A NOTE ON ALEXANDER’S THEOREM

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Abstract. The aim of this note is to extend a result of H. Alexander \cite{1} from the case of scalar functions to the case of functions with values in topological vector spaces.

Let \( B := \{ z \in \mathbb{C}^N ; \| z \| < 1 \} \) be the unit ball in \( \mathbb{C}^N \) with respect to a complex norm \( \| \cdot \| \). Given a subset \( E \) of the unit sphere \( \partial B \), let \( \rho = \rho(E) \) be the radius of the maximal ball \( rB \) contained in the set \( \text{Int}(\bigcap \Omega) \), where the intersection is taken over all balanced domains of holomorphy \( \Omega \) containing \( E \). It is known \cite{3, 4} that \( \rho \) is a Choquet capacity characterizing non-pluripolar complex cones in \( \mathbb{C}^N \). Namely, if \( V \) is a complex cone in \( \mathbb{C}^N \) with vertex at \( 0 \) then \( V \) is pluripolar if and only if \( E := V \cap \partial B \) is pluripolar, if and only if \( \rho(E) = 0 \).

Let \( F \) be a sequentially complete topological vector space over \( \mathbb{C} \). Let \( \Gamma \) be a set of continuous seminorms determining the topology of \( F \).

In 1974 H. Alexander \cite{1} proved (among others) that if \( \{ f_n \} \) is a sequence of holomorphic functions on the unit ball \( B \) such that the restriction of \( \{ f_n \} \) to each complex line \( L \) through the center \( 0 \) of \( B \) is uniformly convergent in a neighborhood of \( 0 \) in \( L \) then \( \{ f_n \} \) converges uniformly in a neighborhood of \( 0 \) in \( B \).

The goal of this note is to extend this result to the case where the target space \( \mathbb{C} \) is replaced by any sequentially complete complex topological vector space \( F \).

The main result of this article is given by the following theorem.

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Theorem A. Let $E$ be a circled non-pluripolar subset of the unit sphere $\partial B$ in $\mathbb{C}^N$. Let $\mathcal{X}$ be a family of $F$-valued holomorphic functions in the unit ball $B$ such that $\forall a \in E \exists r_a > 0 \forall q \in \mathbb{Z}_{M_q > 0}$

(a) $q(f(\lambda a)) \leq M_q$, $|\lambda| \leq r_a$, $f \in \mathcal{X}$.

Then there exists $r > 0$ such that $\forall q \in \mathbb{Z}_{M_q > 0}$ such that

(b) $q(f(z)) \leq M_q$, $\|z\| \leq r$, $f \in \mathcal{X}$.

Corollary 1. Let $V$ be a non-pluripolar complex cone in $\mathbb{C}^N$ with vertex at 0. Then for every family $X$ of $F$-valued holomorphic functions on $B$ such that for every complex line $L \subset V$ with $0 \in L$ the family $\mathcal{X}_L := \{f|_{B \cap L}; f \in \mathcal{X}\}$ of holomorphic functions of a complex variable in the disk $B \cap L$ is uniformly bounded on a neighborhood (dependent on $L$) of $0 \in \mathbb{C}$, then there exists $r > 0$ such that $\mathcal{X}$ is uniformly bounded on the ball $rB$.

This and Vitali’s theorem [2] imply the following Corollary 2 which is the Alexander theorem in the case of functions with values in sequentially complete topological vector spaces.

Corollary 2. Let $V$ be a non-pluripolar complex cone in $\mathbb{C}^N$. If $\mathcal{X} = \{f_n\}$ is a sequence of $F$-valued holomorphic functions in the unit ball $B \subset \mathbb{C}^N$ such that for every complex line $L \subset V$ with $0 \in L$ the sequence $\{f_n|_{L \cap B}\}$ is uniformly convergent on a neighborhood (dependent on $L$) of $0 \in \mathbb{C}$, then there exists $r > 0$ such that the sequence $\mathcal{X}$ is uniformly convergent on the ball $rB$.

Proof of Theorem A. We have

$$f(z) = \sum_{n=0}^{\infty} P_n(z, f), \quad \|z\| < 1, \quad f \in \mathcal{X},$$

where $P_n(z, f) := \sum_{|\alpha|=n} \frac{f^{(\alpha)}(0)}{\alpha!} z^\alpha$ is the $n$th homogeneous polynomial of the Taylor series development of $f$ around 0. In particular, $f(\lambda a) = \sum_{n=0}^{\infty} P_n(a, f) \lambda^n$, $|\lambda| < 1$, $a \in E$, $f \in \mathcal{X}$. Hence, by (1),

$$q(P_n(a, f)) \leq \frac{M_q}{r_a^n}, \quad n \geq 0, \quad a \in E, \quad f \in \mathcal{X}.$$ 

The function

$$\varphi_n(z) := \frac{1}{n} \log \sup_{f \in \mathcal{X}} q(P_n(z, f)), \quad z \in \mathbb{C}^N, \quad n \geq 1,$$

is a continuous PSH function of the Lelong class $\mathcal{L}$.

Put $E_s := \{a \in E; \varphi_n(a) \leq s, \quad n \geq 1\}$. By $\bigcup_{s=1}^{\infty} E_s = E$ and $E_s \subset E_{s+1}$ for all $s \geq 1$. Therefore $\lim_{s \to \infty} \rho(E_s) = \rho \equiv \rho(E)$.
Fix $0 < \theta < 1$ and take $s = s_q$ so large that $\rho(E_{a}) \geq \theta \rho$. Then by the Bernstein–Walsh inequality for the homogeneous functions of Lelong class we get

$$\varphi_n(z) \leq s_q + \log \frac{||z||}{\theta \rho}, \quad n \geq 1, \quad z \in \mathbb{C}^N.$$  

Put $\varphi(z) := \limsup_{n \to \infty} \varphi_n(z)$. The sequence $\{\varphi_n\}$ is locally uniformly upper bounded in $\mathbb{C}^N$. Therefore $\varphi^*$ is a homogeneous function of the Lelong class. By Bedford–Taylor theorem on negligible sets there exists a circled non-pluripolar subset $E_0$ of $E$ such that $\rho(E_0) = \rho(E)$ and $\varphi^*(z) = \varphi(z)$ for all $z \in E_0$. Put $A_s := \{a \in E_0; \varphi(a) \leq s\}$. By (1) there exists $s$ such that $\rho(A_s) \geq \theta \rho$. Hence, by Bernstein–Walsh inequality, we get

$$\varphi(z) \leq \varphi^*(z) \leq s + \log \frac{||z||}{\theta \rho}, \quad z \in \mathbb{C}^N.$$  

Observe that the number $s$ does not depend on $q \in \Gamma$. It depends only on $\theta$ and on the function $E \ni a \to \rho(a) \in (0, \infty)$.

By the Hartogs Lemma for every $q \in \Gamma$ there is $n_q$ such that

$$q(P_n(z, f)) \leq m_q \left( \frac{e^{s+1}||z||}{\theta \rho} \right)^n, \quad 0 \leq n \leq n_q, z \in \mathbb{C}^N, \quad f \in \mathcal{X}.$$  

From (2) and (3) one gets

$$q(P_n(z, f)) \leq m_q \left( \frac{e^{s+1}||z||}{\theta \rho} \right)^n, \quad n \geq 0, \quad f \in \mathcal{X}, \quad z \in \mathbb{C}^N.$$  

It follows that

$$q(f(z)) \leq \frac{m_q}{1 - \theta}, \quad ||z|| \leq \theta^2 \rho e^{-s-1}, \quad f \in \mathcal{X}.$$  

Hence $q(f(z)) \leq M_q$ for all $f \in \mathcal{X}$ and $||z|| \leq r$, where $M_q := m_q / (1 - \theta)$, $r := \theta^2 \rho e^{-s-1}$.

**Corollary from the proof.** If a family $\mathcal{X}$ satisfies (a) with $r_a = r_0 = \text{const}$, $a \in E$ where $0 < r_0 \leq 1$ then the family is locally uniformly bounded in the ball $rB$ with $r := r_0 \rho$, $\rho = \rho(E)$. 


References


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