7. Parshin’s higher local class field theory in characteristic $p$

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Parshin’s theory in characteristic $p$ is a remarkably simple and effective approach to all the main theorems of class field theory by using relatively few ingredients.

Let $F = K_n, \ldots, K_0$ be an $n$-dimensional local field of characteristic $p$.

In this section we use the results and definitions of 6.1–6.5; we don’t need the results of 6.6–6.8.

7.1

Recall that the group $V_F$ is topologically generated by

$$1 + \theta t_n^{i_n} \ldots t_1^{i_1}, \quad \theta \in \mathbb{R}^*, \, p^\downarrow (i_n, \ldots, i_1)$$

(see 1.4.2). Note that

$$i_1 \ldots i_n \{1 + \theta t_n^{i_n} \ldots t_1^{i_1}, t_1, \ldots, t_n\} = \{1 + \theta t_n^{i_n} \ldots t_1^{i_1}, t_1^{i_1}, \ldots, t_n^{i_n}\}$$

$$= \{1 + \theta t_n^{i_n} \ldots t_1^{i_1}, t_1^{i_1}, \ldots, t_n^{i_n}\} = \{1 + \theta t_n^{i_n} \ldots t_1^{i_1}, -\theta, \ldots, t_n^{i_n}\} = 0,$$

since $\theta^{q-1} = 1$ and $V_F$ is $(q-1)$-divisible. We deduce that

$$K_{n+1}^{\text{top}}(F) \simeq \mathbb{F}_q^*, \quad \{\theta, t_1, \ldots, t_n\} \mapsto \theta, \quad \theta \in \mathbb{R}^*.$$
7.2. The structure of $VK_n^\text{top}(F)$

Using the Artin–Schreier–Witt pairing (its explicit form in 6.4.3)

$$(\cdot, \cdot): K_n^\text{top}(F)/p^r \times W_r(F)/(F - 1)W_r(F) \to \mathbb{Z}/p^r, \ r \geq 1$$

and the method presented in subsection 6.4 we deduce that every element of $VK_n^\text{top}(F)$ is uniquely representable as a convergent series

$$\sum a_{\theta, i_n, \ldots, i_1} \{1 + \theta t_{i_n} \ldots t_{i_1}, t_{i_n}, \ldots, t_{i_1}\}, \ a_{\theta, i_n, \ldots, i_1} \in \mathbb{Z}_p,$$

where $\theta$ runs over a basis of the $\mathbb{F}_p$-space $K_0$, $p \nmid \gcd(i_n, \ldots, i_1)$ and $l = \min \{ k : p \nmid i_k \}$. We also deduce that the pairing $(\cdot, \cdot)_r$ is non-degenerate.

**Theorem 1** (Parshin, [P2]). Let $J = \{ j_1, \ldots, j_{m-1} \}$ run over all $(m - 1)$-elements subsets of $\{1, \ldots, n\}$, $m \leq n + 1$. Let $\mathcal{E}_J$ be the subgroups of $V_F$ generated by $1 + \theta t_{i_n} \ldots t_{i_1}$, $\theta \in \mu_{q-1}$ such that $p \nmid \gcd(i_1, \ldots, i_n)$ and $\min \{ l : p \nmid i_l \} \notin J$. Then the homomorphism

$$h: \prod_J \mathcal{E}_J \to VK_m^\text{top}(F), \ \ (\varepsilon, \ldots) \mapsto \sum_{J = \{ j_1, \ldots, j_{m-1} \}} \{ \varepsilon, t_{j_1}, \ldots, t_{j_{m-1}} \}$$

is a homeomorphism.

**Proof.** There is a sequentially continuous map $f: V_F \times F^{\ast \oplus m-1} \to \prod_J \mathcal{E}_J$ such that its composition with $h$ coincides with the restriction of the map $\varphi: (F^\ast)^m \to K_m^\text{top}(F)$ of 6.3 on $V_F \oplus F^{\ast \oplus m-1}$.

So the topology of $\prod_J \ast$-topology $\mathcal{E}_J$ is $\leq \lambda_m$, as follows from the definition of $\lambda_m$.

Let $U$ be an open subset in $VK_m(F)$. Then $h^{-1}(U)$ is open in the $\ast$-product of the topology $\prod_J \mathcal{E}_J$. Indeed, otherwise for some $J$ there were a sequence $\alpha_J^{(i)} \notin h^{-1}(U)$ which converges to $\alpha_J \in h^{-1}(U)$. Then the sequence $\varphi(\alpha_J^{(i)}) \notin U$ converges to $\varphi(\alpha_J) \in U$ which contradicts the openness of $U$.

**Corollary.** $K_m^\text{top}(F)$ has no nontrivial $p$-torsion; $\cap p^r VK_m^\text{top}(F) = \{0\}$. 

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7.3

Put $\overrightarrow{W}(F) = \lim_{\rightarrow} W_r(F)/(F - 1)W_r(F)$ with respect to the homomorphism $V: (a_0, \ldots, a_{r-1}) \rightarrow (0, a_0, \ldots, a_{r-1})$. From the pairings (see 6.4.3)

$$K_n^{\text{top}}(F)/p^r \times W_r(F)/(F - 1)W_r(F) \xrightarrow{i \cdot 1_r} \mathbb{Z}/p^r \rightarrow \frac{\mathbb{Z}}{p^r} \mathbb{Z}$$

one obtains a non-degenerate pairing

$$\langle \cdot, \cdot \rangle: \tilde{K}_n(F) \times \overrightarrow{W}(F) \rightarrow \mathbb{Q}_p/\mathbb{Z}_p$$

where $\tilde{K}_n(F) = K_n^{\text{top}}(F)/\bigcap_{r \geq 1} p^r K_n^{\text{top}}(F)$. From 7.1 and Corollary of 7.2 we deduce

$$\bigcap_{r \geq 1} p^r K_n^{\text{top}}(F) = \text{Tor}_{p^r} K_n^{\text{top}}(F) = T_0 K_n^{\text{top}}(F)$$

where $\text{Tor}_{p^r}$ is prime-to-$p$-torsion.

Hence

$$\tilde{K}_n(F) = K_n^{\text{top}}(F)/\text{Tor}_n^{\text{top}}(F).$$

7.4. The norm map on $K^{\text{top}}$-groups in characteristic $p$

Following Parshin we present an alternative description (to that one in subsection 6.8) of the norm map on $K^{\text{top}}$-groups in characteristic $p$.

If $L/F$ is cyclic of prime degree $l$, then it is more or less easy to see that

$$K_n^{\text{top}}(L) = \langle \{ L^* \} \cdot i_{F/L} K_n^{\text{top}}(F) \rangle$$

where $i_{F/L}$ is induced by the embedding $F^* \rightarrow L^*$. For instance, if $f(L|F) = l$ then $L$ is generated over $F$ by a root of unity of order prime to $p$; if $e_i(L|F) = l$, then there is a system of local parameters $t_1, \ldots, t_i, \ldots, t_n$ of $L$ such that $t_1, \ldots, t_i, \ldots, t_n$ is a system of local parameters of $F$.

For such an extension $L/F$ define $[P2]$

$$N_{L/F}: K_n^{\text{top}}(L) \rightarrow K_n^{\text{top}}(F)$$

as induced by $N_{L/F}: L^* \rightarrow F^*$. For a separable extension $L/F$ find a tower of subextensions

$$F = F_0 - F_1 - \cdots - F_{r-1} - F_r = L$$

such that $F_i/F_{i-1}$ is a cyclic extension of prime degree and define

$$N_{L/F} = N_{F_1/F_0} \circ \cdots \circ N_{F_r/F_{r-1}}.$$
To prove correctness use the non-degenerate pairings of subsection 6.4 and the properties
\[(N_{L/F}\alpha, \beta)_{F,r} = (\alpha, i_{F/L}\beta)_{L,r}\]
for \(p\)-extensions;
\[t \left(N_{L/F}\alpha, \beta\right)_F = t(\alpha, i_{F/L}\beta)_L\]
for prime-to-\(p\)-extensions (\(t\) is the tame symbol of 6.4.2).

7.5. Parshin’s reciprocity map

Parshin’s theory [P2], [P3] deals with three partial reciprocity maps which then can be glued together.

**Proposition** ([P3]). Let \(L/F\) be a cyclic \(p\)-extension. Then the sequence
\[
0 \rightarrow \tilde{K}_n(F) \xrightarrow{i_{F/L}} \tilde{K}_n(L) \xrightarrow{1-\sigma} \tilde{K}_n(L) \xrightarrow{N_{L/F}} \tilde{K}_n(F)
\]
is exact and the cokernel of \(N_{L/F}\) is a cyclic group of order \([L:F]\).

**Proof.** The sequence is dual (with respect to the pairing of 7.3) to
\[
\tilde{W}(F) \rightarrow \tilde{W}(L) \xrightarrow{1-\sigma} \tilde{W}(L) \xrightarrow{\text{Tr}_{L/F}} \tilde{W}(F) \rightarrow 0.
\]
The norm group index is calculated by induction on degree. \(\square\)

Hence the class of \(p\)-extensions of \(F\) and \(\tilde{K}_n(F)\) satisfy the classical class formation axioms. Thus, one gets a homomorphism \(\tilde{K}_n(F) \rightarrow \text{Gal}(F^{abp}/F)\) and
\[
\Psi_F^{(p)}: K_n^{\text{top}}(F) \rightarrow \text{Gal}(F^{abp}/F)
\]
where \(F^{abp}\) is the maximal abelian \(p\)-extension of \(F\). In the one-dimensional case this is Kawada–Satake’s theory [KS].

The valuation map \(v\) of 6.4.1 induces a homomorphism
\[
\Psi_F^{(ur)}: K_n^{\text{top}}(F) \rightarrow \text{Gal}(F_{ur}/F),
\]
\[
\{t_1, \ldots, t_n\} \rightarrow \text{the lifting of the Frobenius automorphism of } K_0^{\text{sep}}/K_0;
\]
and the tame symbol \(t\) of 6.4.2 together with Kummer theory induces a homomorphism
\[
\Psi_F^{(p')}: K_n^{\text{top}}(F) \rightarrow \text{Gal}(F(\sqrt[p]{1}, \ldots, \sqrt[p]{n})/F).
\]

The three homomorphisms \(\Psi_F^{(p)}, \Psi_F^{(ur)}, \Psi_F^{(p')}\) agree [P2], so we get the reciprocity map
\[
\Psi_F: K_n^{\text{top}}(F) \rightarrow \text{Gal}(F^{ab}/F)
\]
with all the usual properties.
Remark. For another rather elementary approach [F1] to class field theory of higher local fields of positive characteristic see subsection 10.2. For Kato’s approach to higher class field theory see section 5 above.

References


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