H-space structure on pointed mapping spaces

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Abstract We investigate the existence of an $H$-space structure on the function space, $\mathcal{F}_*(X,Y,\ast)$, of based maps in the component of the trivial map between two pointed connected CW-complexes $X$ and $Y$. For that, we introduce the notion of $H(n)$-space and prove that we have an $H$-space structure on $\mathcal{F}_*(X,Y,\ast)$ if $Y$ is an $H(n)$-space and $X$ is of Lusternik-Schnirelmann category less than or equal to $n$. When we consider the rational homotopy type of nilpotent finite type CW-complexes, the existence of an $H(n)$-space structure can be easily detected on the minimal model and coincides with the differential length considered by Y. Kotani. When $X$ is finite, using the Haefliger model for function spaces, we can prove that the rational cohomology of $\mathcal{F}_*(X,Y,\ast)$ is free commutative if the rational cup length of $X$ is strictly less than the differential length of $Y$, generalizing a recent result of Y. Kotani.

AMS Classification 55R80, 55P62, 55T99

Keywords Mapping spaces, Haefliger model, Lusternik-Schnirelmann category

1 Introduction

Let $X$ and $Y$ be pointed connected CW-complexes. We study the occurrence of an $H$-space structure on the function space, $\mathcal{F}_*(X,Y,\ast)$, of based maps in the component of the trivial map. Of course when $X$ is a co-$H$-space or $Y$ is an $H$-space this mapping space is an $H$-space. Here, we are considering weaker conditions, both on $X$ and $Y$, which guarantee the existence of an $H$-space structure on the function space. In Definition 3 we introduce the notion of $H(n)$-space designed for this purpose and prove:

Proposition 1 Let $Y$ be an $H(n)$-space and $X$ be a space of Lusternik-Schnirelmann category less than or equal to $n$. Then the space $\mathcal{F}_*(X,Y,\ast)$ is an $H$-space.
The existence of an $H(n)$-structure and the Lusternik-Schnirelmann category (LS-category in short) are hard to determine. We first study some properties of $H(n)$-spaces and give some examples. Concerning the second hypothesis, we are interested in replacing $\text{cat}(X) \leq n$ by an upper bound on an approximation of the LS-category (see [9, Chapter 2]). We succeed in Proposition 7 with an hypothesis on the dimension of $X$ but the most interesting replacement is obtained in the rational setting which constitutes the second part of this paper.

We use Sullivan minimal models for which we refer to [6]. We recall here that each finite type nilpotent CW-complex $X$ has a unique minimal model $(\wedge V, d)$ that characterises all the rational homotopy type of $X$. We first prove that the existence of an $H(n)$-structure on a rational space $X_0$ can be easily detected from its minimal model. It corresponds to a valuation of the differential of this model, introduced by Y. Kotani in [11]: The differential $d$ of the minimal model $(\wedge V, d)$ can be written as $d = d_1 + d_2 + \cdots$ where $d_i$ increases the word length by $i$. The differential length of $(\wedge V, d)$, denoted $\text{dl}(X)$, is the least integer $n$ such that $d_{n-1}$ is non zero. As a minimal model of $X$ is defined up to isomorphism, the differential length is a rational homotopy type invariant of $X$, see [11, Theorem 1.1]. Proposition 8 establishes a relation between $\text{dl}(X)$ and the existence of an $H(n)$-structure on the rationalisation of $X$.

Finally, recall that the rational cup-length $\text{cup}_0(X)$ of $X$ is the maximal length of a nonzero product in $H^{>0}(X; \mathbb{Q})$. In [11], by using this cup-length and the invariant $\text{dl}(Y)$, Y. Kotani gives a necessary and sufficient condition for the rational cohomology of $\mathcal{F}_*(X,Y,*)$ to be free commutative when $X$ is a rational formal space and when the dimension of $X$ is less than the connectivity of $Y$. We show here that a large part of the Kotani criterium remains valid, without hypothesis of formality and dimension. We prove:

**Theorem 2** Let $X$ and $Y$ be nilpotent finite type CW-complexes, with $X$ finite.

1. The cohomology algebra $H^*(\mathcal{F}_*(X,Y,*); \mathbb{Q})$ is free commutative if $\text{cup}_0(X) < \text{dl}(Y)$.

2. If $\text{dim}(X) \leq \text{conn}(Y)$, then the cohomology algebra $H^*(\mathcal{F}_*(X,Y,*); \mathbb{Q})$ is free commutative if, and only if, $\text{cup}_0(X) < \text{dl}(Y)$.

As an application, we describe in Theorem 12 the Postnikov tower of the rationalisation of $\mathcal{F}_*(X,Y,*)$ where $X$ is a finite nilpotent space and $Y$ a finite type CW-complex whose connectivity is greater than the dimension of $X$. Our description implies the solvability of the rational Pontrjagin algebra of $\Omega(\mathcal{F}_*(X,Y,*))$. 

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Section 2 contains the topological setting and the proof of Proposition 1. The link with rational models is done in Section 3. Our proof of Theorem 2 uses the Haefliger model for mapping spaces. In order to be self-contained, we recall briefly Haefliger’s construction in Section 4. The proof of Theorem 2 is contained in Section 5. Finally, Section 6 is devoted to the description of the Postnikov tower.

In this text, all spaces are supposed of the homotopy type of connected pointed CW-complexes and we will use cdga for commutative differential graded algebra. A quasi-isomorphism is a morphism of cdga’s which induces an isomorphism in cohomology.

## 2 Structure of $H(n)$-space

First we recall the construction of Ganea fibrations, $p_n^X : G_n(X) \to X$.

- Let $F_0(X) \xrightarrow{i_0} G_0(X) \xrightarrow{p_0^X} X$ denote the path fibration on $X$, $\Omega X \to PX \to X$.

- Suppose a fibration $F_n(X) \xrightarrow{i_n} G_n(X) \xrightarrow{p_n^X} X$ has been constructed. We extend $p_n^X$ to a map $q_n : G_n(X) \cup C(F_n(X)) \to X$, defined on the mapping cone of $i_n$, by setting $q_n(x) = p_n^X(x)$ for $x \in G_n(X)$ and $q_n([y, t]) = *$ for $[y, t] \in C(F_n(X))$.

- Now convert $q_n$ into a fibration $p_{n+1}^X : G_{n+1}(X) \to X$.

This construction is functorial and the space $G_n(X)$ has the homotopy type of the $n$th-classifying space of Milnor [12]. We quote also from [8] that the direct limit $G_\infty(X)$ of the maps $G_n(X) \to G_{n+1}(X)$ has the homotopy type of $X$. As spaces are pointed, one has two canonical applications $i_n^l : G_n(X) \to G_n(X \times X)$ and $i_n^r : G_n(X) \to G_n(X \times X)$ obtained from maps $X \to X \times X$ defined respectively by $x \mapsto (x, *)$ and $x \mapsto (*, x)$.

**Definition 3** A space $X$ is an $H(n)$-space if there exists a map $\mu_n : G_n(X \times X) \to X$ such that $\mu_n \circ i_n^l = \mu_n \circ i_n^r = p_n^X : G_n(X) \to X$.

Directly from the definition, we see that an $H(\infty)$-space is an $H$-space and that any space is a $H(1)$-space. Recall also that any co-$H$-space is of LS-category 1. Then, Proposition 1 contains the trivial cases of a co-$H$-space $X$ and of an $H$-space $Y$.
Proof of Proposition\[1\] From the hypothesis, we have a section \(\sigma: X \to G_n(X)\) of the Ganea fibration \(p_n^X\) and a map \(\mu_n: G_n(Y \times Y) \to Y\) extending the Ganea fibration \(p_n^Y\), as in Definition\[3\]. If \(f\) and \(g\) are elements of \(\mathcal{F}_n(X,Y,*)\), we set
\[
f \cdot g = \mu_n \circ G_n(f \times g) \circ G_n(\Delta_X) \circ \sigma,
\]
where \(\Delta_X\) denotes the diagonal map of \(X\). One checks easily that \(f \cdot * \simeq * \cdot f \simeq f\).

In the rest of this section, we are interested in the existence of \(H(n)\)-structures on a given space. For the detection of an \(H(n)\)-space structure, one may replace the Ganea fibrations \(p_n^X\) by any functorial construction of fibrations \(\hat{p}_n: \hat{G}_n(X) \to X\) such that one has a functorial commutative diagram,
\[
\begin{array}{ccc}
\hat{G}_n(X) & \xrightarrow{\hat{p}_n} & G_n(X) \\
\downarrow & & \downarrow \hat{p}_n^X \\
X & \xleftarrow{p_n^X} & G_n(X)
\end{array}
\]
Such maps \(\hat{p}_n\) are called fibrations à la Ganea in [13] and substitutes to Ganea fibrations here. Moreover, as we are interested in product spaces, the following filtration of the space \(G_\infty(X) \times G_\infty(Y)\) plays an important role:
\[
(G(X) \times G(Y))_n = \bigcup_{i+j=n} G_i(X) \times G_j(Y).
\]
In [10], N. Iwase proved the existence of a commutative diagram
\[
\begin{array}{ccc}
(G(X) \times G(Y))_n & \xleftarrow{\cup (p_i^X \times p_j^Y)} & G_n(X \times Y) \\
\downarrow & & \downarrow p_n^{X \times Y} \\
X \times Y & \xrightarrow{p_n^{X \times Y}} & G_n(X \times Y)
\end{array}
\]
and used it to settle a counter-example to the Ganea conjecture. Therefore, in Definition\[3\] we are allowed to replace the Ganea space \(G_n(X \times X)\) by \((G(X) \times G(X))_n\). Moreover, if \(\hat{p}_n: \hat{G}_n(X) \to X\) are substitutes to Ganea fibrations as above, we may also replace \(G_n(X \times X)\) by
\[
(\hat{G}(X) \times \hat{G}(Y))_n = \bigcup_{i+j=n} \hat{G}_i(X) \times \hat{G}_j(Y).
\]
We will use this possibility in the rational setting.

In the case \(n = 2\), we have a cofibration sequence,
\[
\Sigma(G_1(X) \wedge G_1(X)) \xrightarrow{Wh} G_1(X) \vee G_1(X) \xrightarrow{} G_1(X) \times G_1(X),
\]
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coming from the Arkowitz generalisation of a Whitehead bracket, \([2]\). Therefore, the existence of an \(H(2)\)-structure on a space \(X\) is equivalent to the triviality of \((p_1^X \vee p_1^X) \circ \text{Wh}\). As the loop \(p_1^X\) of the Ganea fibration \(p_1^X : G_1(X) \to X\) admits a section, we get the following necessary condition:

– if there is an \(H(2)\)-structure on \(X\), then the homotopy Lie algebra of \(X\) is abelian, i.e. all Whitehead products vanish.

**Example 4** In the case \(X\) is a sphere \(S^n\), the existence of an \(H(2)\)-structure on \(S^n\) implies \(n = 1, 3\) or \(7\), \([1]\). Therefore, only the spheres which are already \(H\)-spaces endow a structure of \(H(2)\)-space. One can also observe that, in general, if a space \(X\) is both of category \(n\) and an \(H(2n)\)-space, then it is an \(H\)-space. The law is given by

\[
X \times X \xrightarrow{\sigma} G_{2n}(X \times X) \xrightarrow{\mu_{2n}} X,
\]

where the existence of the section \(\sigma\) to \(p_{2n}^{X \times X}\) comes from \(\text{cat}(X \times X) \leq 2 \text{cat}(X)\).

**Example 5** If we restrict to spaces whose loop space is a product of spheres or of loop spaces on a sphere, the previous necessary condition becomes a criterion. For instance, it is proved in \([3]\) that all Whitehead products are zero in the complex projective 3-space. This implies that \(\mathbb{C}P^3\) is an \(H(2)\)-space. (Observe that \(\mathbb{C}P^3\) is not an \(H\)-space.) From \([3]\), we know also that the homotopy Lie algebra of \(\mathbb{C}P^2\) is not abelian. Therefore \(\mathbb{C}P^2\) is not an \(H(2)\)-space.

The following example shows that we can find \(H(n)\)-spaces, for any \(n > 1\).  

**Example 6** Denote by \(\varphi_r : K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2r)\) the map corresponding to the class \(x^r \in H^{2r}(K(\mathbb{Z}, 2); \mathbb{Z})\), where \(x\) is the generator of \(H^2(K(\mathbb{Z}, 2); \mathbb{Z})\). Let \(E\) be the homotopy fibre of \(\varphi_r\). We prove below that \(E\) is an \(H(r-1)\)-space.

First we derive, from the homotopy long exact sequence associated to the map \(\varphi_r\), that \(\Omega E\) has the homotopy type of \(S^1 \times K(\mathbb{Z}, 2r - 2)\). Therefore, the only obstruction to extend \(G_{r-1}(E) \vee G_{r-1}(E) \to E\) to \((G(E) \times G(E))_{r-1} = \cup_{i+j=r-1} G_i(E) \times G_j(E)\) lies in

\[
\text{Hom}(H_{2r}((G(E) \times G(E))_{r-1}; \mathbb{Z}), \pi_{2r-2}(E)).
\]

If \(A\) and \(B\) are CW-complexes, we denote by \(A \sim_n B\) the fact that \(A\) and \(B\) have the same \(n\)-skeleton. If we look at the Ganea total spaces and fibres, we get:

\[
\Sigma \Omega E \sim_{2r} S^2 \vee S^{2r-1} \vee S^{2r}, \quad F_1(E) = \Omega E * \Omega E \sim_{2r} S^3 \vee S^{2r} \vee S^{2r},
\]
and more generally, \( F_{s}(E) \sim_{2r} S^{2s+1} \), for any \( s, \ 2 \leq s \leq r - 1 \). Observe also that \( H_{2r}(F_{E}; \mathbb{Z}) \to H_{2r}(G_{1}(E); \mathbb{Z}) \) is onto. (As we have only spherical classes in this degree, this comes from the homotopy long exact sequence.)

As a conclusion, we have no cell in degree \( 2r \) in \( (G(E) \times G(E))_{r-1} \) and \( E \) is an \( H(r-1) \)-space.

We end this section with a reduction to a more computable invariant than the LS-category. Consider \( \rho_{n}^{X} : X \to G_{[n]}(X) \) the homotopy cofibre of the Ganea fibration \( p_{n}^{X} \). Recall that, by definition, \( \text{wcat}_{G}(X) \leq n \) if the map \( \rho_{n}^{X} \) is homotopically trivial. Observe that we always have \( \text{wcat}_{G}(X) \leq \text{cat}(X) \), see [3 Section 2.6] for more details on this invariant.

**Proposition 7** Let \( X \) be a CW-complex of dimension \( k \) and \( Y \) be a CW-complex \((c-1)\)-connected with \( k \leq c - 1 \). If \( Y \) is an \( H(n) \)-space such that \( \text{wcat}_{G}(X) \leq n \), then \( \mathcal{F}_{*}(X,Y,*) \) is an \( H \)-space.

**Proof** Let \( f \) and \( g \) be elements of \( \mathcal{F}_{*}(X,Y,*) \). Denote by \( \tilde{\iota}^{X}_{n} : \tilde{F}_{n}(X) \to X \) the homotopy fibre of \( \rho_{n}^{X} : X \to G_{[n]}(X) \). This construction is functorial and the map \( (f,g) : X \to Y \times Y \) induces a map \( \tilde{F}_{n}(f,g) : \tilde{F}_{n}(X) \to \tilde{F}_{n}(Y \times Y) \) such that \( \tilde{\iota}^{Y \times Y}_{n} \circ \tilde{F}_{n}(f,g) = (f,g) \circ \tilde{\iota}^{X}_{n} \).

By hypothesis, we have a homotopy section \( \tilde{\sigma} : X \to \tilde{F}_{n}(X) \) of \( \tilde{\iota}^{X}_{n} \). Therefore, one gets a map \( X \to \tilde{F}_{n}(Y \times Y) \) as \( \tilde{F}_{n}(f,g) \circ \tilde{\sigma} \).

Recall now that, if \( A \to B \to C \) is a cofibration with \( A \) \((a-1)\)-connected and \( C \) \((c-1)\)-connected, then the canonical map \( A \to F \) in the homotopy fibre of \( B \to C \) is an \((a + c - 2)\)-equivalence. We apply it in the following situation:

\[
\begin{array}{ccc}
G_{n}(Y \times Y) & \xrightarrow{p_{n}^{Y \times Y}} & Y \times Y \\
\downarrow j_{n}^{Y \times Y} & & \downarrow \rho_{n}^{Y \times Y} \\
\tilde{F}_{n}(Y \times Y) & \xrightarrow{\tilde{\iota}^{Y \times Y}_{n}} & G_{[n]}(Y \times Y)
\end{array}
\]

The space \( G_{n}(Y \times Y) \) is \((c-1)\)-connected and \( G_{[n]}(Y \times Y) \) is \(c\)-connected. Therefore the map \( j_{n}^{Y \times Y} \) is \((2c-1)\)-connected. From the hypothesis, we get \( k \leq c - 1 < 2c - 1 \) and the map \( j_{n}^{Y \times Y} \) induces a bijection

\[ [X,G_{n}(Y \times Y)] \xrightarrow{\cong} [X,\tilde{F}_{n}(Y \times Y)]. \]

Denote by \( g_{n} : X \to G_{n}(Y \times Y) \) the unique lifting of \( \tilde{F}_{n}(f,g) \circ \tilde{\sigma} \). The composition \( g \circ f \) is defined as \( \mu_{n} \circ g_{n} \) where \( \mu_{n} \) is the \( H(n) \)-structure on \( Y \).
If we set \( g = \ast \), then \( \tilde{F}_n(f, g) \) is obtained as the composite of \( \tilde{F}_n(f) \) with the map \( \tilde{F}_n(Y) \to \tilde{F}_n(Y \times Y) \) induced by \( y \mapsto (y, \ast) \). As before, one has an isomorphism

\[
[X, G_n(Y)] \cong [X, \tilde{F}_n(Y)].
\]

A chase in the following diagram shows that \( f \cdot \ast = f \) as expected,

\[
\begin{array}{ccc}
G_n(Y) & \to & G_n(Y \times Y) \\
\downarrow & & \downarrow \\
\tilde{F}_n(X) & \to & \tilde{F}_n(Y \times Y) \\
\downarrow & \phi & \downarrow \\
X & \xrightarrow{f} & Y.
\end{array}
\]

\( \square \)

### 3 Rational characterisation of \( H(n) \)-spaces

Define \( m_H(X) \) as the greatest integer \( n \) such that \( X \) admits an \( H(n) \)-structure and denote by \( X_0 \) the rationalisation of a nilpotent finite type CW-complex \( X \). Recall that \( dl(X) \) is the valuation of the differential of the minimal model of \( X \), already defined in the introduction.

**Proposition 8** Let \( X \) be a nilpotent finite type CW-complex of rationalisation \( X_0 \). Then we have:

\[
m_H(X_0) + 1 = dl(X).
\]

**Proof** Let \( (\wedge V, d) \) be the minimal model of \( X \). Recall from [7] that a model of the Ganea fibration \( p^n_X \) is given by the following composition,

\[
(\wedge V, d) \to (\wedge V/ \wedge >^n V, \bar{d}) \leftarrow (\wedge V/ \wedge >^n V, \bar{d}) \oplus S,
\]

where the first map is the natural projection and the second one the canonical injection together with \( S \cdot S = S \cdot V = 0 \) and \( d(S) = 0 \). As the first map is functorial and the second one admits a left inverse over \( (\wedge V, d) \), we may use the realisation of \( (\wedge V, d) \to (\wedge V/ \wedge >^n V, d) \) as substitute for the Ganea fibration.

Suppose \( dl(X) = r \). We consider the cdga \( (\wedge V', d') \otimes (\wedge V'', d'')/I_r \) where \( (\wedge V', d') \) and \( (\wedge V'', d'') \) are copies of \( (\wedge V, d) \) and where \( I_r \) is the ideal \( I_r = \oplus_{i+j \geq r} \wedge^i V' \otimes \wedge^j V'' \). Observe that this cdga has a zero differential and that the morphism

\[
\varphi : (\wedge V, d) \to (\wedge V', d') \otimes (\wedge V'', d'')/I_r
\]

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defined by $\varphi(v) = v' + v''$ satisfies $\varphi(dv) = 0$. Therefore $\varphi$ is a morphism of cdga’s and its realisation induces an $H(n)$-structure on the rationalisation $X_0$. That shows: $m_H(X_0) + 1 \geq dl(X)$.

Suppose now that $m_H(X_0) + 1 > dl(X) = r$. By hypothesis, we have a morphism of cdga’s $\varphi: (\wedge V, d) \to (\wedge V', d') \otimes (\wedge V'', d'')/I_{r+1}$.

By construction, in this quotient, a cocycle of wedge degree $r$ cannot be a coboundary. Since the composition of $\varphi$ with the projection on the two factors is the natural projection, we have $\varphi(v) - v' - v'' \in \wedge^+ V' \otimes \wedge^+ V''$. Now let $v \in V$, of lowest degree with $d_r(v) \neq 0$. From $d_r(v) = \sum_{i_1,i_2,\ldots,i_r} c_{i_1i_2\ldots i_r} v_{i_1} v_{i_2} \cdots v_{i_r}$, we get

$$\varphi(dv) = \sum_{i_1,i_2,\ldots,i_r} c_{i_1i_2\ldots i_r} (v'_{i_1} + v''_{i_1}) \cdot (v'_{i_2} + v''_{i_2}) \cdots (v'_{i_r} + v''_{i_r}).$$

This expression cannot be a coboundary and the equation $d\varphi(x) = \varphi(dx)$ is impossible. We get a contradiction, therefore one has $m_H(X_0) + 1 = dl(X)$. □

4 The Haefliger model

Let $X$ and $Y$ be finite type nilpotent CW-complexes with $X$ of finite dimension. Let $(\wedge V, d)$ be the minimal model of $Y$ and $(A, d_A)$ be a finite dimensional model for $X$, which means that $(A, d_A)$ is a finite dimensional cdga equipped with a quasi-isomorphism $\psi$ from the minimal model of $X$ into $(A, d_A)$. Denote by $A^\vee$ the dual vector space of $A$, graded by

$$(A^\vee)^{-n} = \text{Hom}(A^n, \mathbb{Q}).$$

We set $A^+ = \bigoplus_{i=1}^{\infty} A^i$, and we fix an homogeneous basis $(a_1, \ldots, a_p)$ of $A^+$. The dual basis $(a^s)_{1 \leq s \leq p}$ is a basis of $B = (A^+)^\vee$ defined by $\langle a^s; a_t \rangle = \delta_{st}$.

We construct now a morphism of algebras

$$\varphi: \wedge V \to A \otimes \wedge (B \otimes V)$$

by

$$\varphi(v) = \sum_{s=1}^{p} a_s \otimes (a^s \otimes v).$$

In [9] Haefliger proves that there is a unique differential $D$ on $\wedge(B \otimes V)$ such that $\varphi$ is a morphism of cdga’s, i.e. $(d_A \otimes D) \circ \varphi = \varphi \circ d$. 

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In general, the cdga \((\wedge (B \otimes V), D)\) is not positively graded. Denote by \(D_0: B \otimes V \rightarrow B \otimes V\) the linear part of the differential \(D\). We define a cdga \((\wedge Z, D)\) by constructing \(Z\) as the quotient of \(B \otimes V\) by \(\bigoplus_{j \leq 0} (B \otimes V)^j\) and their image by \(D_0\). Haefliger proves:

**Theorem 9** \([9]\) The commutative differential graded algebra \((\wedge Z, D)\) is a model of the mapping space \(\mathcal{F}_*(X, Y, \star)\).

## 5 Proof of Theorem 2

**Proof** We start with an explicit description of the Haefliger model, keeping the notation of Section 4. The cdga \((\wedge V, d)\) is a minimal model of \(Y\) and we choose for \(V\) a basis \((v^k)\), indexed by a well-ordered set and satisfying \(d(v^k) \in \wedge (v^r)_{r<k}\) for all \(k\). As homogeneous basis \((a_s)_{1 \leq s \leq p}\) of \(A\), we choose elements \(h^j, e_j\) and \(b_j\) such that:

- the elements \(h^j\) are cocycles and their classes \([h^j]\) form a linear basis of the reduced cohomology of \(A\);
- the elements \(e_j\) form a linear basis of a supplement of the vector space of cocycles in \(A\), and \(b_j = d_A(e_j)\).

We denote by \(h^j, e_j\) and \(b^j\) the corresponding elements of the basis of \(B = (A^+)^Y\). By developing \(D_0(\sum_s a_s \otimes (a^s \otimes v)) = 0\), we get a direct description of the linear part \(D_0\) of the differential \(D\) of the Haefliger model:

\[
D_0(b^j \otimes v) = -(-1)^{|b^j|} e^j \otimes v \quad \text{and} \quad D_0(h^i \otimes v) = 0, \quad \text{for each } v \in V.
\]

A linear basis of the graded vector space \(Z\) is therefore given by the elements:

\[
\begin{align*}
    b^j \otimes v_k, & \quad \text{with } |b^j \otimes v_k| \geq 1, \\
    e^j \otimes v_k, & \quad \text{with } |e^j \otimes v_k| \geq 2, \\
    h^i \otimes v_k, & \quad \text{with } |h^i \otimes v_k| \geq 1.
\end{align*}
\]

Now, from \(\varphi(df) = (D - D_0)\varphi(v)\) and \(d(v) = \sum c_{i_1i_2...i_r} v_{i_1} v_{i_2} \cdots v_{i_r}\), we deduce:

\[
(D - D_0)(a^s \otimes v) = \\
\pm \sum c_{i_1i_2...i_r} \sum_{a_{i_1} a_{i_2} \cdots a_{i_r}} \langle a^s; a_{i_1} a_{i_2} \cdots a_{i_r} \rangle \cdot (a_{i_1} \otimes v_{i_1}) \cdot (a_{i_2} \otimes v_{i_2}) \cdots (a_{i_r} \otimes v_{i_r})
\]

where, as usual, the sign \(\pm\) is entirely determined by a strict application of the Koszul rule for a permutation of graded objects.

**Suppose first that** \(\text{cup}_0(X) < \text{dl}(Y)\).
We prove, by induction on $k$, that in $(\wedge Z, D)$ the ideal $I_k$ generated by the elements
\[
\begin{cases}
  b^j \otimes v_s, & s \leq k, \text{ with degree at least 1}, \\
  e^j \otimes v_s, & s \leq k, \text{ with degree at least 2},
\end{cases}
\]
is a differential ideal and that the elements $h^i \otimes v_s$, with $s \leq k$ and $|h^i \otimes v_s| \geq 1$, are cocycles in the quotient $((\wedge Z)/I_k, D)$. Note that this ideal is acyclic as shown by the formula given for $D_0$. Therefore the quotient map $\rho: (\wedge Z, D) \to ((\wedge Z)/I_k, D)$ is a quasi-isomorphism. The induction will prove that the differential is zero in the quotient, which is the first assertion of Theorem 2.

Begin with the induction. One has $dv_1 = 0$ which implies $(D - D_0)(a^s \otimes v_1) = 0$. Therefore, we deduce $D(b^j \otimes v_1) = -(-1)^{|b^j|}e^j \otimes v_1$ and $D(h^i \otimes v_1) = 0$. That proves the assertion for $k = 1$.

We suppose now that the induction step is true for the integer $k$. Taking the quotient by the ideal $I_k$ gives a quasi-isomorphism $\rho: (\wedge Z, D) \to (\wedge T, D) := ((\wedge Z)/I_k, D)$.

As the elements $b^j \otimes v_s$ and $e^j \otimes v_s$, $s \leq k$, have disappeared and as $\text{cup}_0(X) \leq \text{dl}(Y)$, we have $\rho \circ \varphi(dv_{k+1}) = 0$. Therefore $D(b^j \otimes v_{k+1}) = -(-1)^{|b^j|}e^j \otimes v_{k+1}$ and $D(h^i \otimes v_{k+1}) = 0$. The induction is thus proved.

We consider now the case $\text{cup}_0(X) \geq \text{dl}(Y)$ in the case $\dim(X) \leq \text{conn}(Y)$.

We choose first in the lowest possible degree $q$ an element $y \in V$ that satisfies $dy = d_r y + \cdots$ with $d_r(y) \neq 0$ and $r \leq \text{cup}_0(X)$. As above we can kill all the elements $e^j \otimes v$ and $b^i \otimes v$ with $|v| < q$ and keep a quasi-isomorphism $\rho: (\wedge Z, D) \to (\wedge T, D) := (\wedge Z/I_{q-1}, D)$.

Next we choose cocycles, $h_1, h_2, \cdots, h_m$, such that the class $[\omega]$, associated to the product $\omega = h_1 \cdot h_2 \cdots h_m$, is not zero. We choose $m \geq r$ and suppose that $\omega$ is in the highest degree for such a product. Let $\omega'$ be an element in $A^q$ such that $\langle \omega'; \omega \rangle = 1$. Then, by the Haefliger formula, the differential $D$ is zero in $\wedge T$ in degrees strictly less than $|\omega' \otimes y|$. Observe that $|\omega' \otimes y| \geq 2$ and that the $D_r$ part of the differential $D(\omega' \otimes y)$ is a nonzero sum. This proves that the cohomology is not free.

**Example 10** Let $X$ be a space with $\text{cup}_0(X) = 1$, which means that all products are zero in the reduced rational cohomology of $X$. Then, for any nilpotent finite type CW-complex $Y$, the rational cohomology $H^*(\mathcal{F}_*(X, Y, *); \mathbb{Q})$ is a free commutative graded algebra. For instance, this is the case for the (non-formal) space $X = S^3 \vee S^3 \cup \omega e^8$, where the cell $e^8$ is attached along a sum of triple Whitehead products.
Example 11 When the dimension of $X$ is greater than the connectivity of $Y$, the degrees of the elements have some importance. The cohomology can be commutative free even if $\text{cup}_0(X) \geq \text{dl}(Y)$. For instance, consider $X = S^5 \times S^{11}$ and $Y = S^8$. One has $\text{cup}_0(X) = \text{dl}(Y) = 2$ and the function space $\mathcal{F}_*(X,Y,\ast)$ is a rational $H$-space with the rational homotopy type of $K(Q, 3) \times K(Q, 4) \times K(Q, 10)$, as a direct computation with the Haefliger model shows.

6 Rationalisation of $\mathcal{F}_*(X,Y,\ast)$ when $\dim(X) \leq \text{conn}(Y)$

Let $X$ be a finite nilpotent space with rational LS-category equal to $m - 1$ and let $Y$ be a finite type nilpotent CW-complex whose connectivity $c$ is greater than the dimension of $X$. We set $r = \text{dl}(Y)$ and denote by $s$ the maximal integer such that $m/r^s \geq 1$, i.e. $s$ is the integral part of $\log_r m$.

Theorem 12 There is a sequence of rational fibrations $K_k \to F_k \to F_{k-1}$, for $k = 1, \ldots, s$, with $F_0 = \ast$, $F_0$ is the rationalisation of $\mathcal{F}_*(X,Y,\ast)$ and each space $K_k$ is a product of Eilenberg-MacLane spaces. In particular, the rational loop space homology of $\mathcal{F}_*(X,Y,\ast)$ is solvable with solvable index less than or equal to $s$.

Proof By a result of Cornea [4], the space $X$ admits a finite dimensional model $A$ such that $m$ is the maximal length of a nonzero product of elements of positive degree. We denote by $(\wedge, d)$ the minimal model of $Y$.

We consider the ideals $I_k = A^{\geq m/r^k}$, and the short exact sequences of cdga’s

$$I_k/I_{k-1} \to A/I_{k-1} \to A/I_k.$$  

These short exact sequences realise into cofibrations $T_k \to T_{k-1} \to Z_k$ and the sequences

$$(\wedge((A^+ / I_k)^\vee \otimes V), D) \to (\wedge((A^+ / I_{k-1})^\vee \otimes V), D) \to (\wedge((I_k/I_{k-1})^\vee \otimes V), D)$$

are relative Sullivan models for the fibrations

$$\mathcal{F}_*(Z_k, Y, \ast) \to \mathcal{F}_*(T_{k-1}, Y, \ast) \to \mathcal{F}_*(T_k, Y, \ast).$$

Now since the cup length of the space $Z_k$ is strictly less than $r$, the function spaces $\mathcal{F}_*(Z_k, Y, \ast)$ are rational $H$-spaces, and this proves Theorem 12. □
References


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Received: 13 February 2005