Seifert fibered contact three-manifolds via surgery

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Abstract Using contact surgery we define families of contact structures on certain Seifert fibered three-manifolds. We prove that all these contact structures are tight using contact Ozsvath-Szabo invariants. We use these examples to show that, given a natural number \( n \), there exists a Seifert fibered three-manifold carrying at least \( n \) pairwise non-isomorphic tight, not fillable contact structures.

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1 Introduction and statement of results

The classification problem for tight contact structures on closed oriented three-manifolds is one of the driving forces in present day contact topology. Contact surgery along Legendrian links provides a powerful tool for constructing contact three-manifolds. Tightness of these structures is, however, hard to prove, unless the structures can be shown to be fillable, i.e., can be viewed as living on the boundary of a symplectic four-manifold satisfying appropriate compatibility conditions. The question whether any tight contact structure is fillable was open for some time, until the first tight, non-fillable contact three-manifolds were found by Etnyre and Honda [6], followed by infinitely many such examples [12, 13]. The tightness of those examples was proved using a delicate topological method called state traversal (see [9]). In this paper we prove tightness by applying the Heegaard Floer theory recently developed by Ozsvath and Szabo [17, 18, 21]. According to our main result, tight, not fillable contact structures are more common than one would expect:

Theorem 1.1 For any \( n \geq 2 \) \( \mathbb{N} \) there is a Seifert fibered 3-manifold \( M_n \) carrying at least \( n \) pairwise non-isomorphic tight, not fillable contact structures.
The construction of the contact structures in Theorem 1.1 relies on contact surgery. We verify non-llability via the Seiberg-Witten equations, following the approach of [12, 13]. In order to state precisely our results we need a little preparation.

Contact surgery

In a given contact three-manifold \((Y; \xi)\) a knot \(K \subset (Y; \xi)\) is Legendrian if \(K\) is everywhere tangent to \(\xi\). The framing of \(K\) naturally induced by \(\xi\) is called the contact framing. Given a Legendrian knot \(K\) in a contact three-manifold \((Y; \xi)\) and a rational number \(r \in \mathbb{Q}, r \neq 0\), one can perform contact \(r\) surgery along \(K\) to obtain a new contact three-manifold \((Y^0, \xi)\) [1, 2]. Here \(Y^0\) is the three-manifold obtained by smooth \(r\) surgery along \(K\), where the surgery coefficient is measured with respect to the contact framing defined above, not with respect to the framing induced by a Seifert surface (which, in general, does not exist). The contact structure \(\xi^0\) is constructed by extending from the complement of a standard neighborhood of \(K\) to a tight contact structure on the glued-up solid torus. If \(r \neq 0\) such an extension always exists, and for \(r = -1\) the corresponding contact surgery coincides with Legendrian surgery along \(K\) [5, 8, 22].

Below we outline an algorithm for replacing a contact \(r\) surgery on a Legendrian knot \(K\) with a sequence of contact \((-1)\) surgeries on a suitable Legendrian link. By [2, Proposition 3], contact \(r\) surgery along \(K\) with \(r < 0\) is equivalent to Legendrian surgery along a Legendrian link \(L = \bigcup_{i=0}^{m} L_i\) which is determined via the following simple algorithm by the Legendrian knot \(K\) and the contact surgery coefficient \(r\). The algorithm to obtain \(L\) is the following.

Let

\[
[a_0 + 1; \ldots; a_m; \ a_0; \ldots; a_m \ -2]
\]

be the continued fraction expansion of \(r\). To obtain the first component \(L_0\), push \(K\) using the contact framing and stabilize it \(-a_0 - 2\) times. Then, push \(L_0\) and stabilize it \(-a_1 - 2\) times. Repeat the above scheme for each of the remaining pivots of the continued fraction expansion. Since there are \(-a_i - 1\) inequivalent ways to stabilize a Legendrian knot \(-a_i - 2\) times, this construction yields \(\frac{m}{a_0}(-a_i - 1)\) potentially different contact structures. According to [2, Proposition 7], a contact \(r = \frac{p}{q}\) surgery \((p; q \geq 2 \in \mathbb{N})\) on a Legendrian knot \(K\) is equivalent to a contact \(\frac{p}{kq}\) surgery on \(K\) followed by a contact \(\frac{p}{q}\) surgery on a Legendrian push of \(K\) for any integer \(k \geq 2 \in \mathbb{N}\) such that \(q - kp < 0\). Therefore, the latter surgery can be turned into a sequence.
of Legendrian surgeries, as described above. By [1, Proposition 9], a contact $\frac{1}{k}$ surgery on a Legendrian knot $K$ can be replaced by $k$ contact $(+1)$ surgeries on $k$ Legendrian pushos of $K$.

In conclusion, any contact rational $r$ surgery ($r \neq 0$) can be replaced by contact $(1)$ surgery along a Legendrian link (which is not necessarily uniquely specified); for a related discussion see also [3].

**Statement of results**

In the following, we shall denote by

$$M(g; n; (1; 1); \ldots; (k; k))$$

the Seifert bered 3-manifold obtained by performing $(-\gamma_1); \ldots; (-\gamma_k)$ surgeries along $k$ bers of the circle bundle $Y_{g,n}$ over the genus $g$ surface with Euler number $e(Y_{g,n}) = n$. The Seifert invariants

$$(g; n; (1; 1); \ldots; (k; k))$$

are said to be in normal form if

$$i > i_1, \quad i = 1; \ldots; k.$$ 

Using Rolfsen twists (hence changing $n$ if necessary), any tuple

$$(g; n; (1; 1); \ldots; (k; k))$$

can be transformed into normal form.

Consider the family of contact 3-manifolds defined by the contact surgery diagrams of Figure 1 (the box is repeated $(g - 1)$ times, $g \neq 1$).

Throughout the paper we shall assume

$$g \geq 1; \quad \frac{1}{2} r_1 < 1; \quad r_i < 0; \quad i = 2; \ldots; k \quad (r_i \in \mathbb{Q})$$ \quad (1.1)$$

Under the assumptions (1.1) one can write the coefficients as:

$$r_1 = \frac{(n - 2g + 1)}{(n - 2g + 2)} \frac{1}{1} + \frac{1}{1}; \quad r_i = \frac{i - i_1}{i}$$ \quad (1.2)$$

where

$$n \geq 2g; \quad i > i_1; \quad i > i_1; \quad i = 2; \ldots; k.$$
Converting the contact surgery coefficients into smooth coefficients, after $(n - 2g + 1)$ Rolfsen twists on the $r_1$-framed unknot we conclude that the 3-manifolds underlying the contact structures given by Figure 1 are of the form:

\[ M(g; n; (1; 1); \ldots (k; k)); \quad n \geq 2g. \]  

Moreover, if $r_1 > 0$ the Seifert invariants are in normal form. Observe that for $r_1 = 0$ the $(-1,1)$ surgery is trivial.

Conversely, given a Seifert fibered 3-manifold $M$ as in (1.3), Figure 1 provides a contact structure on $M$ as long as the coefficients $r_i$ defined by (1.2) satisfy the conditions (1.1).
Let $1;:::;t$ denote the contact structures obtained by turning the diagrams of Figure 1 into contact ($-1$) surgeries in all possible ways according to the algorithm described in the previous subsection. This paper is devoted to the study of $1;:::;t$. Using the contact Ozsváth-Szabó invariants [21] we prove:

**Theorem 1.2** Fix $k = 1$, $g = 1$, $\frac{1}{2} < r_1 < 1$ and $r_i < 0$ for $i = 2;:::;k$. Then, all the contact structures defined by Figure 1 are tight.

It is unclear from the construction whether the contact structures $1;:::;t$ are all distinct up to isotopy. Observe that for $k = 1$ and $r_1 = \frac{d+1}{2d+1}$ the 3-manifold underlying Figure 1 is $M(g;2g;(-;1))$.

**Theorem 1.3** Given $g = 1$ and $n \in \mathbb{N}$, there is an $2 \mathbb{N}$ such that at least $n$ of the contact structures defined by Figure 1 for $k = 1$ and $r_1 = \frac{d+1}{2d+1}$ are pairwise non-isomorphic.

In fact, a more detailed analysis shows that the contact structures defined by Figure 1 on $M(g;2g;(-;1))$ are all distinct up to isotopy (see Section 4). This leads us to:

**Conjecture 1** All the tight contact structures defined by Figure 1 and satisfying the assumptions (1.1) are distinct up to isotopy.

Recall that a contact 3-manifold $(Y;\xi)$ is symplectically fillable, or simply fillable, if there exists a compact symplectic four-manifold $(W;\omega)$ such that (i) $\partial W = Y$ as oriented manifolds (here $W$ is oriented by $\omega^+$) and (ii) $\omega^+|_Y \neq 0$ at every point of $Y$. Our next result concerns fillability properties of some of the contact structures under examination.

**Theorem 1.4** Fix $2 \mathbb{N}$ and $g = 1$ such that $d(d+1) \geq 2g$ for some positive integer $d$. Then, the tight contact structures defined by Figure 1 for $k = 1$ and $r_1 = \frac{d+1}{2d+1}$ are not symplectically fillable.

As we show in Section 4, there is some evidence supporting the following:

**Conjecture 2** No contact structure defined by Figure 1 and satisfying conditions (1.1) is fillable.

The above results immediately imply Theorem 1.1:
Proof of Theorem 1.1 Fix $n \geq 2N$ and $g = 1$. Choose $2N$ such that the statement of Theorem 1.3 holds. The contact structures $\xi_1; \ldots; \xi_t$ defined by Figure 1 on $M(1; 2; ( \frac{1}{2}; 1))$ are tight by Theorem 1.2 and there are at least $n$ pairwise non-isomorphic among them by Theorem 1.3. By Theorem 1.4 applied with $d = 1$ they are also not fillable. This concludes the proof.

Our results seem to suggest (see Section 4) that a Seifert fibered 3-manifold $M(g; n; (\frac{1}{1}; \ldots; \frac{1}{1}); (\frac{k}{k}; \ldots; \frac{k}{k}))$ with Seifert invariants in normal form should support a tight, not fillable contact structure if $n \geq 2g > 0$. This should be contrasted with the result of Gompf [8], who showed that a Seifert fibered 3-manifold with base genus $g \geq 1$ always carries a Stein fillable contact structure.

Section 2 is devoted to the proof of Theorem 1.2, while Theorems 1.3 and 1.4 will be proved in Section 3. In Section 4 we give further evidence supporting Conjectures 1 and 2.

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2 Proof of Theorem 1.2

In a remarkable series of papers [17, 18, 19, 21] Ozsvath and Szabo defined new invariants of many low-dimensional objects including contact structures on closed 3-manifolds. In this section we apply these invariants to prove Theorem 1.2.

Heegaard Floer theory associates abelian groups $HF^+(Y; t)$ and $\hat{HF}(Y; t)$ to a closed, oriented Spin$^c$ 3-manifold $(Y; t)$, and homomorphisms

$$F^+_W: HF^+(Y_1; t_1) \to HF^+(Y_2; t_2); \quad \hat{F}^+_W: \hat{HF}(Y_1; t_1) \to \hat{HF}(Y_2; t_2)$$

to a Spin$^c$ cobordism $(W; s)$ between two Spin$^c$ 3-manifolds $(Y_1; t_1)$ and $(Y_2; t_2)$.

Throughout this paper we shall assume that $\mathbb{Z} = \mathbb{Z}$ coefficients are being used in the complexes defining the $HF^+$ and $\hat{HF}$ groups.
Let $Y_{g;-2g}$ be a circle bundle over the genus $g$ surface with Euler number $-2g$ ( $g \geq 1$), and let $D_{g;-2g}$ denote the corresponding disk bundle. Since $H^2(D_{g;-2g};\mathbb{Z})$ has no 2-torsion, each Spin$^c$ structure on $D_{g;-2g}$ is uniquely determined by its first Chern class. Let $s$ be the unique Spin$^c$ structure on $D_{g;-2g}$ with $c_1(s) = 0$, and denote by $t$ the restriction of $s$ to $Y_{g;-2g}$.

Let $W$ denote the cobordism from $#_{2g}(S^1 \times S^2)$ to $Y_{g;-2g}$ given by the attachment of a 4-dimensional 2-handle along the $(-2g)$-framed knot $K$ of Figure 2. Let $t_0$ be the unique Spin$^c$ structure on $#_{2g}(S^1 \times S^2)$ with vanishing first Chern class. In [20, Lemma 9.17] it is proved that there is an isomorphism

$$HF^+(#_{2g}(S^1 \times S^2); t_0) \cong HF^+(Y_{g;-2g}; t)$$

which can be written as a sum of maps $\varphi_{W,s}$ over the set of Spin$^c$ structures on $W$ which restrict to $t_0$ and $t$. Application of the 5-lemma to the long exact sequence connecting $HF^+(Y_{g;-2g}; t)$ and $HF(Y_{g;-2g}; t)$ immediately yields the following:

**Lemma 2.1** The homomorphism

$$\varphi_{W,s}: HF(#_{2g}(S^1 \times S^2); t_0) \rightarrow HF(Y_{g;-2g}; t)$$

is an isomorphism.

**Contact Ozsvath-Szabo invariants**

Let $(Y; \xi)$ be a closed contact $3$-manifold oriented by $\xi$, and let $t$ be the Spin$^c$ structure induced by $\xi$. In [21], Ozsvath and Szabo define an
invariant
\[ c(Y; t) \cap F(-Y; t) \]
whose main properties are summarized in the following two theorems.

**Theorem 2.2** [21] If \((Y; t)\) is overtwisted, then \(c(Y; t) = 0\). If \((Y; t)\) is Stein fillable then \(c(Y; t) \neq 0\). In particular, for the standard contact structure \((S^3; st)\) we have \(c(S^3; st) \neq 0\).

**Theorem 2.3** Suppose that \((Y_2; 2)\) is obtained from \((Y_1; 1)\) by a contact \((+1)\) surgery. Then we have
\[ F_{-W}(c(Y_1; 1)) = c(Y_2; 2); \]
where \(-W\) is the cobordism induced by the surgery with reversed orientation and \(F_{-W}\) is the sum of \(\mathbb{P}_{-W; s}\) over all Spin\(^c\) structures \(s\) extending the Spin\(^c\) structures induced on \(-Y_i\) by \(K_i\), \(i = 1; 2\). In particular, if \(c(Y_2; 2) \neq 0\) then \((Y_1; 1)\) is tight.

**Proof** Let us assume that we are performing contact \((+1)\) surgery along the Legendrian knot \(K\) along \((Y_1; 1)\). Then, there is an open book decomposition \((F; 1)\) on \(Y_1\) compatible with \(1\) in the sense of Giroux and such that \(K\) lies on a page. In fact, the proof of [7, Theorem 3] shows that the 1-skeleton of any contact cellular decomposition of \((Y_1; 1)\) is contained in a page of a compatible open book. Since \(K\) can be assumed to lie in the 1-skeleton of a contact cellular decomposition of \((Y_1; 1)\), the conclusion follows. Moreover, up to re-arranging the decomposition, we may assume that \(K\) is not homotopic to the boundary of the page. Then, an open book for \((Y_2; 2)\) is given by \((F; 0)\), where \(0 = R^{-1}_K\) and \(R_k\) is the right-handed Dehn twist along \(K\). The first part of the statement now follows applying [21, Theorem 4.2]. The second part of the statement follows immediately from the fact that the invariant of an overtwisted contact structure vanishes.

Theorem 2.3 immediately yields:

**Corollary 2.4** If \(c(Y_2; 2) \neq 0\) and \((Y_1; 1)\) is obtained from \((Y_2; 2)\) by Legendrian surgery along a Legendrian knot, then \(c(Y_1; 1) \neq 0\). In particular, \((Y_1; 1)\) is tight.

**Proof** Let \(K\) be the Legendrian knot along which the Legendrian surgery is performed. A Legendrian push- of \(K\) gives rise to a Legendrian
knot $\mathcal{K}$ in $(Y_1; 1)$. By [1, Proposition 8], contact $(+1)$-surgery on $(Y_1; 1)$ along $\mathcal{K}$ gives back $(Y_2; 2)$. Therefore, by Theorem 2.3 $c(Y_2; 2) \neq 0$ implies $c(Y_1; 1) \neq 0$.

Let $(Z_j; j)$ be the contact 3-manifold obtained by performing contact $(+1)$-surgery on the standard contact three-sphere along the $j$-component Legendrian unlink depicted in Figure 3.

![Figure 3: The contact 3-manifold $(Z_j; j)$](image)

**Lemma 2.5** The contact 3-manifold $(Z_j; j)$ given by Figure 3 has non-vanishing contact Ozsvath-Szabo invariant for every $j \neq 0$.

**Proof** Notice first that $Z_j$ is diffeomorphic to $\#_j(S^1 \times S^2)$. We will argue by induction on $j$. For $j = 0$ we have the standard contact 3-sphere, which has non-vanishing contact Ozsvath-Szabo invariant by Theorem 2.2. Now consider $j - 1$ and add the $j$-th component of the Legendrian unlink to it with contact framing $(+1)$. Let $-W$ be the corresponding cobordism with reversed orientation. By [18, Theorem 9.16] the homomorphism $F_{-W}$ fits into an exact triangle:

$$
\begin{array}{c}
\hat{H}F(\#_j(S^1 \times S^2)) \\
\hat{H}F(\#_{j-1}(S^1 \times S^2)) \\
\hat{H}F(\#_{j-1}(S^1 \times S^2))
\end{array}
\xrightarrow{F_{-W}} \begin{array}{c}
\hat{H}F(\#_{j-1}(S^1 \times S^2)) \\
\hat{H}F(\#_{j-1}(S^1 \times S^2)) \\
\hat{H}F(\#_{j-1}(S^1 \times S^2))
\end{array}
$$

In [18, Subsection 3.1 and Proposition 6.1] it is proved that

$$\dim_{\mathbb{Z}[\mathbb{Q}]} \hat{H}F(\#_j(S^1 \times S^2)) = 2^j$$

Therefore, the exactness of the triangle implies that the map $F_{-W}$ is injective. Since by Theorem 2.3 we have

$$F_{-W}(c(Z_{j-1}; j-1)) = c(Z_j; j)$$
and by the inductive assumption $c(Z_{j-1}^*; j-1)) \not\in 0$, this concludes the proof.  

Note that when $k = 1$ and $r_1 = \frac{1}{2}$, Figure 1 specifies a unique contact structure $g$ for every $g$ because the contact surgery coefficients are of the form $\frac{1}{k}, k \geq 2$. Denote the resulting contact 3-manifold by $(Y_g; g)$. It is a simple exercise to verify that $Y_g$ is an $S^1$-bundle over a genus-$g$ surface with Euler number $e(Y_g) = 2g$.

**Proposition 2.6** The contact Ozsváth-Szabó invariant of $(Y_g; g)$ is nonzero.

**Proof** Let $(Y_g^0; g)$ be the contact 3-manifold given by Figure 1 with $k = 1$ and $r_1 = 1$, and perform contact (+1)-surgery on a pusho of the $r_1$-framed Legendrian knot $K$. According to the algorithm described in Section 1, the resulting contact structure is $(Y_g; g)$. Note that $Y_g^0$ is diffeomorphic to $\#_{2g}(S^1 \times S^2)$. Combining Lemma 2.5 and Corollary 2.4 we conclude $c(Y_g^0; g) \not\in 0$. In fact, $(Y_g^0; g)$ must be the only tight, hence Stein fillable contact structure on $\#_{2g}(S^1 \times S^2)$. The cobordism given by the handle attachment induced by the surgery along $K$ can be easily identified (after reversing orientation) with the cobordism appearing in Lemma 2.1, therefore the non-vanishing of $c(Y_g^0; g)$ implies, by Theorem 2.3, that $c(Y_g; g) \not\in 0$.

**Remark 2.7** The tightness of the contact structures $g$ was first proved by Honda [9] (see also [13]).

**Proof of Theorem 1.2** Let $K_1; K_2$ denote two Legendrian pushos of the $r_1$-framed Legendrian unknot $K$ of Figure 1. According to the algorithm of Section 1 all contact structures of Figure 1 can be given as negative contact surgery on the diagram obtained erasing the $r_i$-framed circles $(i = 2; \ldots; k)$ from Figure 1 and performing contact (+1)-surgery on $K; K_1$ and contact $\frac{r_1}{1-2r_1}$-surgery on $K_2$. (Here we use the assumption $r_1 < 0$ for $i = 2; \ldots; k$.) Since $r_1 = \frac{1}{2}$, the surgery coefficient of $K_2$ is also negative (or infinity), therefore all the contact structures defined by Figure 1 (obeying the restrictions on the $r_i$) can be given as Legendrian surgery on $(Y_g; g)$ for an appropriate $g \geq 1$. Since negative contact surgery can be replaced by a sequence of Legendrian surgeries, Corollary 2.4 and Proposition 2.6 imply that these contact structures have non-vanishing contact Ozsváth-Szabó invariants, hence by Theorem 2.2 they are tight. This concludes the proof of the theorem.
3 The proof of non-illability

Suppose that \((Y; \xi)\) is given by a contact \((-1)\)-surgery diagram and denote the corresponding 4-manifold by \(X\). Then, the \(\text{Spin}^c\) structure of the 0-handle of \(X\) extends to a \(\text{Spin}^c\) structure \(\mathbf{s} \in \text{Spin}^c(X)\) with the property that \(\mathbf{s}|_{\partial X} = \mathbf{t}\) and \(c_1(\mathbf{s})\) evaluates on a homology class \([K]\) given by an oriented surgery curve \(K\) as \(\text{rot}(K)\). This statement was proved for \((-1)\)-surgeries by Gompf \[8\] in this case the complex structure of \(D^4\) also extends over the 2-handles and in \[13\] for the case of \((+1)\)-surgeries; see also \[3\].

Consider the diagram obtained from Figure 1 for \(k = 1\) and \(r_1 = -1\); this diagram represents contact structures on \(M(g; 2g; (1))\). According to the algorithm outlined in Section 1, these contact structures are also representable by replacing the Legendrian knot \(K\) with three Legendrian pushos \(K_1; K_2; K_3\) having contact surgery coefficients \((+1), (+1)\) and \((-1, +1)\), respectively. This last diagram can be turned into a contact \((-1)\)-surgery diagram by stabilizing the Legendrian curve \(K_3\) times. There are \((-1)\) different ways to do this. Choose an orientation for \(K_3\) and define \(r\) as the result of the surgery along the diagram with \(\text{rot}(K_3) = r\). (Notice that \(r \equiv 0 \pmod{2}\) and \(-r \equiv 1 \pmod{2}\).)

The above observation regarding \(\text{Spin}^c\) structures yields:

**Lemma 3.1** Let \(\mathbf{s} \in \text{Spin}^c(X)\) be the unique \(\text{Spin}^c\) structure such that \(c_1(\mathbf{s}); [K_3] = r\) and \(c_1(\mathbf{s}); [J] = 0\) on the 2-homology classes defined by the remaining surgery circles. Then, the restriction of \(\mathbf{s}\) to \(\partial X\) is the \(\text{Spin}^c\) structure \(\mathbf{t} \in \text{Spin}^c(M(g; 2g; (1)))\) induced by the contact structure \(r\).

Recall that, since \(X\) is simply connected, the Chern class \(c_1(\mathbf{s})\) uniquely specifies the \(\text{Spin}^c\) structure \(\mathbf{s} \in \text{Spin}^c(X)\). For \(M = M(g; 2g; (1))\) let \(2\) \(H_1(M; \mathbb{Z})\) denote the homology class of the normal circle to the knot \(K_3\) or, equivalently, the homology class represented by the singular fiber of the Seifert bration. Then, Lemma 3.1 implies that

\[ c_1(\mathbf{t}, r) = c_1(\mathbf{t}, r) = r \text{PD}(\xi\mathbf{t}) \]

In particular, since the order of \(r\) in \(H_1(M; \mathbb{Z})\) is equal to \(2g + 1\), \(\mathbf{t}, r\) is a torsion \(\text{Spin}^c\) structure for all \(r\).

**Proof of Theorem 1.3** By the classical Dirichlet’s theorem on primes in arithmetic progressions, there are infinitely many primes of the form \(2gm + 1\) as \(m\) varies among the natural numbers. Therefore, we can choose natural numbers \(a_1; \ldots; a_n\) so that

\[ p_1 = 2ga_1 + 1; \ldots; p_n = 2ga_n + 1 \]
are distinct odd primes. Define so that
\[ 2ga + 1 = p_1 \cdots p_n. \]
If \( a \) is odd, let \( \gamma_i = a \), otherwise let \( \gamma_i = a(2g+1) + 1 \). With this choice \( 2g + 1 \) is divisible by \( p_1 \cdots p_n \) and \( n \) is odd. Therefore,
\[ p_i \mod 2; \quad i = 1; \cdots; n; \]
and we can choose the stabilizations of \( K_3 \) so that \( c_i(\gamma_i) = p_i \). This implies that the order of \( c_i(\gamma_i) \) is \( \frac{2g+1}{p_i} \), and since the \( p_i \)'s are all distinct, the orders of the \( c_i(\gamma_i) \)'s are all different for \( i = 1; \cdots; n \). This shows that the contact structures \( \gamma_i, i = 1; \cdots; n \), are pairwise non-isomorphic, concluding the proof.

The proof of Theorem 1.4 will follow the approach used in [10] and further exploited in [12]. Fix a Seifert fibration
\[ M = M(g; n; (1; 1); \cdots; (k; k)) \]
over the orbifold \( g \). The surface \( g \) can be thought of as the underlying space of an orbifold with \( k \) marked points of multiplicities \( 1; \cdots; k \). An orbifold line bundle \( L \) on \( g \) can be pulled back to an honest line bundle \( L \) on \( M \) with torsion first Chern class, and if the invariants \( \gamma_i \) are mutually coprime, all line bundles on \( M \) with torsion first Chern class arise in this way. An orbifold line bundle \( L \) on \( g \) can be described by its Seifert data \( (c; \gamma_1; \cdots; \gamma_k) \), where \( c \) is the background degree of \( L \) and the numbers \( \gamma_i \) determine the orbifold bundle around the orbifold points of \( g \) (see [14, x2] for further details). For example, the orbifold canonical bundle \( K \) has Seifert data \( (2g - 2; 1 - 1; \cdots; k - 1) \). The degree of the orbifold line bundle \( L \) is equal by definition to the rational number
\[ \deg(L) = b + \sum_{i=1}^{k} \frac{\gamma_i}{i}. \]

For more about Seifert fibred three-manifolds and line bundles on them see [14, 16].

**Theorem 3.2** [14]. The moduli space of Seiberg-Witten solutions for the Seifert fibred 3-manifold \( M = M(g; 2g; (1; 1)) \) and Spin\(^c\) structure \( t \), \( 2 \text{ Spin}^c(M) \) contains only reducible solutions, for all of which the associated Dirac operator has trivial kernel.
Proof. We need to express the Spin\(^c\) structure \(t_r\), in the coordinates used in [14] and then appeal to the description of the Seiberg-Witten moduli spaces on Seifert fibered 3-manifolds as given in [14, Theorem 5.19]. In that paper the Spin\(^c\) structures are parametrized by their twisting relative to the canonical Spin\(^c\) structure \(t_{can}\) induced by any tangent 2-plane field transverse to the \(S^1\) fibration. As explained in [14, x3], the orbifold disk bundle associated to \(M\) can be desingularized to a smooth complex surface \(X\) with \(\partial X = M\). The group \(H_2(X;\mathbb{Z})\) is generated by the classes of a genus-\(g\) smooth complex curve \(C\) and a smooth rational curve \(R\), satisfying:

- \(C \cdot C = 2g\)
- \(C \cdot R = 1\)
- \(R \cdot R = -1\)

The restriction to \(\partial X\) of the complex bundle \(TX\) is isomorphic to the pull-back of

\[
\begin{align*}
&\mathbb{C} \to K^{-1} \overset{g}{\to} \\
&\text{where } \mathbb{C} \text{ is the trivial complex line bundle and } K \text{ is the orbifold canonical bundle of } g.
\end{align*}
\]

Therefore, denoting by \(s^c\) the Spin\(^c\) structure on \(X\) induced by the complex structure, we have \(s^c_{|\partial X} = t_{can}\) (cf. text following [14, Lemma 5.10]). The adjunction formula gives:

- \(h^c_1(X); Ci = 2\)
- \(h^c_1(X); Ri = 2 - \)

Thus, if \(\Gamma_r \in H^2(X;\mathbb{Z})\) is a cohomology class satisfying

\[
H_{\Gamma_r}; Ci = -1 \quad H_{\Gamma_r}; Ri = \frac{1}{2}(r + - 2);
\]

setting \(s_r = s^c + \Gamma_r\), we have \(s^c_{|\partial X} = t_r\). This implies:

\[
t_r = t_{can} + \Gamma_r = t_{can} + \frac{1}{2}(r - - 2) \text{ PD}( );
\]

(3.1)

Now [14, Theorem 5.19] can be restated in the following form, more convenient for our present purposes. Fix a torsion Spin\(^c\) structure

\[
\begin{align*}
\mathbf{t}_k = t_{can} + k \text{ PD}( ) \text{ 2 Spin}^c(M);\end{align*}
\]

Let \(L_k\) be an orbifold line bundle which pulls back to a line bundle \(L_k\) on \(M\) with \(c_1(L_k) = k \text{ PD}( )\). Then, the moduli space \(\mathcal{M}_k\) of Seiberg-Witten solutions on \(M\) in the Spin\(^c\) structure \(\mathbf{t}_k\) has a component of reducible solutions (homeomorphic to the Jacobian torus of \(g\)), and by [14, Corollary 5.17] the associated Dirac operators have trivial kernels if and only if either \(r\) is even or

\[
\deg L_k \mathbb{B} \frac{1}{2} \deg K + (2g + \frac{1}{2}) \in \mathbb{Q};
\]

(3.2)

In addition, $M_k$ contains irreducible solutions if and only if there exists some orbifold line bundle $L$! satisfying:

$$\deg L \geq 0, \deg K \geq \frac{1}{2} \deg K \geq \deg L_k + (2g + \frac{1}{2}) \mathbb{Z}; \quad (3.3)$$

In view of (3.1), in our case we have:

$$k = \frac{1}{2}(r - 2) - 2g + \frac{1}{2}(r - ) \mod (2g + 1);$$

Therefore, since $r 2 [- ; ]$, 

$$\deg K = 2g - 1 - \frac{1}{2} < \deg L_k = 2g + \frac{1}{2}(r - ) < 2g + 1;$$

It follows that $L_k$ satisfies (3.2) and there is no orbifold line bundle $L$! satisfying (3.3). Hence, $M_k$ consists entirely of reducible solutions with associated Dirac operators having trivial kernels.

**Corollary 3.3** Let $(W; !)$ be a weak filling of the contact 3-manifold $(M; r)$. Then, $b_2^g(W) = 0$ and the homomorphism $H^2(W; \mathbb{R}) \rightarrow H^2(\partial W; \mathbb{R})$ induced by the inclusion $\partial W \rightarrow W$ is the zero map.

**Proof** The statement follows from Theorem 3.2 in exactly the same way as [12, Proposition 4.2] follows from [12, Lemma 4.1].

**Proof of Theorem 1.4** Let $r$ be one of the contact structures on $M = M(g; 2g; (1))$ given by Figure 1. We shall argue as in [12, Theorem 1.1], therefore we shall need to nd a 4-manifold $Z = Z(g; 2g; (1))$ with $b_2^g(Z) = 0$, $\partial Z = -M$ and such that the intersection form $Q_Z$ does not embed into the diagonal lattice $D_m = (Z^m; m(-1))$ for any $m$.

We shall use a construction similar to the one given in [12, Proposition 4.4]. To this end, let $\mathbb{CP}^2$ be a smooth complex curve of degree $d + 2$, and let $\mathbb{CP}^2$ be the blow-up of $\mathbb{CP}^2$ at $(d + 2)^2 - 2g - 1$ distinct points of $C$. Denote by $\mathbb{CP}^2$ the proper transform of $C$. Let $\mathbb{CP}^2$ be a smooth, oriented surface obtained by adding $g - \frac{1}{2}d(d + 1)$ fake handles to $\mathbb{CP}^2$. Blow up $\mathbb{CP}^2$ at one more point of $\mathbb{CP}^2$, then blow up repeatedly at distinct points of the last exceptional sphere until the corresponding proper transform in the resulting rational surface $X$ is an embedded sphere $S$ with self-intersection $-1$. Define $Z$ as the complement in $X$ of a tubular neighborhood of $\mathbb{CP}^2 | S$. 

The group $H_2(X;\mathbb{Z})$ is generated by classes $h; e_1; e_2; \ldots; e_t$, where $h$ corresponds to the standard generator of $H_2(CP^2;\mathbb{Z})$ and the $e_i$'s are the classes of the exceptional curves. Let $q$ be a positive integer such that $2q \leq t$, and define $H_q = (H_q; Q_q)$ as the intersection lattice given by the subgroup

$$H_q = h e_1 - e_2 - e_3 - \ldots - e_{2q}; h - e_1 - e_2 - \ldots - e_q$$

$H_2(X;\mathbb{Z})$ together with the restriction $Q_q$ of the intersection form $Q_X$.

As in the proof of [12, Proposition 4.4], the inequality $2g d (d + 2) - 1$ guarantees that $2(d + 2) \leq t$, hence the lattice $d+2 = (H_{d+2}; Q_{d+2})$ embeds into $(H_2(Z;\mathbb{Z}); Q_2)$. Since by [12, Lemma 4.3] $d+2 = (H_{d+2}; Q_{d+2})$ does not embed into any diagonal lattice $D_m$, the same holds for $(H_2(Z;\mathbb{Z}); Q_2)$.

By Corollary 3.3, an filling $(W;!)$ would give rise to a negative definite closed 4-manifold $V = W \setminus Z$ with nonstandard intersection form, contradicting Donaldson’s famous diagonalizability result [4].

4 Concluding remarks

With a little more work, essentially the same proof as the one given in Section 3 yields nonfillability for all structures defined by Figure 1 on $M(g;n; ( ; ))$ and satisfying

$$d(d+1) < 2g d (d + 2) - 1$$

for $g \geq 1$ and some integer $d$. In fact, a slightly more general argument in the computation of the Spin$^c$ structures allows one to check that the statement of Theorem 3.2 still holds.

In another direction, Theorem 1.4 generalizes to all $M(g;n; ( ; )1)$ with $n 2g > 0$. In this case, one needs to consider Figure 1 for $k = 1$ and

$$r_1 = \frac{(n - 2g + 1)}{(n - 2g + 2)} + 1.$$ 

According to the algorithm described in Section 1, the corresponding contact surgery can be expressed as a contact $(1)$ surgery by replacing the $r_1$ framed unknot $K$ with two pushos of $K$, $n - 2g$ pushos of a stabilization $K$ of $K$, and one pusho of $K$ stabilized $-1$ times. Depending on the choice of stabilization of $K$, the result looks either like Figure 4 or Figure 5. Denoting by $r$ the rotation number of the last knot (after a choice of orientation), this gives a contact structure $\xi^+$ for every $- < r = -1$ and a contact structure $\xi^-$ for every $-1 < r < \mod 2$ in both cases.
A computation as in Section 3 gives
\[ t_r = t_{can} + \frac{1}{2}(r - 2)(n - 2g) - (n - 2g) \text{PD}(\cdot) \]
This already shows that the contact structures defined on \( M((g;n;( ;1)) \) by Figure 1 are all distinct up to homotopy, providing further evidence for Conjecture 1.

One can also compute the 3-dimensional invariant \( d_3([r]) \) of the homotopy class \([r]\) of tangent 2-plane fields containing the contact structure \( r \) (as discussed in [13]), obtaining:
\[ d_3([r]) = \frac{1}{4(n + 1)}((n - 2g)^2 - r^2n - 2(n - 2g)r) + \frac{2g - 1}{2}; \]
On the other hand, the statement of Theorem 3.2 holds for all contact structures defined on \( M = M((g;n;( ;1)) \) by Figure 1 for \( n \geq 2g \). Therefore, the argument of [11, Theorem 2.1] and [13, Theorem 4.1] applies, showing that there is a unique homotopy class \( (t_r) \) of 2-plane fields inducing the Spin\(^c\) structure \( t_r \) and which might potentially contain a fillable contact structure. The proof of this observation rests on the fact that, assuming Theorem 3.2 to hold, the 3-dimensional invariant of \( (t_r) \) is determined by some topological terms plus an \( \{\text{invariant of } (M; t_r)\} \) as follows.

By the formula preceding [15, Section 3] (when \( L \neq 0 \), which always holds in our case), the dimension \( d_1 \) of the Seiberg-Witten moduli space with fixed boundary limit can be expressed as
\[ d_1 = d_3((t_r)) + !_{red}(t_r) - (2g - 1); \]

Figure 5: The contact structures

where $d_3(t_r)$ is the 3-dimensional invariant of $(t_r)$ and $\!_{\text{red}}(t_r)$ is given, in the notations of [15], by the formula:

$$
\frac{2g-1}{2} - \frac{1 - \text{sign}(l)}{4} + l \left( 1 - \frac{1}{2} \right) - \frac{1 - 2g}{2} (1 - 2) + S(1; \gamma) + F(1; \gamma) + 2S(1; \gamma) : 
$$

In our situation we have:

$$
l = n + \frac{1}{2}; \quad \text{sign}(l) = 1; \quad = \frac{(n - 2g) - r + 1}{2n + 2};
$$

$$
\gamma = \frac{1}{2}(r + 2); \quad S(1; \gamma) = \frac{2 + 2}{12} - \frac{1}{4}; \quad F(1; \gamma) = \frac{\gamma + 1}{12};
$$

$$
S(1; \gamma) = \frac{2 - 3(1 + 2\gamma) + 2(1 + 3\gamma + 3\gamma^2)}{12};
$$

This shows that

$$
\!_{\text{red}}(t_r) = -\frac{1}{4(n + 1)} ((n - 2g)^2 - r^2n - 2(2n - 2g)r) + \frac{2g - 1}{2};
$$

On the other hand, by the argument of [11, Theorem 2.1] we have

$$
d_1 = -1 - b_2(M) = -1 - 2g;
$$

therefore

$$
d_3(t_r) = -\!_{\text{red}}(t_r) - 2;
$$

yielding

$$
d_3(t_r) = \frac{1}{4(n + 1)} ((n - 2g)^2 - r^2n - 2(2n - 2g)r) - \frac{2g + 3}{2};
$$
Since
\[ d_3([x]) - d_3(t) = 2g + 1 \neq 0; \]
none of the contact structures defined by Figure 1 on \( M(g;n;[1]) \) (\( n > 2g \)) are symplectically fillable.

We believe that the same idea should work for all the tight contact structures given by Figure 1 (with the constraints (1.1)). The verification of non-fillability, however, seems to be much more tedious in the general case. The difficulty is number-theoretic in nature: it is hard to see that \( d_3([x]) \neq d_3(t) \), because the formulas involve sums which are hard to write in closed form.

References


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