Abstract  The goal of this paper is to demonstrate that, at least for non-
simply connected 4–manifolds, the Seiberg–Witten invariant alone does not
determine diffeomorphism type within the same homeomorphism type.

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Dedicated to Robion C Kirby on the occasion of his 60th birthday

1 Introduction

The goal of this paper is to demonstrate that, at least for nonsimply con-
nected 4–manifolds, the Seiberg–Witten invariant alone does not determine
diffeomorphism type within the same homeomorphism type. The first exam-
"ples which demonstrate this phenomenon were constructed by Shuguang Wang
[13]. These are examples of two homeomorphic 4–manifolds with \( \pi_1 = \mathbb{Z}_2 \) and
trivial Seiberg–Witten invariants. One of these manifolds is irreducible and the
other splits as a connected sum. It is our goal here to exhibit examples among
symplectic 4–manifolds, where the Seiberg–Witten invariants are known to be
nontrivial. We shall construct symplectic 4–manifolds with \( \pi_1 = \mathbb{Z}_p \) which have
the same nontrivial Seiberg–Witten invariant but whose universal covers have
different Seiberg–Witten invariants. Thus, at the very least, in order to deter-
mine diffeomorphism type, one needs to consider the Seiberg–Witten invariants
of finite covers.

Recall that the Seiberg–Witten invariant of a smooth closed oriented 4–manifold
\( X \) with \( b_2^+ (X) > 1 \) is an integer-valued function which is defined on the set of
spin\(^c\) structures over \( X \) (cf [14]). In case \( H_1(X; \mathbb{Z}) \) has no 2–torsion there is a
natural identification of the spin \(^c\) structures of \(X\) with the characteristic elements of \(H_2(X, \mathbb{Z})\) (i.e., those elements \(k\) whose Poincaré duals \(\hat{k}\) reduce mod 2 to \(w_2(X)\)). In this case we view the Seiberg–Witten invariant as

\[
\text{SW}_X: \{k \in H_2(X, \mathbb{Z}) | \hat{k} \equiv w_2(TX) \pmod{2}\} \to \mathbb{Z}.
\]

The sign of \(\text{SW}_X\) depends on an orientation of \(H^0(X, \mathbb{R}) \otimes \det H^1_+(X, \mathbb{R}) \otimes \det H^1(X, \mathbb{R})\). If \(\text{SW}_X(\beta) \neq 0\), then \(\beta\) is called a basic class of \(X\). It is a fundamental fact that the set of basic classes is finite. Furthermore, if \(\beta\) is a basic class, then so is \(-\beta\) with \(\text{SW}_X(-\beta) = (-1)^{(e + \text{sign}(X))/4}\text{SW}_X(\beta)\) where \(e(X)\) is the Euler number and \(\text{sign}(X)\) is the signature of \(X\).

Now let \(\{\pm \beta_1, \ldots, \pm \beta_n\}\) be the set of nonzero basic classes for \(X\). Consider variables \(t_\beta = \exp(\beta)\) for each \(\beta \in H^2(X; \mathbb{Z})\) which satisfy the relations \(t_{\alpha + \beta} = t_\alpha t_\beta\). We may then view the Seiberg–Witten invariant of \(X\) as the Laurent polynomial

\[
\text{SW}_X = \text{SW}_X(0) + \sum_{j=1}^n \text{SW}_X(\beta_j) \cdot (t_{\beta_j} + (-1)^{(e + \text{sign}(X))/4} t_{\beta_j}^{-1}).
\]

### 2 The Knot and Link Surgery Construction

We shall need the knot surgery construction of [3]: Suppose that we are given a smooth simply connected oriented 4–manifold \(X\) with \(b^+ > 1\) containing an essential smoothly embedded torus \(T\) of self-intersection 0. Suppose further that \(\pi_1(X \setminus T) = 1\) and that \(T\) is contained in a cusp neighborhood. Let \(K \subset S^3\) be a smooth knot and \(M_K\) the 3–manifold obtained from 0–framed surgery on \(K\). The meridional loop \(m\) to \(K\) defines a 1–dimensional homology class \([m]\) both in \(S^3 \setminus K\) and in \(M_K\). Denote by \(T_m\) the torus \(S^1 \times m \subset S^1 \times M_K\). Then \(X_K\) is defined to be the fiber sum

\[
X_K = X \#_{T=T_m} S^1 \times M_K = (X \setminus N(T)) \cup (S^1 \times (S^3 \setminus N(K))),
\]

where \(N(T) \cong D^2 \times T^2\) is a tubular neighborhood of \(T\) in \(X\) and \(N(K)\) is a neighborhood of \(K\) in \(S^3\). If \(\lambda\) denotes the longitude of \(K\) (\(\lambda\) bounds a surface in \(S^3 \setminus K\)) then the gluing of this fiber sum identifies \(\{\text{pt}\} \times \lambda\) with a normal circle to \(T\) in \(X\). The main theorem of [3] is:

**Theorem** [3] With the assumptions above, \(X_K\) is homeomorphic to \(X\), and

\[
\text{SW}_{X_K} = \text{SW}_X \cdot \Delta_K(t)
\]

where \(\Delta_K\) is the symmetrized Alexander polynomial of \(K\) and \(t = \exp(2[T])\).
In case the knot \( K \) is fibered, the 3–manifold \( M_K \) is a surface bundle over the circle; hence \( S^1 \times M_K \) is a surface bundle over \( T^2 \). It follows from \([12]\) that \( S^1 \times M_K \) admits a symplectic structure and \( T_m \) is a symplectic submanifold. Hence, if \( T \subset X \) is a torus satisfying the conditions above, and if in addition \( X \) is a symplectic 4–manifold and \( T \) is a symplectic submanifold, then the fiber sum \( X_K = X \# T = T_m S^1 \times M_K \) carries a symplectic structure \([4]\). Since \( K \) is a fibered knot, its Alexander polynomial is the characteristic polynomial of its monodromy \( \varphi \); in particular, \( M_K = S^1 \times \varphi \Sigma \) for some surface \( \Sigma \) and \( \Delta_K(t) = \det(\varphi_* - tI) \), where \( \varphi_* \) is the induced map on \( H_1 \).

There is a generalization of the above theorem in this case due to Ionel and Parker \([7]\) and to Lorek \([8]\).

**Theorem** \([7, 8]\) Let \( X \) be a symplectic 4–manifold with \( b^+ > 1 \), and let \( T \) be a symplectic self-intersection 0 torus in \( X \) which is contained in a cusp neighborhood. Also, let \( \Sigma \) be a symplectic 2–manifold with a symplectomorphism \( \varphi: \Sigma \rightarrow \Sigma \) which has a fixed point \( \varphi(x_0) = x_0 \). Let \( m_0 = S^1 \times \varphi \{ x_0 \} \) and \( T_0 = S^1 \times m_0 \subset S^1 \times (S^1 \times \varphi \Sigma) \). Then \( X_\varphi = X \# T_0 = T_m S^1 \times (S^1 \times \varphi \Sigma) \) is a symplectic manifold whose Seiberg–Witten invariant is

\[
SW_{X_\varphi} = SW_X \cdot \Delta(t)
\]

where \( t = \exp(2[T]) \) and \( \Delta(t) \) is the obvious symmetrization of \( \det(\varphi_* - tI) \).

Note that in case \( K \) is a fibered knot and \( M_K = S^1 \times \varphi \Sigma \), Moser’s theorem \([9]\) guarantees that the monodromy map \( \varphi \) can be chosen to be a symplectomorphism with a fixed point.

There is a related link surgery construction which starts with an oriented \( n \)–component link \( L = \{ K_1, \ldots, K_n \} \) in \( S^3 \) and \( n \) pairs \( (X_i, T_i) \) of smoothly embedded self-intersection 0 tori in simply connected 4–manifolds as above. Let

\[
\alpha_L: \pi_1(S^3 \setminus L) \rightarrow \mathbb{Z}
\]

denote the homomorphism characterized by the property that it send the meridian \( m_i \) of each component \( K_i \) to 1. Let \( N(L) \) be a tubular neighborhood of \( L \). Then if \( \ell_i \) denotes the longitude of the component \( K_i \), the curves \( \gamma_i = \ell_i + \alpha_L(\ell_i) m_i \) on \( \partial N(L) \) given by the \( \alpha_L(\ell_i) \) framing of \( K_i \) form the boundary of a Seifert surface for the link. In \( S^1 \times (S^3 \setminus N(L)) \) let \( T_{m_i} = S^1 \times m_i \) and define the 4–manifold \( X(X_1, \ldots X_n; L) \) by

\[
X(X_1, \ldots X_n; L) = (S^1 \times (S^3 \setminus N(L)) \cup \bigcup_{i=1}^n (X_i \setminus (T_i \times D^2))
\]
where $S^1 \times \partial N(K_i)$ is identified with $\partial N(T_i)$ so that for each $i$:

$$[T_{m_i}] = [T_i], \quad \text{and} \quad [\gamma_i] = [pt \times \partial D^2].$$

**Theorem** [3] If each $T_i$ is homologically essential and contained in a cusp neighborhood in $X_i$ and if each $\pi_1(X \setminus T_i) = 1$, then $X(X_1, \ldots, X_n; L)$ is simply connected and its Seiberg–Witten invariant is

$$SW_{X(X_1, \ldots, X_n; L)} = \Delta_L(t_1, \ldots, t_n) \cdot \prod_{j=1}^{n} SW_{E(1)^{#r=T_j} X_j}$$

where $t_j = \exp(2[T_j])$ and $\Delta_L(t_1, \ldots, t_n)$ is the symmetric multivariable Alexander polynomial.

### 3 2–bridge knots

Recall that 2–bridge knots, $K$, are classified by the double covers of $S^3$ branched over $K$, which are lens spaces. Let $K(p/q)$ denote the 2–bridge knot whose double branched cover is the lens space $L(p,q)$. Here, $p$ is odd and $q$ is relatively prime to $p$. Notice that $L(p,q) \cong L(p,q - p)$; so we may assume at will that either $q$ is even or odd. We are first interested in finding a pair of distinct fibered 2–bridge knots $K(p/q_i), i = 1, 2$ with the same Alexander polynomial. Since 2–bridge knots are alternating, they are fibered if and only if their Alexander polynomials are monic [2]. There is a simple combinatorial scheme for calculating the Alexander polynomial of a 2–bridge knot $K(p/q)$; it is described as follows in [10]. Assume that $q$ is even and let $b(p/q) = (b_1, \ldots, b_n)$ where $p/q$ is written as a continued fraction:

$$\frac{p}{q} = \frac{1}{2b_1 + \frac{1}{-2b_2 + \frac{1}{2b_3 + \cdots \frac{1}{\pm 2b_n}}}}$$

There is then a Seifert surface for $K(p/q)$ whose corresponding Seifert matrix is:

$$V(p/q) = \begin{pmatrix}
\begin{array}{cccccc}
b_1 & 0 & 0 & 0 & 0 & \cdots \\
1 & b_2 & 1 & 0 & 0 & \cdots \\
0 & 0 & b_3 & 0 & 0 & \cdots \\
0 & 0 & 1 & b_4 & 1 & \cdots \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
\end{array}
\end{pmatrix}$$

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Thus the Alexander polynomial for $K(p/q)$ is
\[ \Delta_{K(p/q)}(t) = \det(t \cdot V(p/q) - V(p/q)^{tr}). \]

Using this technique we calculate:

**Proposition 3.1** The 2-bridge knots $K(105/64)$ and $K(105/76)$ share the Alexander polynomial
\[ \Delta(t) = t^4 - 5t^3 + 13t^2 - 21t + 25 - 21t^{-1} + 13t^{-2} - 5t^{-3} + t^{-4}. \]
In particular, these knots are fibered.

**Proof** The knots $K(105/64)$ and $K(105/76)$ correspond to the vectors
\[ b(105/64) = (1, 1, -1, -1, -1, 1, 1, 1) \]
\[ b(105/76) = (1, 1, 1, -1, -1, 1, 1, 1). \]

4 The examples

Consider any pair of inequivalent fibered 2-bridge knots $K_i = K(p/q_i), i = 1, 2,$ with the same Alexander polynomial $\Delta(t)$. Let $\bar{K}_i = \pi_i^{-1}(K_i)$ denote the branch knot in the 2-fold branched covering space $\pi_i: L(p, q_i) \to S^3$, and let $\bar{m}_i = \pi_i^{-1}(m_i)$, with $m_i$ the meridian of $K_i$. Then $M_{K_i} = S^1 \times \varphi_i \Sigma$ with double cover $\bar{M}_{K_i} = \tilde{S^1} \times \varphi_i \Sigma$.

Let $X$ be the $K3$-surface and let $F$ denote a smooth torus of self-intersection 0 which is a fiber of an elliptic fibration on $X$. Our examples are
\[ X_{K_i} = X \#_{F=T_{\bar{m}_i}} (S^1 \times \bar{M}_{K_i}). \]
The gluing is chosen so that the boundary of a normal disk to $F$ is matched with the lift $\bar{\ell}_i$ of a longitude to $K_i$. A simple calculation and our above discussion implies that $X_{K_1}$ and $X_{K_2}$ are homeomorphic [5] and have the same Seiberg–Witten invariant:

**Theorem 4.1** The manifolds $X_{K_i}$ are homeomorphic symplectic rational homology $K3$-surfaces with fundamental groups $\pi_1(X_{K_i}) = \mathbb{Z}_p$. Their Seiberg–Witten invariants are
\[ SW_{X_{K_i}} = \det(\varphi_i^2 - \tau^2I) = \Delta(\tau) \cdot \Delta(-\tau) \]
where $\tau = \exp([F])$.

5 Their universal covers

The purpose of this final section is to prove our main theorem.

**Theorem 5.1** \(X_{K(105/64)} \text{ and } X_{K(105/76)}\) are homeomorphic but not diffeomorphic symplectic 4–manifolds with the same Seiberg–Witten invariant.

Let \(K_1 = K(105/64)\) and \(K_2 = K(105/76)\). We have already shown that \(X_{K_1}\) and \(X_{K_2}\) are homeomorphic symplectic 4–manifolds with the same Seiberg–Witten invariant. Suppose that \(f: X_{K_1} \to X_{K_2}\) is a diffeomorphism. It then satisfies \(f_* (SW_{X_{K_1}}) = SW_{X_{K_2}}\). Since these are both Laurent polynomials in the single variable \(\tau = \exp ([F])\), and \([F] = [T_{m_1}]\) in \(X_{K_1}\), after appropriately orienting \(T_{m_2}\), we must have

\[
f_* [T_{m_1}] = [T_{m_2}].
\]

We study the induced diffeomorphism \(\hat{f}: \hat{X}_{K_1} \to \hat{X}_{K_2}\) of universal covers. The universal cover \(\hat{X}_{K_1}\) of \(X_{K_1}\) is obtained as follows. Let \(\varphi_1: S^3 \to L(p, q_1)\) be the universal covering \((p = 105, q_1 = 64, q_2 = 76)\) which induces the universal covering \(\varphi_1: \hat{X}_{K_1} \to X_{K_1}\), and let \(\hat{L}_i\) be the \(p\)-component link \(\hat{L}_i = \varphi_i^{-1}(K_1)\). The composition of the maps \(\varphi \circ \varphi_1: S^3 \to S^3\) is a dihedral covering space branched over \(K_1\), and the link \(\hat{L}_i = \hat{L}(p, q_i)\) is classically known as the ‘dihedral covering link’ of \(K(p/q_i)\). This is a symmetric link, and in fact, the deck transformations \(\tau_{i, k}\) of the cover \(\varphi_i: S^3 \to L(p, q_i)\) permute the link components. The collection of linking numbers of \(\hat{L}_i\) (the dihedral linking numbers of \(K(p/q_i)\)) classify the 2–bridge knots [2]. The universal cover \(\hat{X}_{K_i}\) is obtained via the construction \(\hat{X}_{K_i} = X(X_1, \ldots X_p; L_i)\) of section 2, where each \((X_i, T_i) = (K3, F)\). Hence it follows from section 2 that

\[
SW_{\hat{X}_{K_i}} = \Delta_{L_i}(t_{i,1}, \ldots, t_{i,p}) \cdot \prod_{j=1}^{p} SW_{E(1)^{#_F} K3} = \Delta_{\hat{L}_i}(t_{i,1}, \ldots, t_{i,p}) \cdot \prod_{j=1}^{p} (t_{i,j}^{1/2} - t_{i,j}^{-1/2})
\]

where \(t_{i,j} = \exp([2T_{i,j}])\) and \(T_{i,j}\) is the fiber \(F\) in the \(j\)th copy of \(K3\). Let \(L_{i,1}, \ldots, L_{i,p}\) denote the components of the covering link \(\hat{L}_i\) in \(S^3\), and let \(m_{i,j}\) denote a meridian to \(L_{i,j}\). Then \([T_{i,j}] = [S^1 \times m_{i,j}]\) in \(H_2(\hat{X}_{K_i}; \mathbb{Z})\), and so \(\varphi_{i, [T_{i,j}] = [T_i]}\).

Now we have \(\hat{f}_* (SW_{\hat{X}_{K_1}}) = SW_{\hat{X}_{K_2}}\) as elements of the integral group ring of \(H_2(\hat{X}_{K_2}; \mathbb{Z})\). The formula given for \(SW_{\hat{X}_{K_1}}\) shows that each basic class may be
written in the form $\beta = \sum_{j=1}^{p} a_j[T_{i,j}]$. Thus if $\beta$ is a basic class of $\tilde{X}_{K_1}$, then

$$\hat{f}_*(\beta) = \hat{f}_*(\sum_{j=1}^{p} a_j[T_{1,j}]) = \sum_{j=1}^{p} b_j[T_{2,j}]$$

for some integers, $b_1, \ldots, b_p$. But since $f_*[T_1] = [T_2]$ in $H_2(X_{K_2}; \mathbb{Z})$ we have

$$\left(\sum_{j=1}^{p} a_j\right)[T_2] = f_*\left(\sum_{j=1}^{p} a_j[T_1]\right) = f_* \hat{\vartheta}_{1*}(\beta) = \hat{\vartheta}_{2*} f_*(\beta) = \hat{\vartheta}_{2*}\left(\sum_{j=1}^{p} b_j[T_{2,j}]\right) = \sum_{j=1}^{p} b_j[T_2].$$

Hence $\sum_{j=1}^{p} a_j = \sum_{j=1}^{p} b_j$.

Form the 1-variable Laurent polynomials $P_i(t) = \Delta_{L_i}(t, \ldots, t) \cdot (t^{1/2} - t^{-1/2})^p$ by equating all the variables $t_{i,j}$ in $SW_{\tilde{X}_{K_i}}$. The coefficient of a fixed term $t^k$ in $P_i(t)$ is

$$\sum \{SW_{\tilde{X}_{K_i}}(\sum_{j=1}^{p} a_j[T_{i,j}]) \mid \sum_{j=1}^{p} a_j = k\}.$$ 

Our argument above (and the invariance of the Seiberg–Witten invariant under diffeomorphisms) shows that $\hat{f}_*$ takes $P_1(t)$ to $P_2(t)$; ie, $P_1(t) = P_2(t)$ as Laurent polynomials.

The reduced Alexander polynomials $\Delta_{L_i}(t, \ldots, t)$ have the form

$$\Delta_{L_i}(t, \ldots, t) = (t^{1/2} - t^{-1/2})^{p-2} \cdot \nabla_{L_i}(t),$$

where the polynomial $\nabla_{L_i}(t)$ is called the Hosokawa polynomial [6]. Consider the matrix:

$$\Lambda(p/q) = \begin{pmatrix}
\sigma & \varepsilon_1 & \cdots & \varepsilon_{p-1} \\
\varepsilon_{p-1} & \sigma & \cdots & \varepsilon_{p-2} \\
\vdots & \vdots & \ddots & \vdots \\
\varepsilon_1 & \varepsilon_2 & \cdots & \sigma
\end{pmatrix}$$

(Burde has shown that this is the linking matrix of $L(p/q)$.)

It is a theorem of Hosokawa [6] that $\nabla_{L(p/q)}(1)$ can be calculated as the determinant of any $(p-1)$ by $(p-1)$ minor $\Lambda'(p/q)$ of $\Lambda(p/q)$. In particular, we have
the following Mathematica calculations. (Note that \( K(105/64) = K(105/−41) \)
and \( K(105/76) = K(105/−29) \).)

\[
\begin{align*}
\det(\Lambda'(105/−41))/105 &= 13^2 \cdot 61^2 \cdot 127^2 \cdot 463^4 \cdot 631^4 \cdot 1358281^4 \\
\det(\Lambda'(105/−29))/105 &= 139^4 \cdot 211^4 \cdot 491^2 \cdot 8761^2 \cdot 100054514.
\end{align*}
\]

This means that \( \nabla_{L_1}^t(1) \neq \nabla_{L_2}^t(1) \). However, if we let \( Q(t) = (t^{1/2}−t^{−1/2})^{2p−2} \),
then \( P_1(t) = \nabla_{L_1}^{t}Q(t) \). For \(|u−1|\) small enough, \( P_1(u)/Q(u) \neq P_2(u)/Q(u) \).
Hence for \( u \neq 1 \) in this range, \( P_1(u) \neq P_2(u) \). This contradicts the existence
of the diffeomorphism \( f \) and completes the proof of Theorem 5.1.

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