperconvex domain in $\mathbb{C}^n$ and if $\Omega \ni z_j \to \partial \Omega$, $j \to \infty$, then the pluricomplex Green function $g_{\Omega}(z_j, \cdot)$ tends to 0 in capacity, as $j \to \infty$.

A bounded open connected set $\Omega \subset \mathbb{C}^n$ is called hyperconvex if there exists negative plurisubharmonic function $\psi \in PSH(\Omega)$ such that $\{ z \in \Omega : \psi(z) < c \} \subset \subset \Omega$ for all $c < 0$. Such $\psi$ is called an exhaustion function for $\Omega$. It was proved in [6] that for every hyperconvex domain there exists smooth exhaustion function $\psi$ such that $\lim_{z \to \zeta} \psi(z) = 0$, for all $\zeta \in \partial \Omega$.

Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$. Let $z \in \Omega$. Recall that the pluricomplex Green function with the pole at $z$ is defined as follows:

$$g_{\Omega}(z, w) = \sup \{ u(w) : u \in PSH(\Omega), u \leq 0, |u(\xi) - \log |\xi - z|| \leq C \text{ near } z \}.$$  

It is well known that $g_{\Omega}(z, \cdot) \in PSH(\Omega) \cap C(\Omega \setminus \{ z \})$, $g_{\Omega}(z, w) = 0$ for $w \in \partial \Omega$ and $(dd^{c}g_{\Omega}(z, \cdot))^{n} = (2\pi)^n \delta_z$, where $\delta_z$ is the Dirac measure at $z$ (see [7]). Carlehed, Cegrell and Wikstöm proved in [4] that for every $z_0 \in \partial \Omega$ there exists a pluripolar set $E \subset \Omega$ such that

$$\limsup_{z \to z_0} g_{\Omega}(z, w) = 0,$$

for every $w \in \Omega \setminus E$. Blocki and Pflug proved in [3] that if $\Omega \ni z_j \to \partial \Omega$ then $g_{\Omega}(z_j, \cdot) \to 0$ in $L^p$ for every $1 \leq p < +\infty$, as $j \to \infty$. By $z_j \to \partial \Omega$ we mean that $\text{dist}(z_j, \partial \Omega) \to 0$. This result was used in [3] to show Bergman completeness of the hyperconvex domain. Herbort proved in [5] that if a

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bounded hyperconvex domain $\Omega \subset \mathbb{C}^n$ admits a Hölder continuous exhaustion function then the pluricomplex Green function $g_{\Omega}(z_j, \cdot)$ tends to zero uniformly on compact subsets of $\Omega$ if the pole $z_j \to z_0 \in \partial \Omega$. We prove the following theorem.

**Theorem 1.** Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $\Omega \ni z_j \to \partial \Omega$, $j \to \infty$. Then $g_{\Omega}(z_j, \cdot) \to 0$ in capacity as $j \to \infty$.

First let us recall the definition of the relative capacity and of convergence in capacity.

**Definition 2.** The relative capacity of the Borel set $E \subset \Omega \subset \mathbb{C}^n$ with respect $\Omega$ is defined in \[ cap(E, \Omega) = \sup \left\{ \int_E (dd^c u)^n : u \in PSH(\Omega), -1 \leq u \leq 0 \right\}. \]

**Definition 3.** Let $u_j, u \in PSH(\Omega)$. We say that a sequence $u_j$ converges to $u$ in capacity if for any $\epsilon > 0$ and $K \subset \subset \Omega$

$$
\lim_{j \to \infty} cap(K \cap \{|u_j - u| > \epsilon\}) = 0.
$$

**Remark.** Convergence in capacity is stronger than convergence in $L^p$ since the Lebesgue measure $(d\lambda)$ is dominated by the relative capacity, i.e. there exists constant $C(n, \Omega) > 0$ depends only on $n$ and $\Omega$ such that

$$
cap(E) \geq C(n, \Omega) \lambda(E).
$$

To prove the last inequality observe that there exist constants $C_1, C_2 > 0$ depending only on $\Omega$ such that $-1 \leq C_1|z|^2 - C_2 \leq 0$ on $\Omega$ and $(dd^c(C_1|z|^2 - C_2))^n = 4^n n! C_1^n d\lambda$. Therefore the above inequality holds with $C(n, \Omega) = 4^n n! C_1^n$. Observe also that uniform convergence on compact sets is stronger then convergence in capacity, since the following inequality holds

$$
cap(K \cap \{|u_j - u| > \epsilon\}) \leq \epsilon^{-1} cap(K) sup_{K} |u_j - u|.
$$

To prove Theorem 1 we will need the following lemma proved in [2].

**Lemma 4.** Let $\Omega$ be a bounded domain $\mathbb{C}^n$. Assume that $u, v$ are bounded negative plurisubharmonic functions such that $\lim_{z \to \zeta} v(z) = 0$, for all $\zeta \in \partial \Omega$. Then

$$
\int_{\Omega} (-v)^n (dd^c u)^n \leq n!(sup \Omega |u|)^{n-1} \int_{\Omega} (-u)(dd^c v)^n.
$$

**Proof of Theorem 1.** Let us denote $u_j = g_{\Omega}(z_j, \cdot)$. Suppose that $u_j$ does not converge in capacity to 0, $j \to \infty$. Then for some $\epsilon > 0$ and $K \subset \subset \Omega$
there exist a subsequence \( u_{j_k} \), and constants \( c > 0 \) and \( N > 0 \) such that for \( j_k \geq N \) we have

\[
\text{cap}(K \cap \{-u_{j_k} > \epsilon\}) \geq c.
\]

From the definition of capacity there exists \( v \in \text{PSH}(\Omega) \) such that \(-1 \leq v \leq 0\) and

\[
\int_{K \cap \{-u_{j_k} > \epsilon\}} (dd^c v)^n \geq \frac{c}{2}.
\]

Now we will show that \( u_j \to 0 \) on \( K \) in \( L^n((dd^c v)^n) \). Since \( \Omega \) is hyperconvex then there exist \( \psi \) a continuous exhaustion function for \( \Omega \) and a constant \( A > 0 \) such that \( A\psi < v \) on \( K \). Define the following bounded plurisubharmonic function \( \varphi = \max(A\psi, v) \). Then \( \lim_{z \to \zeta} \varphi(z) = 0 \), for all \( \zeta \in \partial \Omega \) and

\[
(dd^c \varphi)^n \geq \chi_K(dd^c v)^n,
\]

where \( \chi_K \) is the characteristic function of the set \( K \). Observe that \( \varphi \) is an exhaustion function for \( \Omega \), which implies that \( \varphi(z_j) \to 0 \) if \( \text{dist}(z_j, \partial \Omega) \to 0 \).

Using the monotone convergence theorem and Lemma \ref{lem:monotone} we get

\[
\int_K (-u_j)^n(dd^c v)^n = \int_\Omega (-u_j)^n(dd^c \varphi)^n = \lim_{k \to +\infty} \int_\Omega (-\max(u_j, -k))^n(dd^c \varphi)^n
\]

\[
\leq n!(\sup_\Omega |\varphi|)^{n-1} \lim_{k \to +\infty} \int_\Omega \varphi((dd^c \max(u_j, -k))^n = n!(2\pi)^n(\sup_\Omega |\varphi|)^{n-1}\varphi(z_j),
\]

which means that \( u_j \to 0 \) on \( K \) in \( L^n((dd^c v)^n) \), since \( \varphi(z_j) \to 0 \), as \( j \to \infty \).

Observe that inequality \ref{eq:inequality} implies that

\[
\frac{c}{2} \leq \int_{K \cap \{-u_{j_k} > \epsilon\}} (dd^c v)^n \leq \epsilon^{-n} \int_K (-u_{j_k})^n(dd^c v)^n,
\]

which is impossible since \( u_{j_k} \to 0 \) on \( K \) in \( L^n((dd^c v)^n) \). This means that \( u_j \to 0 \) in capacity as \( j \to \infty \). The proof is finished.

Now we recall the definition of the multipolar Green function introduced by Lelong \cite{Lelong}. Let \( A = \{(z^{(1)}, \nu^{(1)}), \ldots, (z^{(m)}, \nu^{(m)})\} \) be a finite subset of \( \Omega \times \mathbb{R}_+ \).

Let

\[
g_\Omega(A, w) = \sup\{u(w) : u \in \mathcal{L}_A, u \leq 0\},
\]

where \( \mathcal{L}_A \) denotes the family of plurisubharmonic functions on \( \Omega \) having a logarithmic pole with weight \( \nu^{(k)} \) at \( w^{(k)} \), for \( k = 1, \ldots, m \), i.e.

\[
\mathcal{L}_A = \{u \in \text{PSH}(\Omega) : |u(\xi) - \nu^{(j)} \log |\xi - z^{(j)}|| \leq C_j \text{ near } z^{(j)}, 1 \leq j \leq m\}.
\]

We show that it is possible to generalize Theorem \ref{thm:generalized} for the multipolar Green function.
Corollary 5. Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $A_j = \{(z^{(1)}_j, \nu^{(1)}_j), \ldots, (z^{(m)}_j, \nu^{(m)}_j)\}$ be a subset of $\Omega \times \mathbb{R}_+$, for $j = 1, 2, \ldots$, such that $\Omega \ni z^{(k)}_j \to \partial \Omega$, $j \to \infty$ for all $k = 1, \ldots, m$. Then $g_{\Omega}(A_j, \cdot) \to 0$ in capacity as $j \to \infty$.

Proof. Directly from the definition of the multipolar Green function we have
\[
\sum_{k=1}^{m} \nu^{(k)}_j g_{\Omega}(z^{(k)}_j, \cdot) \leq g_{\Omega}(A_j, \cdot) \leq 0.
\]
By Theorem 1 we have that $g_{\Omega}(z^{(k)}_j, \cdot) \to 0$ in capacity as $j \to \infty$ for all $k = 1, \ldots, m$, so also $g_{\Omega}(A_j, \cdot) \to 0$ in capacity as $j \to \infty$. This ends the proof.

References


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