Abelian Subgroups of the Torelli Group

William R. Vautaw

Abstract  Let $S$ be a closed oriented surface of genus $g \geq 2$, and let $T$ denote its Torelli group. First, given a set $E$ of homotopically nontrivial, pairwise disjoint, pairwise nonisotopic simple closed curves on $S$, we determine precisely when a multitwist on $E$ is an element of $T$ by defining an equivalence relation on $E$ and then applying graph theory. Second, we prove that an arbitrary Abelian subgroup of $T$ has rank $2g - 3$.

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1 Introduction

Here we present the notation, definitions, and terminology that will be used in the paper.

1.1 Surfaces

Throughout this work, $S$ will denote a closed, connected, oriented surface. We use the symbols $a; b; c; \varepsilon; h$ to denote simple closed curves on $S$.

The mapping class group, $\text{M}(S)$, of $S$ is the group of isotopy classes of orientation preserving self-homeomorphisms of $S$. In general, we will not distinguish between a map $f : S \to S$ and its isotopy class. The symbol $D_{\varepsilon}$ will denote the right Dehn twist about the simple closed curve $\varepsilon$. Recall that if $a$ and $b$ are simple closed oriented curves on $S$, then in $H_1(S)$, the first homology group of $S$ with integer coefficients, we have

$$D_a(b) = b + h_a; b a$$

where $h_a; b a$ denotes the algebraic intersection number of $a$ and $b$. Also, the Dehn twists $D_{a_1}$ and $D_{a_2}$ commute if and only if the isotopy classes of the curves $a_1$ and $a_2$ have representatives that are disjoint.
The Torelli group, \( T = T(S) \), of \( S \) is the subgroup of the mapping class group consisting of the isotopy classes of those self-homeomorphisms of \( S \) which induce the identity isomorphism on \( H_1(S) \). The Torelli group is torsion-free, and is trivial in the case of the sphere or torus.

1.2 Graphs

We use graph-theoretic terminology consistent with its use in [2]. We remind the reader of the less familiar terms, and give the graph-theoretic definitions of those terms that may be used in different ways in ordinary topology.

Throughout this work, \( G \) will denote a connected, finite linear graph. We include the possibility that \( G \) may contain loops or parallel edges. \( E = E(G) \) will denote the edge set of \( G \), and we use the symbols \( a; b; c; e \) to denote edges of \( G \). For \( E^0 \subseteq E(G) \), \( G - E^0 \) denotes the subgraph obtained from \( G \) by deleting the edges in \( E^0 \), while \( G + E^0 \) is the graph obtained from \( G \) by adding a set of edges \( E^0 \). If \( E = feg \), then we write \( G - e \) and \( G + e \) instead of \( G - feg \) and \( G + feg \). A bond \( E^0 \) in \( G \) is a minimal subset of \( E(G) \) such that \( G - E^0 \) is disconnected. Note that \( G - E^0 \) consists of precisely two components. We say that the edge \( e \) is a cut edge if \( G - e \) is disconnected. We use the symbols \( u; v; x; y \) to denote vertices of \( G \). The degree of a vertex \( v \) is the number of edges incident with \( v \), each loop counting as two edges.

A \( (v_0; v_n) \) walk \( W \) of length \( n \) is a finite nonempty alternating sequence, \( W = v_0e_1v_1e_2v_2\ldots e_nv_n \), of vertices and edges such that the ends of the edge \( e_i \) are the vertices \( v_{i-1} \) and \( v_i \) for \( 1 \leq i \leq n \). If the edges of \( W \) are distinct, \( W \) is called a trail. A cycle in \( G \) is a closed trail of positive length whose origin and internal vertices are distinct. Thus a cycle is an embedded circle in \( G \). For our purposes, to denote a trail or cycle, it will be enough to give its sequence of edges, and we do not distinguish between a closed trail \( W \) and another closed trail whose sequence of edges is a cyclic permutation of \( W \)’s.

A spanning tree \( T \) is a subgraph of \( G \) with the same vertex set as \( G \) such that \( T \) contains no cycles. The number of edges in any spanning tree is equal to one less than the number of vertices of \( G \). Note that if \( T \) is a spanning tree, and \( e \) is an edge of \( G \) not in \( T \), then \( T + e \) contains a unique cycle \( C \), and \( e \) is an edge of \( C \), so the rank of \( H_1(G) \) is equal to the number of edges of \( G \) outside any spanning tree. Every connected graph contains a spanning tree.
2 Reduction Systems and Reduction System Graphs

By a reduction system \( \mathcal{E} \) on \( S \) we mean a collection of simple closed curves on \( S \) that are homotopically nontrivial, pairwise disjoint, and pairwise nonisotopic. We use the symbols \( a; b; c; e \) to denote the elements of a reduction system \( \mathcal{E} \), and \( S_\mathcal{E} \) to denote the natural compactification of \( S \setminus \mathcal{E} \); that is, \( S \) cut along \( \mathcal{E} \).

We partition the set \( \mathcal{E} = \{ e_1; e_2; \ldots; e_n \} \) according to the equivalence relation generated by the rule

\[
\begin{align*}
& e_i \sim e_j \text{ if } \\
& \text{ or } \\
& f(e_i; e_j) \text{ is a minimal separating set in } \mathcal{E}.
\end{align*}
\]

Here, \( f(e_i; e_j) \) is a minimal separating set" means that \( S_{f(e_i; e_j)} \) is disconnected, but both \( S_{f(e_i; e_)} \) and \( S_{f(e_j; e_i)} \) are connected. There are three types of equivalence classes:

(i) Singleton classes \( \{ a_1; a_2; \ldots; a_p \} \) consisting of the separating curves \( a_1; a_2; \ldots; a_p \) in \( \mathcal{E} \). Such a curve will be called an \( a \)-type curve.

(ii) Classes \( \{ b_{11}; b_{12}; \ldots; b_{1q_1}; b_{21}; b_{22}; \ldots; b_{2q_2}; \ldots; b_{r1}; b_{r2}; \ldots; b_{rq} \} \) of cardinality at least 2. Each such class \( f(b_{11}; \ldots; b_{in}; g) \) is characterized by the following three properties:

(a) No curve \( b_{ij} \) is separating.

(b) \( b_{ij} \) is homologous to \( b_{ij}^0 \) for every pair \( b_{ij}, b_{ij}^0 \).

(c) Maximal with respect to (a) and (b).

A curve in such a class will be called a \( b \)-type curve.

(iii) Singleton classes \( \{ c_1; c_2; \ldots; c_s \} \) where each \( c_i \) is non-separating and is homologous to no other curve in \( \mathcal{E} \). Such a curve will be called a \( c \)-type curve.

According to (i), (ii), and (iii) above, we write

\[ \mathcal{E} = \{ a_1; \ldots; a_p; b_{11}; \ldots; b_{1q_1}; b_{21}; \ldots; b_{2q_2}; \ldots; b_{r1}; \ldots; b_{rq}; c_1; \ldots; c_s \} \]

We use \( \mathcal{E} \) to denote a graph \( G_\mathcal{E} \), which we call the reduction system graph of \( \mathcal{E} \), as follows:

The vertices of \( G_\mathcal{E} \) correspond to the components of \( S_\mathcal{E} \).

The edges of \( G_\mathcal{E} \) correspond to the curves in the reduction system \( \mathcal{E} \), with:
Two distinct vertices are connected by the edge $e_i$ if and only if the curve $e_i$ in $E$ is a common boundary curve of the two components of $S_E$ which correspond to the vertices in question.

A vertex has a loop $e_i$ if and only if the curve $e_i$ in $E$ represents two boundary curves of the component of $S_E$ which corresponds to the vertex in question.

Note that $G_E$ is connected, and that any connected graph $G$ is $G_E$ for some surface $S$ and some reduction system $E$ on $S$. However, the genus of $S$ is not determined by $G$, any two possible $S$'s differing by the genera of their complementary components. But, unless $G$ is the graph consisting of a single vertex and either no edges or a single loop, then $\text{genus}(S) = \text{rank}(\chi(G)) + (\text{number of vertices of degree 2})$.

Since $S$ and $E$ will be fixed, we will denote $G_E$ simply by $G$.

The equivalence relation on the curves in $E_0$ induces an equivalence relation on the edge set $E(G) = e_1; e_2; \ldots; e_g$ of $G$. It is generated by

- $e_i = e_j$ or $f e_i; e_j g$ is a bond:

(Again, it should be noted that this equivalence relation may be defined for any graph $G$.) The three types of equivalence classes described above become, for $G$,

(i) Singleton classes $f a_1 g; \ldots; f a_p g$ consisting of the cut edges $a_1; \ldots; a_p$ of $G$. Such an edge will be called an a-type edge.

(ii) Classes $f b_{11}; \ldots; b_{1q} g; f b_{21}; \ldots; b_{2q} g; \ldots; f b_{r1}; \ldots; b_{rq} g$ of cardinality at least 2. Each such class is characterized by the following three properties:

(a) No edge $b_{ij}$ is a cut edge.
(b) $f b_{ij} b_{ij}^0 g$ is a bond for every pair $b_{ij}, b_{ij}^0$.
(c) Maximal with respect to (a) and (b).

An edge in such a class will be called a b-type edge.

(iii) Singleton classes $f c_1 g; \ldots; f c_q g$ where each $c_i$ is not a cut edge, and forms a 2-edge bond with no other edge of $G$. Such an edge will be called a c-type edge.

According to (i), (ii), and (iii) above, we write

$$E(G) = f a_1; \ldots; a_p g; f b_{11}; \ldots; b_{1q} g; f b_{21}; \ldots; b_{2q} g; \ldots; f b_{r1}; \ldots; b_{rq} g; f c_1; \ldots; c_q g$$

A typical example is shown above.

Now let \( h \) be a simple closed curve on \( S \) that intersects each element of \( E \) transversely at most once. Starting at any point on \( h \) and travelling in either direction gives a cyclic ordering of the reduction curves which \( h \) intersects, thus defining a closed trail \( H \) in \( G \). Note that \( H \) is a cycle in \( G \) if and only if \( h \setminus S_i \) is either empty or is a single (that is, connected) arc, for every component \( S_i \) of \( S \). Likewise, given a closed trail \( H \) in \( G \), there is such a curve \( h \) on \( S \) defining \( H \). The fact that the isotopy class of \( h \) is never unique is not important for our purposes.

The following figure shows a typical example. Note that \( h_1 \) and \( h_2 \) are nonisotopic curves which both define the cycle \( H = b_{11}c_2b_{22} \).

The remainder of this section presents some purely graph-theoretic results, concluding with Theorem 2.1, which is used in the following section. So for the remainder of this section, let \( G \) denote an arbitrary connected graph. We explain here the notation and terminology we use. Given a subgraph \( H \) of \( G \), we let \( G \setminus H \) denote the graph obtained by deleting every edge \( e \) of \( H \) and identifying the ends of \( e \). Equivalently, thinking of \( G \) as a CW-complex and \( H \) as a subcomplex, \( G \setminus H \) is the complex obtained from \( G \) by crushing each component of \( H \) to a point. Thus, we have a quotient ("contraction") map \( p: G \to G \setminus H \). Next, by a cut vertex of \( G \), we mean a vertex \( v \) of \( G \) such that when \( v \), and only \( v \), is removed from the topological space \( G \), the resulting
space is disconnected. (This is not the definition used by graph theorists, but is an equivalent topological one.) A block is a connected graph without cut vertices, and a block of a graph is a subgraph that is a block and is maximal with respect to that property. Any graph is the union of its blocks. We leave the proofs of the first two lemmas to the reader.

**Lemma 2.1** If $G$ has no cut edges, then any two vertices of $G$ are connected by two edge-disjoint paths.

**Lemma 2.2** Let $b_1$ and $b_2$ be edges of $G$ such that $f b_1; b_2 g$ is a bond. If $C$ is a cycle in $G$, and $b_1$ is an edge of $C$, then so is $b_2$.

**Lemma 2.3** Let $c$ be a $c$-type edge in $G$ that is not a loop. Then $c$ is contained in two cycles, the intersection of whose edge sets is precisely $c$.

**Proof** Assume that $G$ is a block. If $G$ has exactly two vertices, then each edge of $G$ is a link, and $G$ must have at least three edges, since $c$ is a $c$-type edge. The result is clear in this case. Otherwise, $G$ has at least three vertices and no cut edges. Consider the graph $G - c$. If $G - c$ has a cut edge $e$, then $G - f c; eg$ is not connected, so $f c; eg$ is a bond of $G$. This contradicts the fact that $c$ is a $c$-type edge. So $G - c$ has no cut edges. By Lemma 2.1, there are two edge-disjoint paths $P$ and $P^0$ in $G - c$ connecting the ends of $c$. Then the cycles $C = P + c$ and $C^0 = P^0 + c$ have exactly the edge $c$ in common. In the case that $G$ is not a block, we let $B$ be the block of $G$ containing $c$. It is easy to see that $c$ is a $c$-type edge of $B$, so we apply the first case to $B$ and find two such cycles within $B$.

**Theorem 2.1** Let $G$ have edge set

$$E(G) = f a_1; \ldots; a_p; b_1; \ldots; b_p; c_1; \ldots; c_s g$$

notated according to $a\{, b\{, and $c$-type equivalence classes. Let $w: E(G) \rightarrow \mathbb{Z}$ be a weighting of $G$. Then $w(H) = 0$ for every cycle $H$ in $G$ if and only if

(i) $w(c) = 0$, $1 \leq s$, and

(ii) $w(b_1) + w(b_2) + \ldots + w(b_q) = 0$, $1 \leq j \leq r$.

**Proof** Assume that $w(H) = 0$ for every cycle $H$ in $G$. 

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(i) Let \( c \) be a c{type edge with ends \( u \) and \( v \). If \( c \) is a loop, then \( w(c) = 0 \), by hypothesis. Otherwise, there are two edge-disjoint \( (u;v) \) paths, \( P \) and \( P^0 \), in \( G - c \). We have three cycles: \( P + c \), \( P^0 + c \), and \( P + P^0 \). Thus:

\[
\begin{align*}
\text{w}(P) + w(c) &= w(P + c) = 0 = \text{w}(P^0) + w(c) = w(P^0 + c) = 0, \\
\text{w}(P) + w(P^0) &= w(P + P^0) = 0.
\end{align*}
\]

(ii) Let \( B \) be the equivalence class of the b{type edge \( b \), and \( B = B^{f,bg} \). Let \( p: G \rightarrow G \) be the contraction map. Suppose that \( b \) is a cut edge of \( G \in B \), separating it into two components \( G_1 \) and \( G_2 \). Then the restriction of \( p \) to \( G - b \) maps onto the disconnected space \( G_1 [ G_2 \), and so \( G - b \) is disconnected. This is a contradiction to the hypothesis that \( b \) is a b{type edge of \( G \). We obtain a similar contradiction if we suppose \( f;bg \) is a bond in \( G \in B \). Thus \( b \) is a c{type edge in \( G \in B \). If \( b \) is a loop in \( G \in B \), then \( p^{-1}(b) = B \), which therefore forms a cycle in \( G \). So equation (ii) holds for the equivalence class of \( b \).

If \( b \) is not a loop in \( G \in B \), then by Lemma 2.3 there are two cycles \( H \) and \( H^0 \) in \( G \in B \), the intersection of whose edge sets is \( f,bg \). Lemma 2.2 implies that \( p^{-1}(H) \) and \( p^{-1}(H^0) \) are cycles \( H \) and \( H^0 \), respectively, the intersection of whose edge sets is precisely \( B \). Thus we have

\[
0 = w(H) = w(B) + w(H - B),
\]

\[
0 = w(H^0) = w(B) + w(H^0 - B).
\]

And so, \( w(B) = 0 \). Here we have used the fact that the symmetric difference \( H \cup H^0 \) of the cycles \( H \) and \( H^0 \) is a disjoint union of cycles (regarded as sets of edges).

Assume that

(i) \( w(c_i) = 0 \) for each \( i \), and

(ii) \( w(b_{ij}) + w(b_{ij}) + \ldots + w(b_{jq}) = 0 \) for each \( j \).

Let \( H \) be a cycle in \( G \). \( H \) contains no a{type edges, since they are cut edges, and by Lemma 2.2, if \( H \) contains one edge of a b{type class, then it contains the whole class. So the assumptions imply that \( w(H) = 0 \).

\section{Abelian Subgroups in the Torelli Group}

We at first consider a specific type of Abelian subgroup of the Torelli group \( \mathcal{T}(S) \), namely one consisting of multitwists | that is, compositions of left and right Dehn twists about a fixed reduction system on \( S \).
**Theorem 3.1** Let $S$ be a closed, connected, oriented surface, and let

$$
\mathcal{E} = \{a_{1}; \ldots ; a_{p}; b_{11}; \ldots ; b_{1q_{1}}; \ldots ; b_{r_{1}}; \ldots ; b_{r_{q_{r}}}; c_{1}; \ldots ; c_{s}\}
$$

be a reduction system on $S$, notated by $a_{i}$, $b_{j}$, and $c_{k}$, type equivalence classes as in section 2. Let $D_{\mathcal{E}}$ be the multitwist group on $\mathcal{E}$, and let

$$
f = D_{e_{1}} \cdot D_{e_{p}} \cdot D_{b_{11}} \cdot D_{b_{1q_{1}}} \cdot D_{b_{r_{1}}} \cdot D_{b_{r_{q_{r}}}} \cdot D_{e_{1}} \cdot D_{e_{s}}
$$

be an element of $D_{\mathcal{E}}$. Then $f$ is an element of $D_{\mathcal{E}} \setminus T_{\mathcal{E}}$, which we call the Torelli multitwist group of $\mathcal{E}$, if and only if

(i) $\gamma_{i} = 0, 1$ i s, and

(ii) $j_{1} + j_{2} + \ldots + j_{q} = 0, 1 \ j \ r$.

Consequently, $T_{\mathcal{E}}$ is a free Abelian group of rank

$$p + (q_{1} - 1) + (q_{2} - 1) + \ldots + (q_{r} - 1) = p + q_{1} + q_{2} + \ldots + q_{r} - r:
$$

**Remark** A set of equivalence class representatives of the curves in $\mathcal{E}$ is in general not linearly independent in $H_{1}(S)$, so the nondegeneracy of the algebraic intersection $\mathcal{h}; \mathcal{i}$ is not sufficient to prove the theorem.

**Proof** Assume that $f \not\in T_{\mathcal{E}}$.

Let $G$ be the reduction system graph of $\mathcal{E}$ with edge set $E(G)$. We weight each edge of $G$ according to the exponent in $f$ of the twist about its corresponding curve in $\mathcal{E}$, giving $w: E(G) \rightarrow \mathbb{Z}$.

Let $H = e_{1}e_{2}; \ldots ; e_{n}$ be a cycle in $G$. Then, as in section 2, $H$ is defined by any simple closed curve $h$ on $S$ that intersects each of the corresponding curves $e_{1}; e_{2}; \ldots ; e_{n}$ of $\mathcal{E}$ exactly once, and does not intersect any of the other curves of $\mathcal{E}$. Orient $h$. Then orient the curves $e_{1}; e_{2}; \ldots ; e_{n}$ so that $h_{i}; h_{j} = 1$. So we have

$$0 = h_{i}; h_{j} = h_{i}; f(h_{j}) = h_{i}; h_{j} + e_{1} + 2e_{2} + \ldots + ne_{n}i = 1 + 2 + \ldots + n;$$

where $i = w(e)$. Hence the weight of every cycle in $G$ is zero. The conclusion follows from Theorem 2.1.

(i) $\gamma_{i} = 0, 1$ i s, and

(ii) $j_{1} + j_{2} + \ldots + j_{q} = 0, 1 \ j \ r$.  

Since \( H_1(S) \) has a basis consisting of simple closed curves, in order to prove that \( f \in T \), it suffices to show that in \( H_1(S) \), we have \( f(h) = h \) for any simple closed curve \( h \) on \( S \). Note that for any such \( h \), we have \( h_i; h_j = 0, 1 \) if \( p \), and after orienting \( h \) and then each \( h_{ij} \) so that \( h_{ij}; h_i = h_{ij}; h_j \), we have
\[
\begin{align*}
\text{weighting } w & \quad \text{Given a pair, } e \quad \text{Let } \\
\text{Theorem 3.2} & \quad \text{Let } \\
\text{Since } & \quad \text{Abelian Subgroups of the Torelli Group} \\
\text{E} & \quad 165 \\
\text{Proof} & \quad \text{By the definition of } \\
\text{Proof} & \quad \text{Let } \\
\text{Dehn twists about separating curves in } & \quad \text{E} \\
\text{closed curves on } & \quad \text{E} \\
\text{Corollary 3.1} & \quad \text{Let } \\
\text{twists about separating simple closed curves.} & \quad \text{E} \\
\text{T} & \quad \text{BP maps about bounding pairs in } \\
\text{[5] has shown that the Torelli group} & \quad \text{E} \\
\text{and after orienting } h & \quad \text{E} \\
\text{h} & \quad \text{E} \\
\text{Theorems 2.1 and 3.1 show the conditions to be equivalent.} & \quad \text{E} \\
\text{Proof} & \quad \text{Partition } e \text{ into } \\
\text{Theorems 2.1 and 3.1 show the conditions to be equivalent.} & \quad \text{E} \\
\text{Given a pair, } e_1 \text{ and } e_2 \text{, of disjoint, non-separating, but homologous simple closed curves on } S \text{, we call } D_{e_1}D_{e_2}^{-1} \text{ a bounding-pair map or BP map. Powell [5] has shown that the Torelli group } T \text{ is generated by BP maps and Dehn twists about separating simple closed curves.} & \quad \text{E} \\
\text{Corollary 3.1} & \quad \text{Let } S, e, D_e, \text{ and } T_e \text{ be as in Theorem 3.1. Let } D^0 \text{ be the subgroup of } M(S) \text{ generated by } \\
\text{Let } D^0 & \quad \text{be the subgroup of } M(S) \text{ generated by } \\
\text{(i) BP maps about bounding pairs in } e, \text{ and} & \quad \text{E} \\
\text{(ii) Dehn twists about separating curves in } e. & \quad \text{E} \\
\text{Then } D^0 = D_e \setminus T = T_e. & \quad \text{E} \\
\text{Proof} & \quad \text{By the definition of } D_e, \text{ it is clear that every generator of } D^0 \text{ is in } D_e. \text{ By Powell’s result noted above, every generator of } D^0 \text{ is in } T. \text{ Thus } D^0 \subseteq D_e \setminus T = T_e. \text{ We must show that } D_e \setminus T \subseteq D^0. & \quad \text{E}
Let \( f \in D_\mathcal{E} \setminus T \). By Theorem 3.1, we know that
\[
f = D_{a_1} \cdot D_{a_p} D_{b_{11}} \cdot D_{b_{1q} D_{b_{21}}} \cdot D_{b_{2q_2}} \cdot D_{b_{r_1}} \cdot D_{b_{r_{q_r}}},
\]
where \( i_1 + i_2 + \ldots + i_q = 0, 1 \leq r \). Since each \( D_{a_i} \) is a product of type(iii) generators of \( D_0 \), we will be done if we write \( D_{a_1} D_{a_2} \ldots D_{a_{i_q}} \) as a product of BP maps. We do this:
\[
D_{b_{11}} D_{b_{12}} D_{b_{1q_1}} = (D_{b_{21}} D_{b_{11}}^{-1}) \cdot (D_{b_{2q_2}} D_{b_{11}}^{-1}) \cdot (D_{b_{r_{q_r}} D_{b_{11}}^{-1}) \cdot D_{b_{1q_1}} D_{b_{11}}^{-1}) \cdot q_i;
\]
where we note that \( -i_2 - i_3 - -iq = i_1 \).

**Corollary 3.2** Let \( S \) be a closed, connected, oriented surface, and let
\[
\mathcal{E} = f a_1 ; \ldots ; a_p ; b_{11} ; \ldots ; b_{1q_1} ; \ldots ; b_{r_{q_r}} ; c_1 ; \ldots ; c_g
\]
be a reduction system on \( S \), notated by \( a_i \), \( b_i \), and \( c_i \) type equivalence classes as in section 2. Let \( D_\mathcal{E} \) be the multitwist group on \( \mathcal{E} \), and let
\[
f = D_{a_1} \cdot D_{a_p} D_{b_{11}} \cdot D_{b_{1q_1}} D_{b_{21}} \cdot D_{b_{r_{q_r}}} D_{c_1} \cdot D_{c_g}
\]
be an element of \( D_\mathcal{E} \). Let \( m \geq 2 \) be an integer.

Then \( f \in \Gamma_S(m) \) acts trivially on \( H_1(S; \mathbb{Z}_m) \) if and only if
\[
(i) \quad \gamma_i \equiv 0 \pmod{m}, 1 \leq s, \text{ and}
(ii) \quad j_1 + j_2 + \ldots + j_q \equiv 0 \pmod{m}, 1 \leq r.
\]

Let \( S \) be the surface of genus \( g \geq 2 \) and \( \mathcal{E} \) the reduction system on \( S \) shown below. Since \( \mathcal{E} \) consists of \( 2g - 3 \) type curves, \( \text{rank}(T_\mathcal{E}) = 2g - 3 \). This example, along with Theorem 4.1 below, shows that the maximal rank of an Abelian subgroup of the Torelli group is attained by a multitwist group.
Remark One particular naively-expected symplectic analogue of Theorem 3.1 is not true:

\textbf{Conjecture} Let $(V; h)$ be a symplectic lattice of rank $2g$, where $g \geq 2$. Let $v_1; v_2; \ldots; v_n \in V$ be a set of primitive vectors in $V$ that are pairwise linearly independent and symplectically orthogonal. Let $T_i$ be the transvection corresponding to the vector $v_i$. Thus $T_i(w) = w + hv_i$ for any $w \in V$. Let $m_1; m_2; \ldots; m_n$ be integers. Then the multitransvection $T = T_1^{m_1} T_2^{m_2} \cdots T_n^{m_n}$ is the identity on $V$ if and only if $m_i = 0$ for all $i$.

But now let $a_1; b_2; \ldots; a_g; b_2g$ be the standard symplectic basis for $V$, and for $i = 1, 2, 3, 4, 5$, let $v_i = a_1 + ia_2$. Let $m_1 = 1$, $m_2 = -3$, $m_3 = 3$, and $m_4 = -1$. One can verify that $T = T_1^{m_1} T_2^{m_2} T_3^{m_3} T_4^{m_4} = \text{id}_V$. This shows the conjecture to be false.

Now we prove that for any closed oriented surface of genus $g \geq 2$, the general Abelian subgroup of its Torelli group has rank $2g - 3$. We first give two lemmas.

**Lemma 3.1** Let $S$ be a closed, connected, oriented surface, and $\mathcal{C}$ a reduction system on $S$ with reduction system graph $G$. Let $\mathcal{T}_\mathcal{C}$ be the Torelli multitwist group on $\mathcal{C}$, as in Theorem 3.1. Then $\text{rank}(\mathcal{T}_\mathcal{C}) = -1$, where $\text{card}(G)$ is the number of vertices of $G$, or, equivalently, the number of components of $S_{\mathcal{C}}$.

**Proof** Let $G$ have edge set

$$E(G) = f a_1; \ldots; a_p; b_1; \ldots; b_{pq}; b_2; \ldots; b_{pq}; \ldots; b_1; \ldots; b_g; c_1; \ldots; c_g.$$

Let $E^0 = f b_1; \ldots; b_{(q-1)}; b_2; \ldots; b_{(q-1)}; \ldots; b_1; \ldots; b_{(q-1)}; \ldots; b_1; \ldots; b_g; E(G)$, and let $G^0 = G[E^0]$. Then $G^0$ contains no cycles, since any cycle containing one edge of a b-type class contains the whole class. Therefore, $G^0$ is contained in a spanning tree $T$ of $G$. Since each $a_i$ is a cut edge, $T$ contains $a_i$, $1 \leq i \leq p$.

So $T$ contains the set of edges $E^0[ f a_1; a_2; \ldots; a_g].$ But by Theorem 3.1, the cardinality of this set is equal to the rank of $T_\mathcal{C}$. This gives us

$$-1 = \text{card}(E(T)) = p + (q - 1) + (q - 1) = \text{rank}(T_\mathcal{C}).$$

**Lemma 3.2** Let $S$ be a closed, connected, oriented surface of genus $g \geq 2$, and let $\mathcal{C}$ be a reduction system on $S$. Let $\Omega$ denote the number of components of $S_{\mathcal{C}}$ not homeomorphic to a pair of pants or a one-holed torus. Let $T_\mathcal{C}$ be the Torelli multitwist group on $\mathcal{C}$. Then $\text{rank}(T_\mathcal{C}) + \Omega = 2g - 3.$
Proof Let $G$ be the reduction system graph of $\mathcal{E}$. We use the following notation:

- $\Gamma$ is the maximum genus of any component of $S_{\mathcal{E}}$.
- is the maximum degree of any vertex of $G$, or, equivalently, the maximum number of boundary curves of any component of $S_{\mathcal{E}}$.
- $b$ is the number of vertices of $G$ of degree $b$, or, equivalently, the number of components of $S_{\mathcal{E}}$ with $b$ boundary curves.
- $\gamma^y_b$ is the number of components of $S_{\mathcal{E}}$ of genus $\gamma$ having $b$ boundary curves, or, equivalently, the number of vertices of $G$ of degree $b$ corresponding to a component of $S_{\mathcal{E}}$ of genus $\gamma$.

So we have:

$$b = \sum_{\gamma=0}^{\Gamma} \frac{X^\gamma}{\gamma}$$

But the assumption that each element of $\mathcal{E}$ is homotopically nontrivial means $\sum_{b=0}^{2} b = 0$, and the assumption that the elements of $\mathcal{E}$ are pairwise nonisotopic means $\sum_{b=0}^{2} b = 0$. So, in fact, $b = 1^1 + 2^1 + 3^1 + 4^1 + \cdots$. Now, $1^1$ is the number of one-holed tori, and $2^2$ is the number of pairs of pants, so by the definition of $\Omega$, we have $\Omega = 1^1 + 2^1 + 3^1 + 4^1 + \cdots$. Hence

$$2g - 2 = - (\mathbf{S}) - (\mathbf{V})$$

By Lemma 3.1, $\text{rank}(T_{\mathcal{E}}) = -1$, so we have

$$\text{rank}(T_{\mathcal{E}}) + \Omega + \Omega - 1$$

By Lemma 3.1, $\text{rank}(T_{\mathcal{E}}) = -1$, so we have

$$= (1^1 + 2^1 + \cdots) + (2^2 + 2^1 + 3^1 + 4^1 + \cdots) - 1$$

$$= \left( \sum_{b=0}^{2} b \right) + 1^1 + 2^1 + 3^1 + 4^1 + \cdots - 1$$

$$= \left( \sum_{b=1}^{\infty} b \right) - 2 - 1$$

$$= 2g - 3$$
Theorem 3.3  Let $S$ be a closed, connected, oriented surface of genus $g \geq 2$, and let $A$ be an Abelian subgroup of $T$, the Torelli group of $S$. Then $\text{rank}(A) = 2g - 3$.

Proof  This proof is an adaptation of an analogous proof in [1]. That paper also introduces the reduction homomorphism and essential reduction system which we refer to here.

Let $f \in A$, $f \neq 0$. By Thurston's classification, $f$ is either reducible, pseudo-Anosov, or of finite order. Since $T$ is torsion-free, $f$ cannot be of finite order. We consider the other two possibilities.

Case 1  $f$ is pseudo-Anosov.

Let $h \in A$ denote the cyclic subgroup of $A$ generated by $f$, and let $C = C_M(S)(h)$, the centralizer of $h$ in $M(S)$. Then $A \cap C$ and $A$ is torsion-free. We conclude by a theorem of McCarthy ([4], Corollary 3) that $A$ is infinite cyclic. Hence $\text{rank}(A) = 2g - 3$.

Case 2  $f$ is reducible.

Given $h \in A$, let $\mathcal{E}_h$ denote the essential reduction system of $h$, and let

$$
\mathcal{E} = \begin{bmatrix} \mathcal{E}_h \\ h \in A \end{bmatrix}
$$

Then $\mathcal{E}$ is an adequate reduction system for $A$ ([1], Lemma 3.1(1)), and if reducible implies $\mathcal{E} \in \mathcal{E}_e$; so every element of $A$ is reducible.

Let $M = M_e(S)$ denote the stabilizer of $\mathcal{E}$ in $M(S)$, and let $j : M_e(S) \to M(S_e)$ be the reduction homomorphism. Then $\ker(j) = D_e$, the multitwist group on $\mathcal{E}$, and thus

$$
\ker(j_A) = \ker(j) \setminus A = D_e \setminus A = D_e \setminus T \setminus A = T_e \setminus A:
$$

We now have a short exact sequence

$$
0 \to T_e \setminus A \to A - j_A(A) \to 0
$$

of free Abelian groups, which shows that

$$
\text{rank}(A) = \text{rank}(T_e \setminus A) + \text{rank}(j_A(A)) - \text{rank}(T_e) + \text{rank}(A):
$$

We will be done, by applying Lemma 3.2, once we show that $\text{rank}(j_A(A)) = \Omega$, the number of components of $S_e$ not homeomorphic to a pair of pants or a one-holed torus.

A theorem of Ivanov ([3], Theorem 1.2) implies that \((f)\) restricts to each component \(S_1; S_2; \ldots ; S_e\) of \(S\), giving "projections" \(p_i : (A) \rightarrow M(S_i)\) induced by restricting representatives. Set \(A_i = p_i((A)) \cap M(S_i)\). Then \((A) \simeq A_i\), so \(\text{rank}(A) \leq \sum \text{rank}(A_i)\). We make the following observations:

(i) If \(S_i\) is a pair of pants, then \(M(S_i)\) is finite, so \(\text{rank}(A_i) = 0\).

(ii) If \(S_i\) is a one-holed torus, then the homomorphism \(H_1(S_i) \rightarrow H_1(S)\) induced by inclusion is injective. Any homeomorphism \(f\) representing an element of \(A\) maps a circle \(c\) in \(S_i\) to a circle \(c^0\) in \(S\), so \(A_i\) lies within the Torelli group of \(S_i\), which is trivial in this case.

(iii) If \(S_i\) is neither a pair of pants nor a one-holed torus, then \(A_i\) is either trivial or is an adequately reduced torsion-free Abelian subgroup of \(M(S_i)\). So again by McCarthy's theorem, \(\text{rank}(A_i) \leq 1\).

These observations tell us that

\[
\text{rank}(A) = \sum_{i=1}^{e} \text{rank}(A_i) \leq \Omega.
\]

References