CHARACTERIZATION OF NON-DEGENERATE PLANE CURVE SINGULARITIES

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Abstract. We characterize plane curve germs (non-degenerate in Kouch-nirenko’s sense) in terms of characteristics and intersection multiplicities of branches.

1. Introduction. In this paper we consider (reduced) plane curve germs $C, D, \ldots$ centered at a fixed point $O$ of a complex nonsingular surface. Two germs $C$ and $D$ are equisingular if there exists a bijection between their branches which preserves characteristic pairs and intersection numbers. Let $(x, y)$ be a chart centered at $O$. Then a plane curve germ has a local equation of the form $\sum c_{\alpha, \beta} x^{\alpha} y^{\beta} = 0$. Here $\sum c_{\alpha, \beta} x^{\alpha} y^{\beta}$ is a convergent power series without multiple factors. The Newton diagram $\Delta_{x,y}(C)$ is defined to be the convex hull of the union of quadrants $(\alpha, \beta) + (\mathbb{R}_+)^2, c_{\alpha, \beta} \neq 0$. Recall that the Newton boundary $\partial \Delta_{x,y}(C)$ is the union of the compact faces of $\Delta_{x,y}(C)$. A germ $C$ is called non-degenerate with respect to the chart $(x, y)$ if the coefficients $c_{\alpha, \beta}$, where $(\alpha, \beta)$ runs over integral points lying on the faces of $\Delta_{x,y}(C)$, are generic (see Preliminaries to this Note for the precise definition). It is a well-known fact that the equisingularity class of a germ $C$ non-degenerate with respect to $(x, y)$ depends exclusively on the Newton polygon formed by the faces of $\Delta_{x,y}(C)$: if $(r_1, s_1), (r_2, s_2), \ldots, (r_k, s_k)$ are subsequent vertices of $\partial \Delta_{x,y}(C)$, then the germs $C$ and $C'$ with local equation $x^{r_1} y^{s_1} + \cdots + x^{r_k} y^{s_k} = 0$ are equisingular. Our aim is to give an explicit description of the non-degenerate plane curve germs in terms of characteristic pairs and intersection numbers of branches. In particular, we show that if two germs $C$ and $D$ are equisingular,
then \( C \) is non-degenerate if and only if \( D \) is non-degenerate. The proof of our result is based on a refined version of Kouchnirenko’s formula for the Milnor number and on the concept of contact exponent.

2. Preliminaries. Let \( \mathbb{R}_+ = \{ x \in \mathbb{R} : x \geq 0 \} \). For any subsets \( A, B \) of the quarter \( \mathbb{R}_+^2 \), we consider the arithmetic sum \( A + B = \{ a + b : a \in A \) and \( b \in B \} \). If \( S \subset \mathbb{N}^2 \), then \( \Delta(S) \) is the convex hull of the set \( S + \mathbb{R}_+^2 \). The subset \( \Delta \) of \( \mathbb{R}_+^2 \) is a Newton diagram if \( \Delta = \Delta(S) \) for a set \( S \subset \mathbb{N}^2 \) (see [11, 12]). Following Teissier we put \( \{ \frac{a}{b} \} = \Delta(S) \) if \( S = \{(a, 0), (0, b)\} \), \( \{ \frac{a}{b} \} = (a, 0) + \mathbb{R}_+^2 \), and \( \{ \frac{a}{b} \} = (0, b) + \mathbb{R}_+^2 \) for any \( a, b > 0 \) and call such diagrams elementary Newton diagrams. The Newton diagrams form a semigroup \( \mathcal{N} \) with respect to the arithmetic sum. The elementary Newton diagrams generate \( \mathcal{N} \). If \( \Delta = \sum_{i=1}^{r} \left( \frac{a_i}{b_i} \right) \), then \( a_i/b_i \) are the inclinations of edges of the diagram \( \Delta \) (by convention, \( \frac{a}{\infty} = 0 \) and \( \frac{\infty}{b} = \infty \) for \( a, b > 0 \)). We also put \( a + \infty = \infty \), \( a \cdot \infty = \infty \), \( \inf \{ a, \infty \} = a \) if \( a > 0 \) and \( 0 \cdot \infty = 0 \).

Minkowski’s area \( \Delta, \Delta' \in \mathbb{N} \cup \{ \infty \} \) of two Newton diagrams \( \Delta, \Delta' \) is uniquely determined by the following conditions:

\[
\begin{align*}
(m_1) & \quad [\Delta_1 + \Delta_2, \Delta'] = [\Delta_1, \Delta'] + [\Delta_2, \Delta'], \\
(m_2) & \quad [\Delta, \Delta'] = [\Delta', \Delta], \\
(m_3) & \quad \left[ \left\{ \frac{a}{b} \right\}, \left\{ \frac{a'}{b'} \right\} \right] = \inf \{ ab', a'b \}.
\end{align*}
\]

We define the Newton number \( \nu(\Delta) \in \mathbb{N} \cup \{ \infty \} \) by the following properties:

\[
\begin{align*}
(v_1) & \quad \nu(\sum_{i=1}^{k} \Delta_i) = \sum_{i=1}^{k} \nu(\Delta_i) + 2 \sum_{1 \leq i < j \leq k} [\Delta_i, \Delta_j] - k + 1, \\
(v_2) & \quad \nu\left( \left\{ \frac{a}{b} \right\} \right) = (a - 1)(b - 1), \quad \nu\left( \left\{ \frac{1}{\infty} \right\} \right) = 0.
\end{align*}
\]

A diagram \( \Delta \) is convenient (resp., nearly convenient) if \( \Delta \) intersects both axes (resp., if the distances of \( \Delta \) to the axes are \( \leq 1 \)). Note that \( \Delta \) is nearly convenient if and only if \( \nu(\Delta) \neq \infty \). Fix a complex nonsingular surface, i.e., a complex holomorphic variety of dimension 2. Throughout this paper, we consider reduced plane curve germs \( C, D, \ldots \) centered at a fixed point \( O \) of this surface. We denote by \((C, D)\) the intersection multiplicity of \( C \) and \( D \) and by \( m(C) \) the multiplicity of \( C \). There is \((C, D) \geq m(C)m(D)\); if \((C, D) = m(C)m(D)\), then we say that \( C \) and \( D \) intersect transversally. Let \((x, y)\) be a chart centered at \( O \). Then a plane curve germ \( C \) has a local equation \( f(x, y) = \sum c_{\alpha \beta} x^\alpha y^\beta \in \mathbb{C}(x, y) \) without multiple factors. We put \( \Delta_{x,y}(C) = \Delta(S) \), where \( S = \{(\alpha, \beta) : \alpha, \beta \in \mathbb{N}^2 : c_{\alpha \beta} \neq 0\} \). Clearly, \( \Delta_{x,y}(C) \) depends on \( C \) and \((x, y)\). We note two fundamental properties of Newton diagrams:
\((N_1)\) If \((C_i)\) is a finite family of plane curve germs such that \(C_i\) and \(C_j\) \((i \neq j)\) have no common irreducible component, then

\[
\Delta_{x,y} \left( \bigcup_i C_i \right) = \sum_i \Delta_{x,y}(C_i).
\]

\((N_2)\) If \(C\) is an irreducible germ (a branch) then

\[
\Delta_{x,y}(C) = \begin{cases} (C, y = 0) & \text{if } x \neq 0 \\ (C, x = 0) & \text{if } y \neq 0 \end{cases}.
\]

For the proof, we refer the reader to \[1\], pp. 634–640.

The topological boundary of \(\Delta_{x,y}(C)\) is the union of two half-lines and a finite number of compact segments (faces). For any face \(S\) of \(\Delta_{x,y}(C)\) we let

\[
\frac{\partial f_S}{\partial x}(x, y) = \frac{\partial f_S}{\partial y}(x, y) = 0
\]

has no solutions in \(\mathbb{C}^* \times \mathbb{C}^*\). We say that the germ \(C\) is non-degenerate if there exists a chart \((x, y)\) such that \(C\) is non-degenerate with respect to \((x, y)\).

For any reduced plane curve germs \(C\) and \(D\) with irreducible components \((C_i)\) and \((D_j)\), we put

\[
d(C, D) = \inf_{i,j} \{(C_i, D_j)/(m(C_i)m(D_j))\}
\]

call \(d(C, D)\) the order of contact of germs \(C\) and \(D\). Then for any \(C, D, E\):

\[
\begin{align*}
(d_1) & \quad d(C, D) = \infty \text{ if and only if } C = D \text{ is a branch,} \\
(d_2) & \quad d(C, D) = d(D, C), \\
(d_3) & \quad d(C, D) \geq \inf\{d(C, E), d(E, D)\}.
\end{align*}
\]

The proof of \((d_3)\) is given in \[2\] for the case of irreducible \(C, D, E\), which implies the general case. Condition \((d_3)\) is equivalent to the following: at least two of three numbers \(d(C, D), d(C, E), d(E, D)\) are equal and the third is not smaller than the other two. For each germ \(C\), we define

\[
d(C) = \sup\{d(C, L) : L \text{ runs over all smooth branches}\}
\]

and call \(d(C)\) the contact exponent of \(C\) (see \[4\], Definition 1.5, where the term “characteristic exponent” is used). Using \((d_3)\) we check that \(d(C) \leq d(C, C)\).

\((d_4)\) For every finite family \((C_i)\) of plane curve germs we have

\[
d(\bigcup_i C_i) = \inf\{\inf d(C^i), \inf d(C^i, C^j)\}.
\]

The proof of \((d_4)\) is given in \[3\] (see Proposition 2.6). We say that a smooth germ \(L\) has maximal contact with \(C\) if \(d(C, L) = d(C)\). Note that \(d(C) = \infty\) if and only if \(C\) is a smooth branch. If \(C\) is singular then \(d(C)\) is a rational
number and there exists a smooth branch $L$ which has maximal contact with $C$ (see [4, 1]).

3. Results. Let $C$ be a plane curve germ. A finite family of germs $(C^{(i)})_i$ is called a decomposition of $C$ if $C = \cup_i C^{(i)}$ and $C^{(i)}, C^{(i_1)}$ ($i \neq i_1$) have no common branch. The following definition will play a key role.

**Definition 3.1.** A plane curve $C$ is Newton’s germ (shortly an $\mathcal{N}$-germ) if there exists a decomposition $(C^{(i)})_{1 \leq i \leq s}$ of $C$ such that the following conditions hold

1. $1 \leq d(C^{(1)}) < \ldots < d(C^{(s)}) \leq \infty$.
2. Let $(C^{(i)})_j$ be branches of $C^{(i)}$. Then
   - (a) if $d(C^{(i)}) \in \mathbb{N} \cup \{\infty\}$ then the branches $(C^{(i)})_j$ are smooth,
   - (b) if $d(C^{(i)}) \notin \mathbb{N} \cup \{\infty\}$ then there exists a pair of coprime integers $(a_i, b_i)$ such that each branch $C^{(i)}_j$ has exactly one characteristic pair $(a_i, b_i)$.
   Moreover, $d(C^{(i)}_j) = d(C^{(i)})$ for all $j$.
3. If $C^{(i)}_l \neq C^{(i_1)}_k$, then $d(C^{(i)}_l, C^{(i_1)}_k) = \inf\{d(C^{(i)}), d(C^{(i_1)})\}$.

A branch is Newton’s germ if it is smooth or has exactly one characteristic pair. Let $C$ be Newton’s germ. The decomposition $(C^{(i)})$ satisfying (1), (2) and (3) is not unique. Take for example a germ $C$ that has all $r > 2$ branches smooth intersecting with multiplicity $d > 0$. Then for any branch $L$ of $C$, we may put $C^{(1)} = C \setminus \{L\}$ and $C^{(2)} = \{L\}$ (or simply $C^{(1)} = C$). If $C$ and $D$ are equisingular germs, then $C$ is an $\mathcal{N}$-germ if and only if $D$ is an $\mathcal{N}$-germ.

Our main result is

**Theorem 3.2.** Let $C$ be a plane curve germ. Then the following two conditions are equivalent

1. The germ $C$ is non-degenerate with respect to a chart $(x, y)$ such that $C$ and $\{x = 0\}$ intersect transversally,
2. $C$ is Newton’s germ.

We give a proof of Theorem 3.2 in Section 5 of this paper. Let us note here

**Corollary 3.3.** If a germ $C$ is unitangent, then $C$ is non-degenerate if and only if $C$ is an $\mathcal{N}$-germ.

Every germ $C$ has the tangential decomposition $(\tilde{C}^i)_{i=1,\ldots,t}$ such that

1. $\tilde{C}^i$ are unitangent, that is for every two branches $\tilde{C}^i_j, \tilde{C}^i_k$ of $\tilde{C}^i$ there is $d(\tilde{C}^i_j, \tilde{C}^i_k) > 1$.
2. $d(\tilde{C}^i, \tilde{C}^{i_1}) = 1$ for $i \neq i_1$. 
We call \((\tilde{C}^i)_{i=1,...,t}\) tangential components of \(C\). Note that \(t(C) = t\) (the number of tangential components) is an invariant of equisingularity.

**Theorem 3.4.** If \((\tilde{C}^i)_{i=1,...,t}\) is the tangential decomposition of the germ \(C\) then the following two conditions are equivalent

1. The germ \(C\) is non-degenerate.
2. All tangential components \(\tilde{C}^i\) of \(C\) are \(N\)-germs and at least \(t(C) - 2\) of them are smooth.

Using Theorem 3.4, we get

**Corollary 3.5.** Let \(C\) and \(D\) be equisingular plane curve germs. Then \(C\) is non-degenerate if and only if \(D\) is non-degenerate.

4. Kouchnirenko’s theorem for plane curve singularities.

Let \(\mu(C)\) be the Milnor number of a reduced germ \(C\). By definition, \(\mu(C) = \dim \mathbb{C}\{x,y\}/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})\), where \(f = 0\) is an equation without multiple factors of \(C\). The following properties are well-known (see e.g. [9]).

\((\mu_1)\) \(\mu(C) = 0\) if and only if \(C\) is a smooth branch.

\((\mu_2)\) If \(C\) is a branch with the first characteristic pair \((a, b)\) then \(\mu(C) \geq (a - 1)(b - 1)\). Moreover, \(\mu(C) = (a - 1)(b - 1)\) if and only if \((a, b)\) is the unique characteristic pair of \(C\).

\((\mu_3)\) If \((C^{(i)})_{i=1,...,k}\) is a decomposition of \(C\), then

\[
\mu(C) = \sum_{i=1}^{k} \mu(C^{(i)}) + 2 \sum_{1 \leq i < j \leq k} (C^{(i)}, C^{(j)}) - k + 1.
\]

Now we can give a refined version of Kouchnirenko’s theorem in two dimensions.

**Theorem 4.1.** Let \(C\) be a reduced plane curve germ. Fix a chart \((x, y)\).

Then \(\mu(C) \geq \nu(\Delta_{x,y}(C))\) with equality holding if and only if \(C\) is non-degenerate with respect to \((x, y)\).

**Proof.** Let \(f = 0, f \in \mathbb{C}\{x,y\}\) be the local equation without multiple factors of the germ \(C\). To abbreviate the notation, we put \(\mu(f) = \mu(C)\) and \(\Delta(f) = \Delta_{x,y}(C)\). If \(f = x^a y^b \varepsilon(x,y)\) in \(\mathbb{C}\{x,y\}\) with \(\varepsilon(0,0) \neq 0\) then the theorem is obvious. Then we can write \(f = x^a y^b f_1\) in \(\mathbb{C}\{x,y\}\), where \(a, b \in \{0,1\}\) and \(f_1 \in \mathbb{C}\{x,y\}\) is an appropriate power series. A simple calculation based on properties \([\nu_2]\), \([\mu_3]\) and \([\mu_1]\), \([\nu_2]\) shows that \(\mu(f) - \nu(\Delta(f)) = \mu(f_1) - \nu(\Delta(f_1))\). Moreover, \(f\) is non-degenerate if and only if if \(f_1\) is non-degenerate and the theorem reduces to the case of an appropriate power series which is proved in [8] (Theorem 1.1).
Remark 4.2. The implication \( \mu(C) = \nu(\Delta_{x,y}(C)) \Rightarrow C \) is non-degenerate” is not true for hypersurfaces with isolated singularity (see [5], Remarque 1.21).

Corollary 4.3. For any reduced germ \( C \), there is \( \mu(C) \geq (m(C) - 1)^2 \).

The equality holds if and only if \( C \) is an ordinary singularity, i.e., such that \( t(C) = m(C) \).

Proof. Use Theorem 4.1 in generic coordinates. \( \square \)

5. Proof of Theorem 3.2. We start with the implication (1) \( \Rightarrow \) (2). Let \( C \) be a plane curve germ and let \( (x, y) \) be a chart such that \( \{ x = 0 \} \) and \( C \) intersect transversally. The following result is well-known ([7], Proposition 4.7).

Lemma 5.1. There exists a decomposition \( \{ C(i) \}_{i=1,\ldots,s} \) of \( C \) such that

1. \( \Delta_{x,y}(C(i)) = \left\{ \left\{ C(i), y = 0 \right\} \right\}_i \)
2. Let \( d_i = \frac{(C(i), y=0)}{m(C(i))} \). Then \( 1 \leq d_1 < \cdots < d_s \leq \infty \) and \( d_s = \infty \) if and only if \( C(s) \) is non-degenerate with respect to the chart \( (x, y) \).
3. Let \( n_i = m(C(i)) \) and \( m_i = n_i d_i = (C(i), y = 0) \). Suppose that \( C \) is non-degenerate with respect to the chart \( (x, y) \). Then \( C(i) \) has \( r_i = \gcd(n_i, m_i) \) branches \( C_j(i) : y^{r_i/m_i} - a_{ij} x^{m_i/r_i} + \cdots = 0 \) \( (j = 1, \ldots, r_i) \) and \( a_{ij} \neq a_{i'j} \), if \( j \neq j' \).

Using the above lemma, we prove that any germ \( C \) which is non-degenerate with respect to \( (x, y) \) is an \( N \)-germ. From \( d_1 \) we get \( d(C(i)) = d_i \). Clearly, each branch \( C_j(i) \) has exactly one characteristic pair \( \left( \frac{n_i}{r_i}, \frac{m_i}{r_i} \right) \) or is smooth. A simple calculation shows that

\[
d(C_j(i), C_j(i_1)) = \frac{(C_j(i), C_j(i_1))}{m(C_j(i)) m(C_j(i_1))} = \inf\{d_i, d_{i_1}\}.
\]

To prove the implication (2) \( \Rightarrow \) (1), we need some auxiliary lemmas.

Lemma 5.2. Let \( C \) be a plane curve germ whose all branches \( C_i \) \( (i = 1, \ldots, s) \) are smooth. Then there exists a smooth germ \( L \) such that \( (C_i, L) = d(C) \) for \( i = 1, \ldots, s \).

Proof. If \( d(C) = \infty \), then \( C \) is smooth and we take \( L = C \). If \( d(C) = 1 \), then we take a smooth germ \( L \) such that \( C \) and \( L \) are transversal. Let \( k = d(C) \) and suppose that \( 1 < k < \infty \). By formula \( d_k \), we get \( \inf\{C_i, C_j \} : i, j = 1, \ldots, s = k \). We may assume that \( (C_1, C_2) = \cdots = (C_1, C_r) = k \) and \( (C_1, C_j) > k \) for \( j > r \) for an index \( r, 1 \leq r \leq s \). There is a system of
coordinates \((x, y)\) such that \(C_j\) \((j = 1, \ldots, r)\) have equations \(y = c_jx^k + \ldots\) It suffices to take \(L : y - cx^k = 0\), where \(c \neq c_j\) for \(j = 1, \ldots, r\).

**Lemma 5.3.** Suppose that \(C\) is an \(N\)-germ and let \((C^{(i)})_{1 \leq i \leq s}\) be a decomposition of \(C\) as in Definition 3.1. Then there is a smooth germ \(L\) such that \(d(C^{(i)}_j, L) = d(C^{(i)})\) for all \(j\).

**Proof.** Step 1. There is a smooth germ \(L\) such that \(d(C^{(s)}_j, L) = d(C^{(s)})\) for all \(j\). If \(d(C^{(s)}) \in \mathbb{N} \cup \{\infty\}\), then the existence of \(L\) follows from Lemma 5.2. If \(d(C^{(s)}) \notin \mathbb{N} \cup \{\infty\}\), then all components \(C^{(s)}_j\) have the same characteristic pair \((a_s, b_s)\). Fix a component \(C^{(s)}_{j_0}\) and let \(L\) be a smooth germ such that \(d(C^{(s)}_{j_0}, L) = d(C^{(s)}_j)\).

Let \(j_1 \neq j_0\). Then \(d(C^{(s)}_{j_1}, L) \geq \inf \{d(C^{(s)}_{j_1}, C^{(s)}_{j_0}), d(C^{(s)}_{j_0}, L)\} = d(C^{(s)}_j)\). On the other hand, \(d(C^{(s)}_{j_1}, L) \leq d(C^{(s)}_{j_1}) = d(C^{(s)}_j)\) and we get \(d(C^{(s)}_{j_1}, L) = d(C^{(s)}_j)\).

Step 2. Let \(L\) be a smooth germ such that \(d(C^{(s)}_j, L) = d(C^{(s)})\) for all \(j\). We will check that \(d(C^{(i)}_j, L) = d(C^{(i)})\) for each \(i\) and \(j\). To this purpose, fix \(i < s\). Let \(C^{(s)}_{j_0}\) be a component of \(C^{(s)}\). Then \(d(C^{(i)}_j, C^{(s)}_{j_0}) = \inf \{d(C^{(i)}_j), d(C^{(s)}_{j_0})\} = d(C^{(i)})\) and \(d(C^{(s)}_{j_0}, C^{(i)}_j) = \inf \{d(C^{(s)}_{j_0}), d(C^{(i)}_j)\} = d(C^{(i)}).

By the above lemma, we get \(d(C^{(i)}_j, L) \geq \inf \{d(C^{(i)}_j, C^{(s)}_{j_0}), d(C^{(s)}_{j_0}, L)\} = \inf \{d(C^{(i)}_j), d(C^{(s)}_{j_0})\} = d(C^{(i)}_j)\). On the other hand, \(d(C^{(i)}_j, L) \leq d(C^{(i)}_j) = d(C^{(i)})\), which completes the proof.

**Remark 5.4.** In the notation of the above lemma we have \((C^{(i)}_j, L) = m(C^{(i)})d(C^{(i)})\) for \(i = 1, \ldots, s\).

Indeed, if \(C^{(i)}_j\) are branches of \(C^{(i)}\), then
\[
(C^{(i)}_j, L) = \sum_j (C^{(i)}_j, L) = \sum_j m(C^{(i)}_j)d(C^{(i)}_j, L) = \sum_j m(C^{(i)}_j)d(C^{(i)}) = m(C^{(i)})d(C^{(i)}).
\]

**Lemma 5.5.** Let \(C\) be an \(N\)-germ and let \((C^{(i)})_{1 \leq i \leq s}\) be a decomposition of \(C\) as in Definition 3.1. Then
\[
\mu(C) = \sum_i (m(C^{(i)}) - 1)(m(C^{(i)})d(C^{(i)}) - 1) + 2 \sum_{i < j} m(C^{(i)})m(C^{(j)}) \inf \{d(C^{(i)}), d(C^{(j)})\} - s + 1.
\]

**Proof.** Use properties \((\mu_1), (\mu_2)\) and \((\mu_3)\) of the Milnor number.
To prove implication (2) ⇒ (1) of Theorem 3.2 suppose that $C$ is an $N$-germ and let $(C^{(i)})_{i=1,...,s}$ be a decomposition of $C$ such as in Definition 3.1. Let $L$ be a smooth branch such that $(C^{(i)}, L) = m(C^{(i)})d(C^{(i)})$ for $i = 1, \ldots, s$ (such a branch exists by Lemma 5.3 and Remark 5.4). Take a system of coordinates such that \{x = 0\} and $C$ are transversal and $L = \{y = 0\}$. Then we get

$$\Delta_{x,y}(C) = \sum_{i=1}^{s} \Delta_{x,y}(C^{(i)}) = \sum_{i=1}^{s} \left\{ \frac{(C^{(i)}, \{y = 0\})}{m(C^{(i)})} \right\} = \sum_{i=1}^{s} \left\{ \frac{m(C^{(i)})d(C^{(i)})}{m(C^{(i)})} \right\}$$

and consequently

$$\nu(\Delta_{x,y}(C)) = \sum_{i=1}^{s} (m(C^{(i)}) - 1)(m(C^{(i)})d(C^{(i)}) - 1)$$

$$+ 2 \sum_{1 \leq i < j \leq s} m(C^{(i)})m(C^{(j)}) \inf\{d(C^{(i)}), d(C^{(j)})\} - s + 1$$

$$= \mu(C)$$

by Lemma 5.5. Therefore, $\mu(C) = \nu(\Delta_{x,y}(C))$ and $C$ is non-degenerate with respect to $(x, y)$ by Theorem 4.1.

6. Proof of Theorem 3.4. The Newton number $\nu(C)$ of the plane curve germ $C$ is defined to be $\nu(C) = \sup \{\nu(\Delta_{x,y}(C)) : (x, y) \text{ runs over all charts centered at } O\}$. Using Theorem 4.1 we get

**Lemma 6.1.** A plane curve germ $C$ is non-degenerate if and only if $\nu(C) = \mu(C)$.

The proposition below shows that we can reduce the computation of the Newton number to the case of unitangent germs.

**Proposition 6.2.** If $C = \bigcup_{k=1}^{t} \tilde{C}^k$ ($t > 1$), where $\{\tilde{C}^k\}_k$ are unitangent germs such that $(\tilde{C}^k, \tilde{C}^l) = m(C^k)m(C^l)$ for $k \neq l$, then

$$\nu(C) - (m(C) - 1)^2 = \max_{1 \leq k < l \leq t} \{ (\nu(\tilde{C}^k) - (m(\tilde{C}^k) - 1)^2) + (\nu(\tilde{C}^l) - (m(\tilde{C}^l) - 1)^2) \}.$$
\[ \nu(\Delta_{x,y}(C)) = \nu(\sum_{k=1}^{t} \Delta_{x,y}(\tilde{C}^{k})) = \nu(\Delta_{x,y}(\tilde{C}^{k_1})) + \nu(\Delta_{x,y}(\tilde{C}^{k_2})) + \sum_{k \neq k_1, k_2} \nu(\Delta_{x,y}(\tilde{C}^{k})) + 2 \sum_{1 \leq k < l \leq t} \left[ \Delta_{x,y}(\tilde{C}^{k}), \Delta_{x,y}(\tilde{C}^{l}) \right] - t + 1 \]

\[ = \nu(\Delta_{x,y}(\tilde{C}^{k_1})) + \nu(\Delta_{x,y}(\tilde{C}^{k_2})) + \sum_{k \neq k_1, k_2} (\tilde{n}_k - 1)^2 + 2 \sum_{1 \leq k < l \leq t} \tilde{n}_k \tilde{n}_l - t + 1 \]

\[ = \nu(\Delta_{x,y}(\tilde{C}^{k_1})) - (\tilde{n}_{k_1} - 1)^2 \]

\[ + \nu(\Delta_{x,y}(\tilde{C}^{k_2})) - (\tilde{n}_{k_2} - 1)^2 + (m(C) - 1)^2. \]

The germs \( \tilde{C}^{k_1} \) and \( \tilde{C}^{k_2} \) are unitangent and transversal. Thus it is easy to see that there exists a chart \((x_1, y_1)\) such that \( \nu(\Delta_{x_1,y_1}(\tilde{C}^{k})) = \nu(\tilde{C}^{k}) \) for \( k = k_1, k_2 \).

If \( \{x = 0\} \) (or \( \{y = 0\} \)) and \( C \) are transversal, then there exists a \( k \in \{1, \ldots, t\} \) such that \( \nu(\Delta_{x,y}(C)) = \nu(\Delta_{x,y}(\tilde{C}^{k})) - (\tilde{n}_k - 1)^2 + (m(C) - 1)^2 \) and the proposition follows from the previous considerations.

Now we can pass to the proof of Theorem 3.4. If \( t(C) = 1 \) then \( C \) is non-degenerate with respect to a chart \((x, y)\) such that \( C \) and \( \{x = 0\} \) intersect transversally and Theorem 3.4 follows from Theorem 3.2. If \( t(C) > 1 \), then by Proposition 6.2 there are indices \( k_1 < k_2 \) such that

\[ (\alpha) \quad \nu(C) - (m(C) - 1)^2 = \nu(\tilde{C}^{k_1}) - (m(\tilde{C}^{k_1}) - 1)^2 + \nu(\tilde{C}^{k_2}) - (m(\tilde{C}^{k_2}) - 1)^2. \]

On the other hand, from basic properties of the Milnor number we get

\[ (\beta) \quad \mu(C) - (m(C) - 1)^2 = \sum_{k} \mu(\tilde{C}^{k}) - (m(\tilde{C}^{k}) - 1)^2. \]

Using (\( \alpha \)), (\( \beta \)) and Lemma 6.1, we check that \( C \) is non-degenerate if and only if \( \mu(\tilde{C}^{k_1}) = \nu(\tilde{C}^{k_1}), \mu(\tilde{C}^{k_2}) = \nu(\tilde{C}^{k_2}) \) and \( \mu(\tilde{C}^{k}) = (m(\tilde{C}^{k}) - 1)^2 \) for \( k \neq k_1, k_2 \). Now Theorem 3.4 follows from Lemma 6.1 and Corollary 4.3.

7. Concluding remark. M. Oka in [6] proved that the Newton number like the Milnor number is an invariant of equisingularity. Therefore, the invariance of non-degeneracy (Corollary 3.5) follows from the equality \( \nu(C) = \mu(C) \) characterizing non-degenerate germs (Lemma 6.1).

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References


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