ON THE BIANCHI IDENTITIES IN A GENERALIZED WEYL SPACE

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Abstract. In this paper, we show that the first Bianchi identity is valid for a generalized Weyl space having a semi-symmetric \(E\)-connection and that the second Bianchi identity is satisfied for a recurrent generalized Weyl space provided that the recurrence vector \(\psi_l\) and the Vranceanu vector \(\Omega_l\) are related by \(\psi_l = \frac{2}{n-1} \Omega_l\).

1. Introduction

An \(n\)-dimensional differentiable manifold \(W^*_n\) having an asymmetric connection \(\nabla^*\) and asymmetric conformal metric tensor \(g^*\) preserved by \(\nabla^*\) is called a \textit{generalized Weyl space} [1]. For a such a space, in local coordinates, we have the compatibility condition

\[
\nabla^*_k g^*_{ij} - 2 T^*_k g^*_{ij} = 0 ,
\]

where \(T^*_k\) are the components of a covariant vector field called the complementary vector field of the generalized Weyl space.

The coefficients \(L^i_{jk}\) of the connection \(\nabla^*\) are obtained from the compatibility condition as [2]

\[
L^i_{jk} = \Gamma^i_{jk} + \frac{1}{2} \left[ \Omega^h_{kl} g^*_{ij} + \Omega^h_{jl} g^*_{hk} + \Omega^h_{jk} g^*_{hl} \right] g^{*(li)}
\]

or, putting

\[
Q^i_{jk} = \frac{1}{2} \left[ \Omega^h_{kl} g^*_{ij} + \Omega^h_{jl} g^*_{hk} + \Omega^h_{jk} g^*_{hl} \right] g^{*(li)}
\]

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we have

\[ L^i_{jk} = \Gamma^i_{jk} + Q^i_{jk} \]  

(1.4)

where \( \Gamma^i_{jk} \) and \( \Omega^i_{jk} \) are, respectively, the coefficients of a Weyl connection and the torsion tensor of \( W^*_n \) given by

\[ \Gamma^i_{jk} = \begin{bmatrix} i \\ jk \end{bmatrix} - (\delta^i_j T_k + \delta^i_k T_j - g^{lj} g_{jk} T_l), \]

(1.5)

and

\[ \Omega^i_{jk} = L^i_{jk} - L^i_{kj} = 2L^i_{[jk]} . \]

(1.6)

According to Norden [3], if under a renormalization of the fundamental tensor \( g \) of the form \( \tilde{g} = \lambda^2 g \), an object \( A \) admitting a transformation of the form \( \tilde{A} = \lambda^{\nu} A \) is called a satellite with weight \( \{ p \} \) of the tensor \( g \). The prolonged covariant derivative of the satellite \( A \) relative to the symmetric connection \( \nabla \), denoted by \( \tilde{\nabla} A \), is defined by [4]

\[ \tilde{\nabla}_k A = \nabla_k A - pT_k A. \]

(1.7)

The prolonged covariant derivative of the satellite \( A \) relative to \( \nabla^* \) will be denoted by \( \tilde{\nabla}^* A \), is defined by

\[ \tilde{\nabla}^* A = \nabla^* A - pT^*_k A. \]

(1.8)

2. Bianchi Identities

Let \( v^i \) be the contravariant components of vector field \( v \) in \( W^*_n \). For the second order covariant derivative of \( v \) relative to \( \nabla^* \) we have

\[ \nabla^*_l \nabla^*_k v^i = \partial_i \partial_l v^i + (\partial_i L^i_{hl}) v^h + (\partial_h v^h) L^i_{hl} \\
+ L^i_{jk} \partial_l v^j + L^i_{jk} L^j_{hl} v^h - L^i_{ik} (\partial_j v^i) - L^i_{kl} L^i_{hj} v^h. \]

(2.1)

Interchanging the indices \( k \) and \( l \) in (2.1) we obtain

\[ \nabla^*_l \nabla^*_k v^i = \partial_i \partial_l v^i + (\partial_i L^i_{hk}) v^h + (\partial_h v^h) L^i_{hk} + L^i_{jl} \partial_k v^j \\
+ L^i_{jl} L^j_{hk} v^h - L^i_{kl} (\partial_j v^i) - L^i_{kl} L^i_{hj} v^h. \]

(2.2)

Subtracting (2.2) from (2.1) we get

\[ \nabla^*_k \nabla^*_i v^i - \nabla^*_l \nabla^*_k v^i = L^i_{hkl} v^h + \Omega^i_{kl} \nabla^*_j v^i, \]

(2.3)

where

\[ L^i_{ijk} = \partial_j L^i_{ik} - \partial_k L^i_{ij} + L^h_{ik} L^j_{ij} - L^j_{ik} L^i_{hj}. \]

(2.4)
This is the curvature tensor corresponding to the connection $\nabla^*$. By cyclic permutation of $i, j$ and $k$ in (2.4)

$$L^l_{j i k} = \partial_k L^l_{j i} - \partial_i L^l_{j k} + L^h_{j i} L^l_{h k} - L^h_{j k} L^l_{h i} ,$$

and summing (2.4), (2.5) and (2.6) side by side we obtain

$$L^l_{i j k} + L^l_{j k i} + L^l_{k i j} = \partial_k \Omega^l_{j i} + \partial_i \Omega^l_{k j} + \partial_j \Omega^l_{i k} + L^h_{h k} \Omega^l_{j i} + L^h_{h i} \Omega^l_{k j} + L^h_{h j} \Omega^l_{i k} ,$$

showing that the first Bianchi identity is not satisfied in general.

If the connection $\nabla^*$ is semi-symmetric, i.e. if

$$\Omega^i_{j k} = \frac{1}{n - 1} (\delta^i_j \Omega^l_k - \delta^i_k \Omega^l_j) ,$$

the identity (2.7) reduces to

$$L^l_{i j k} + L^l_{j k i} + L^l_{k i j} = \frac{1}{n - 1} \left[ \delta^l_j (\Omega^i_{i k} - \Omega^i_{k i}) + \delta^l_k (\Omega^i_{j i} - \Omega^i_{i j}) + \delta^l_i (\Omega^i_{k j} - \Omega^i_{j k}) \right] , \quad n \neq 1$$

where $\Omega^l_{i j}$ is the Vranceanu vector of the connection $\nabla^*$ and $\Omega^i_{i k} = \frac{\partial \Omega^i_k}{\partial u^k}$.

From this we obtain the

**Theorem 2.1.** If the Vranceanu vector is a gradient then the first Bianchi identity is satisfied.

**Definition.** The connection $\nabla^*$ is said to be an $E$-connection if the condition

$$\nabla^*_k \Omega^i_i - \nabla^*_i \Omega^i_k = 0$$

holds [5].

**Theorem 2.2.** For a generalized Weyl space having a semi-symmetric $E$-connection the first Bianchi identity is satisfied.

**Proof:** The covariant derivative of the Vranceanu vector $\Omega^i_i$ with respect to the coordinates $u^k$ is

$$\nabla^*_k \Omega^i_i = \frac{\partial \Omega^i_i}{\partial u^k} - L^h_{i k} \Omega^i_h .$$
Subtracting from (2.11) the equation is obtained by interchanging the indices $i$ and $k$ we find that
\begin{equation}
\nabla_k^* \Omega_i - \nabla_i^* \Omega_k = \frac{\partial \Omega_i}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^i} + 2 \Omega^h_{ki} \Omega_h. \tag{2.12}
\end{equation}

On the other hand, for the generalized Weyl space $W^*_n$ with semi-symmetric $E$-connection (2.12) is reduced to
\begin{equation}
\frac{\partial \Omega_i}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^i} = 0. \tag{2.13}
\end{equation}

Using (2.9), from (2.13) we get
\begin{equation*}
L_{ijk}^l + L_{jki}^l + L_{kij}^l = 0,
\end{equation*}
so the proof is completed. \qed

**Corollary 2.1.** For a generalized Weyl space having a semi-symmetric connection, the Vranceanu vector and the complementary vector are related by
\begin{equation*}
L_{[ik]} - nT_{[i,k]} = \frac{n-2}{n-1} \Omega_{[i,k]} \quad (n \neq 1)
\end{equation*}
where $L_{ik}$ denotes the Ricci tensor of $W^*_n$.

**Proof:** If, in (2.9) a contraction on $l$ and $j$ is made we have
\begin{equation}
L_{ilk}^l + L_{ikl}^l + L_{kli}^l = \frac{n-2}{n-1} \left( \Omega_{i,k} - \Omega_{k,i} \right) \tag{2.14}
\end{equation}
from which we get
\begin{equation*}
L_{[ik]} - nT_{[i,k]} = \frac{n-2}{n-1} \Omega_{[i,k]} , \quad \left( T_{i,k}^* = \frac{\partial T_{i,k}^*}{\partial u^k} , \quad \Omega_{i,k} = \frac{\partial \Omega_i}{\partial u^k} \right)
\end{equation*}
where we have used the facts that
\begin{equation*}
L_{ilk}^l = L_{ik} , \quad L_{kli}^l = -L_{ki}
\end{equation*}
so the proof is completed. \qed

The prolonged covariant derivative of the curvature tensor $L_{ijk}^h$, of weight \{0\}, is
\begin{equation}
\nabla_l^* L_{ijk}^h = \nabla_l^* L_{ijk}^h = \partial_l L_{ijk}^h + L_{im}^h L_{mjk}^m - L_{m}^m L_{ijk}^m - L_{ijkl}^m L_{m}^m - L_{ijkl}^m L_{m}^m . \tag{2.15}
\end{equation}

If the indices $j, k$ and $l$ are changed cyclically in (2.15) the equations
\begin{equation*}
\nabla_j^* L_{ikl}^h = \nabla_j^* L_{ikl}^h = \partial_j L_{ikl}^h + L_{nj}^h L_{mk}^n - L_{nj}^h L_{mk}^n - L_{ijkl}^m L_{m}^m - L_{ijkl}^m L_{m}^m \tag{2.16}
\end{equation*}
and
\[ \hat{\nabla}_k^* L_{ij}^h = \nabla_k^* L_{ij}^h = \partial_k L_{ij}^h + L_{mk}^h L_{ij}^m - L_{ik}^m L_{mj}^h - L_{ik}^m L_{mj}^h - L_{jk}^m L_{im}^h \] (2.17)
are obtained respectively.
Summing (2.15), (2.16) and (2.17) we get
\[ \hat{\nabla}_i^* L_{ijk}^h + \hat{\nabla}_j^* L_{ikl}^h + \hat{\nabla}_k^* L_{ijd}^h = \Omega_{ij}^m L_{imk}^h + \Omega_{jk}^m L_{iml}^h + \Omega_{kl}^m L_{imj}^h . \] (2.18)
This shows that the second Bianchi identity is not valid in $W_n^*$.

The generalized Weyl space $W_n^*$ is called recurrent if its curvature tensor $L_{ijk}^l$ satisfies the condition
\[ \hat{\nabla}_i^* L_{ijk}^h = \psi_i L_{ijk}^h \] (2.19)
where $\psi_i$ is a 1-form called the recurrence vector of $W_n^*$.
Let $W_n^*$ be a recurrent generalized Weyl space having a semi-symmetric connection. Then, using (2.8) and (2.19), the identity (2.18) is reduced to
\[ a_i L_{ijk}^h + a_j L_{ikl}^h + a_k L_{ijd}^h = 0 . \] (2.20)
where
\[ a_i = \psi_i - \frac{2}{n - 1} \Omega_i . \]
Thus we proved the

**Theorem 2.3.** For a generalized recurrent Weyl space having a semi-symmetric connection the second Bianchi identity is satisfied provided that the recurrence vector $\psi_i$ and the Vranceanu vector $\Omega_i$ are related by $\psi_i = \frac{2}{n - 1} \Omega_i$, $\psi_i \neq \Omega_i$.

**References**