TOPOLOGICAL PROPERTIES OF SOME COHOMOGENEITY ON RIEMANNIAN MANIFOLDS OF NONPOSITIVE CURVATURE

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Abstract. In this paper we study some non-positively curved Riemannian manifolds acted on by a Lie group of isometries with principal orbits of codimension one. Among other results it is proved that if the universal covering manifold satisfies some conditions then every non-exceptional singular orbit is a totally geodesic submanifold. When $M$ is flat and is not toruslike, it is proved that either each orbit is isometric to $\mathbb{R}^k \times \mathbb{T}^m$ or there is a singular orbit. If the singular orbit is unique and non-exceptional, then it is isometric to $\mathbb{R}^k \times \mathbb{T}^m$.

1. Introduction

Recently, cohomogeneity one Riemannian manifolds have been studied from different points of view. A. Alekseevsky and D. Alekseevsky in [1] and [2] gave a description of such manifolds in terms of Lie subgroups of a Lie group $G$, Podesta and Spiro in [13] got some nice results in negatively curved case, Searle in [14] provided a complete classification of such manifolds in dimensions less than 6 when they are compact and of positive curvature. The aim of this paper is to deal with some non-positively curved cohomogeneity one Riemannian manifolds. We generalize some of the theorems of [13] to the case where $M$ is a product of negatively curved manifolds. Also in Section 4 we study some cohomogeneity one flat Riemannian manifolds. Our main results are Theorems 3.5, 3.7, 3.10, and 4.4.

2. Preliminaries

Definition 2.0. Let $M$ be a complete Riemannian manifold and $G$ a Lie group of isometries which is closed in the full group of isometries of $M$. We say
that $M$ is of cohomogeneity one under the action of $G$ if $G$ has an orbit of codimension one.

It is known (see [1] and [4, 11]) that the orbit space $\Omega = M/G$ is a topological Hausdorff space homeomorphic to one of the following spaces: $\mathbb{R}$, $S^1$, $\mathbb{R}^+ = [0, +\infty)$ and $[0, 1]$. In the following we will indicate by $k : M \rightarrow \Omega$ the projection to the orbit space. Given a point $x \in M$, the orbit $D = Gx$ is called principal (resp. singular) if the corresponding image in the orbit space is an internal (resp. boundary) point of $\Omega$, and the point $x$ is called a regular (resp. singular) point. We say that a singular orbit is exceptional if it has codimension one. Also note that the principal orbits are diffeomorphic to each other and $M$ is diffeomorphic to $\Omega \times D$ if $M/G = R$.

If $G_p$ is the isotropy subgroup of $G$ at $p, (p \in M)$, then $G_x$ and $G_y$ are conjugate if both $x, y$ are regular, while $G_x$ is conjugate to a subgroup of $G_y$ if $x$ is regular and $y$ is singular.

**Definition 2.1.** A (complete) geodesic $\gamma$ on a Riemannian manifold of cohomogeneity one is called a normal geodesic if it crosses each orbit orthogonally.

We know (see [2]) that a geodesic $\gamma$ is a normal geodesic if and only if it is orthogonal to each orbit $Gx$ at one point $x \in \gamma$, and that each regular point belongs to a unique normal geodesic.

**Definition 2.2.** A differentiable real valued function $F$ on a complete Riemannian manifold $M$ is said to be convex (resp. strictly convex) if for each geodesic $\gamma : \mathbb{R} \rightarrow M$ the composed function $F \circ \gamma : \mathbb{R} \rightarrow \mathbb{R}$ is convex (resp. strictly convex), that is $(F \circ \gamma)'' \geq 0$ (resp. $(F \circ \gamma)'' > 0$).

Let $\varphi$ be an isometry of a simply connected Riemannian manifold $M$, the squared displacement function of $\varphi$ is the function defined by $d^2_\varphi(p) = d^2(p, \varphi(p)), p \in M$, where $d$ denotes the distance on $M$.

In the next proposition we list some known properties of cohomogeneity one Riemannian manifolds, which we will use in the sequel.

**Proposition 2.3.** ([4], [8] and [13]) Let $M$ be a cohomogeneity one Riemannian manifold under the action of a connected Lie group $G$ which is closed in the full isometry group of $M$, then

a) If $M$ is simply connected with nonpositive curvature, there is at most one singular orbit;

b) If $M$ has nonpositive curvature and $B$ is the unique singular orbit of $M$, $\pi_1(M) = \pi_1(B)$;

c) If $M$ is simply connected no exceptional orbit may exist;
d) If $M$ is simply connected and without singular orbit then $\Omega \neq S^1$, i.e. $\Omega = \mathbb{R}$;

e) No exceptional orbit is simply connected;
f) If $\gamma$ is a normal geodesic then the map $k: \gamma \to \Omega$ is surjective and it defines a covering over the set $\Omega^0$ of internal points of $\Omega$. When $\Omega = \mathbb{R}^+$ or $\mathbb{R}$, we can endow $\Omega$ with the metric given by the covering $k$.

The following proposition and theorems will be needed later.

**Proposition 2.4.** (see [3]) Let $M$ be a simply connected and complete Riemannian manifold of nonpositive curvature, then

a) If the minimum point set $C$ of a real valued convex function $F$ defined on $M$ is a submanifold of $M$ then $C$ is totally geodesic in $M$, and each critical point of $F$ belongs to $C$;
b) $d_\varphi^2$ is a convex function for each isometry $\varphi$ of $M$ and if $M$ has negative curvature it is strictly convex except at the minimum point set $C$ which is at most the image of a geodesic.

**Theorem 2.5.** ([15]) Let $M$ be a connected homogeneous Riemannian manifold with nonpositive curvature, then $M$ is diffeomorphic to the product of a torus and a Euclidean space.

**Theorem 2.6.** ([9]) Let $M$ be a homogeneous Riemannian manifold with nonpositive curvature and negative definite Ricci tensor then $M$ is simply connected.

3. Cohomogeneity on UND Manifolds

Throughout the following $M$ will denote a complete Riemannian manifold of dimension $n$ with nonpositive curvature and of cohomogeneity one under the action of $G$, a connected Lie group which is closed in the full group of isometries of $M$. If $M$ is not simply connected then $\tilde{M}$ will denote the universal Riemannian covering manifold of $M$ endowed with the pulled back metric and $\pi: \tilde{M} \to M$ will be the covering projection, with the symbol $\Delta$ we will denote the deck transformation group of the universal covering of $M$. We know (see [4] page 63) that the group $G$ always admits a connected covering group $\tilde{G}$ which acts on $\tilde{M}$ by isometries and of cohomogeneity one, the projection $\tilde{\pi}: \tilde{G} \to G$ is such that $\tilde{\pi}(\tilde{g})(x) = \pi(\tilde{g}(y))$ for all $\tilde{g} \in \tilde{G}$, $x \in M$ and $y \in \pi^{-1}(x)$. Moreover $\Delta$ centralizes $\tilde{G}$ so that it maps $\tilde{G}$-orbits onto $G$-orbits, so for each $\varphi \in \Delta$, $d_\varphi^2$ is constant along orbits.
Definition 3.0. We say that a Riemannian manifold $M$ is universally and negatively decomposable (UND) when its universal covering manifold $	ilde{M}$ decomposes as $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2 \times \cdots \times \tilde{M}_k$ and for each $i$, $\tilde{M}_i$ has negative curvature and each $\varphi \in \Delta$ decomposes as $\varphi = \varphi_1 \times \varphi_2 \times \cdots \times \varphi_k$ where $\varphi_i$ is an isometry of $\tilde{M}_i$.

Lemma 3.1. If $M = M_1 \times M_2$ is a complete simply connected Riemannian manifold of nonpositive curvature such that for a geodesic $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ and for an isometry $\varphi = \varphi_1 \times \varphi_2$, $d_{\varphi_1}^2 \circ \gamma_1 : \mathbb{R} \to \mathbb{R}$ is strictly convex, then $d_{\varphi}^2 \circ \gamma : \mathbb{R} \to \mathbb{R}$ is a strictly convex function.

Lemma 3.2. If $\varphi \in \Delta$ is nontrivial and for a normal geodesic $\gamma$, $d_{\varphi}^2 \circ \gamma : \mathbb{R} \to \mathbb{R}$ does not have any minimum point then, $\varphi$ maps each orbit $\tilde{B}$ onto itself.

Lemma 3.3. Let $M$ be a UND cohomogeneity one Riemannian manifold and let $\varphi \in \Delta$ be nontrivial, then there exists a normal geodesic $\gamma$ on $\tilde{M}$ such that $d_{\varphi}^2 \circ \gamma : \mathbb{R} \to \mathbb{R}$ is a strictly convex function.

Lemma 3.4. Let $\gamma$ be a normal geodesic in $\tilde{M}$ and $\varphi \in \Delta$ be such that $d_{\varphi}^2 \circ \gamma : \mathbb{R} \to \mathbb{R}$ is strictly convex and $t_1 \in \mathbb{R}$ is not a minimum point of the function $F(t) = d_{\varphi}^2 \circ \gamma(t)$, then the orbit $\tilde{B} = \tilde{G}_{\gamma}(t_1)$ is a hypersurface in $\tilde{M}$.

Theorem 3.5. If $M$ is a non-simply connected UND cohomogeneity one Riemannian manifold with only one singular orbit $B$, and $B$ is not exceptional, then it is a totally geodesic submanifold of $M$ diffeomorphic to $\mathbb{R}^k \times \mathbb{T}^m$ and $\pi_1(M) = \mathbb{Z}^m$.

Proof: First note that since $\dim \pi^{-1}(B) = \dim B < n - 1$, each component of $\pi^{-1}(B)$ must be a non-exceptional singular orbit in $\tilde{M}$. Therefore by 2.3(a), $\pi^{-1}(B)$ has only one component $\tilde{B}$. Now let $\varphi \in \Delta$ be a nontrivial deck transformation and $\gamma$ a normal geodesic in $\tilde{M}$ such that $F = d_{\varphi}^2 \circ \gamma : \mathbb{R} \to \mathbb{R}$ is a strictly convex function (see 3.3), then we have two cases.

Case 1: $F$ has only one minimum point $t_0 \in \mathbb{R}$.

In this case since $d_{\varphi}^2$ is constant along orbits, we get that $\tilde{G}_{\gamma}(t_0)$ is the minimum point set of $d_{\varphi}^2$, so by 2.4(a) it is a totally geodesic submanifold of $\tilde{M}$. We show that $\tilde{B} = \tilde{G}_{\gamma}(t_0)$. If not, then $\tilde{B} = \tilde{G}_{\gamma}(t_1)$, $t_1 \neq t_0$, so by 3.4 $\tilde{B}$ must be a hypersurface in $\tilde{M}$, since $\dim \tilde{B} < n - 1$ this is a contradiction, therefore $\tilde{B} = \tilde{G}_{\gamma}(t_0)$ and $\tilde{B}$ is a totally geodesic submanifold of $\tilde{M}$. Consequently $B = \pi(\tilde{B})$ is totally geodesic in $M$, so is of nonpositive curvature. Since $B$ is homogeneous we get by 2.5 that $B$ is diffeomorphic to $\mathbb{R}^k \times \mathbb{T}^m$ and by 2.3(b) we have $\pi_1(M) = \pi_1(B) = \mathbb{Z}^m$.

Case 2: $F$ has not any minimum point.
This case can not occur because by 3.4 each orbit of $\tilde{M}$ must be a hypersurface, so $\tilde{B}$ is a hypersurface, which is in contrast with the fact $\dim \tilde{B} < n - 1$. □

**Lemma 3.6.** If for each deck transformation $\varphi \in \Delta$ and each orbit $\tilde{D}$ in $\tilde{M}$, $\varphi$ maps $\tilde{D}$ onto itself and if there is no singular orbit in $M$, then each orbit $D$ in $M$ is diffeomorphic to $\mathbb{R}^{k_1} \times \mathbb{T}^{m_1}$.

**Proof:** The proof of this lemma is given in a portion of the proof of Theorem 3.7 in [4] and the sketch of the proof is as follows: for an orbit $D$ in $M$, $\pi^{-1}(D)$ has only one component $\tilde{D}$ and $\tilde{D} = \tilde{G}/\tilde{K}$ with $\tilde{K}$ maximal compact in $\tilde{G}$. So there is a solvable subgroup $H$ acting transitively on $\tilde{D}$. Since $D = \tilde{D}/\Delta$ and $\Delta$ centeralizes $\tilde{G}$ (and hence $H$ too), we obtain that $H$ acts transitively on $D$, so $D$ is a solvmanifold and diffeomorphic to a product $\mathbb{R}^{k_1} \times \mathbb{T}^{m_1}$ (see [13], p. 76 and [16]). □

**Theorem 3.7.** If $M$ is a non-simply connected UND cohomogeneity one Riemannian manifold without any singular orbit, then each orbit is diffeomorphic to $\mathbb{R}^{k_1} \times \mathbb{T}^{m_1}$. In this case if $M/G = \mathbb{R}$, then $M$ is diffeomorphic to $\mathbb{R}^k \times \mathbb{T}^m$, $k = k_1 + 1$.

**Proof:** By 3.3 for each nontrivial $\varphi \in \Delta$, there is a normal geodesic $\gamma$ (related to $\varphi$) such that $d^2_{\varphi} \circ \gamma$ is a strictly convex function. We have two cases.

**Case 1:** There exists a $\varphi \in \Delta$ such that $d^2_{\varphi} \circ \gamma$ has a minimum point $t_0 \in \mathbb{R}$.

In this case the orbit $\tilde{B} = \tilde{G}\gamma(t_0)$ is the minimum point set of the function $d^2_{\varphi}$. Therefore by 2.4(a) it is totally geodesic and so $B = \pi(\tilde{B})$ is totally geodesic in $M$, hence is of nonpositive curvature. Since $B$ is homogeneous, it is diffeomorphic to $\mathbb{R}^{k_1} \times \mathbb{T}^{m_1}$ by 2.5. From the fact that the (principal) orbits are diffeomorphic we get that each orbit is diffeomorphic to $\mathbb{R}^{k_1} \times \mathbb{T}^{m_1}$.

**Case 2:** For each nontrivial $\varphi \in \Delta$, $d^2_{\varphi} \circ \gamma$ does not have any minimum point.

In this case by 3.2, $\varphi$ maps each orbit $\tilde{D}$ onto itself. Therefore by 3.6 each orbit $D$ in $M$ is diffeomorphic to $\mathbb{R}^{k_1} \times \mathbb{T}^{m_1}$.

If $M/G = \mathbb{R}$, from the fact that $M$ is diffeomorphic to $M/G \times D$ we get that $M$ is diffeomorphic to $\mathbb{R} \times \mathbb{R}^{k_1} \times \mathbb{T}^{m_1} = \mathbb{R}^k \times \mathbb{T}^m$. □

**Lemma 3.8.** Let $M = M_1 \times M_2$ and $X = X_1 + X_2$, $Z$ be two vectors at the point $p = (p_1, p_2)$ such that $X_1$, $Z$ are tangent to $M_1$ and $X_2$ is tangent to $M_2$, then $K_M(X, Z) = K_{M_1}(X_1, Z)$.

**Lemma 3.9.** If $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2 \times \cdots \times \tilde{M}_k$, where for each $i$, $\tilde{M}_i$ is negatively curved with $\dim \tilde{M}_i \geq 3$, then each totally geodesic hypersurface $S$ of $\tilde{M}$ has negative definite Ricci tensor.
Theorem 3.10. If $M$ is a nonsimply connected UND cohomogeneity one Riemannian manifold and $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2 \times \cdots \times \tilde{M}_k$, where for each $i$, $\dim \tilde{M}_i \geq 3$, then

a) There is at most one singular orbit;

b) If there is a singular orbit $B$, it is non-exceptional and diffeomorphic to $\mathbb{R}^{K_1} \times T^m$ and $\pi_1(M) = \mathbb{Z}^m$.

Proof: We prove the theorem in two steps.

Step 1: $M$ does not have two exceptional singular orbits.

If $M$ has two exceptional singular orbits, then the dimension of each orbit of $M$ (and so the dimension of each orbit of $\tilde{M}$) is $n - 1$, so by 2.3(c, e), $\tilde{M}$ does not have any singular orbit and $\tilde{M}/\tilde{G} = \mathbb{R}$. Therefore each normal geodesic $\gamma$ in $\tilde{M}$ intersects an orbit $\tilde{D}$ exactly once. But since $M/G = [0, 1]$, the normal geodesic $\pi \circ \gamma$ intersects a principal orbit $D$ in $M$ infinitely many times, so $\pi^{-1}(D)$ has more than one connected component. Therefore if $\tilde{D}$ is a component of $\pi^{-1}(D)$, there exist a nontrivial $\varphi \in \Delta$ such that $\varphi(\tilde{D}) \neq \tilde{D}$ thus by Lemmas 3.2, 3.3, for a normal geodesic $\gamma$, $d^2_{\varphi} \circ \gamma$ is strictly convex with a minimum point $t_0 \in \mathbb{R}$, and since $d^2_{\varphi}$ is constant along orbits, $\tilde{B} = \tilde{G}\gamma(t_0)$ is the minimum point set of $d^2_{\varphi}$. So it is totally geodesic by 2.4(a). Now since each factor of the decomposition of $\tilde{M}$ is negatively curved with $\dim \tilde{M}_i \geq 3$, we get by 3.9 that every totally geodesic hypersurface of $\tilde{M}$ has negative definite Ricci tensor, so $\tilde{B}$ (hence $B = \pi(\tilde{B})$) has negative definite Ricci tensor, thus by 2.6, $B$ is simply connected. Since $\dim B = n - 1$, we get by 2.3(d) that $B$ is not a singular orbit. As $B$ is simply connected, $B = G/K$ ($K = G_x$, $x \in B$), where $K$ is maximal compact subgroup of $G$ (see [10], Vol II, p. 112), which is in contrast with the fact that there exists singular orbit.

Step 2: $M$ does not have two singular orbits, at least one orbit non-exceptional.

Let $B_1$ be a non-exceptional singular orbit of $M$ then $\tilde{B} = \pi^{-1}(B_1)$ is the unique singular orbit of $\tilde{M}$. Because of dimensional reasons for each $\varphi \in \Delta$ we have $\varphi(\tilde{B}) = \tilde{B}$. The isometry $\varphi$ induces an isometry $\varphi^*$ on the orbits pace $\mathbb{R}^+$ of $\tilde{M}$ such that for each orbit $\tilde{D}$ we have $\varphi^*(k(\tilde{D})) = k(\varphi(\tilde{D}))$. Since $\varphi(\tilde{B}) = \tilde{B}$, we get that $\varphi^*(0) = \varphi^*(k(\tilde{B})) = k_{\varphi}(\tilde{B}) = k(\tilde{B}) = 0$, so for each $t \in \mathbb{R}^+$ we have $\varphi^*(t) = t$. Thus $\varphi(\tilde{D}) = \tilde{D}$. Now we have a contradiction because a normal geodesic $\gamma$ in $\tilde{M}$ intersects each principal orbit in two points ($\tilde{M}/\tilde{G} = \mathbb{R}^+$) while $\pi \circ \gamma$ intersects a principal orbit infinitely many times ($M/G = [0, 1]$). So there exists $\varphi \in \Delta$ such that $\varphi(\tilde{D}) \neq \tilde{D}$.

We need only to show that $B$ can not be an exceptional orbit, the other parts of the claim is a simple consequence of Theorem 3.5. To prove the claim observe that if it were the case, $\tilde{M}$ would admit only principal orbits and a normal geodesic intersects each orbit in $\tilde{M}$ exactly in one point while since
$M/G = \mathbb{R}^+$, a normal geodesic in $M$ intersects each principal orbit in two points, and a contradiction arises as in the Step 1. □

4. Cohomogeneity One Flat Manifolds

In this section we study cohomogeneity one flat Riemannian manifolds which are not toruslike.

It is known that every isometry $\varphi \in \text{Iso}(\mathbb{R}^n)$ is of the form $\varphi = (A, b)$, $A \in O(n)$, $b \in \mathbb{R}^n$ that is, $\varphi(x) = Ax + b$, $x \in \mathbb{R}^n$. We say that $\varphi$ is an ordinary translation when $A = \text{Id}$ (Id is the identity map on $\mathbb{R}^n$).

Note that $\mathbb{R}^n$ is the universal Riemannian covering manifold of each flat manifold $M$ of dimension $n$.

**Definition 4.1.** We say that a flat Riemannian manifold $M$ is “toruslike” if each deck transformation of the universal covering manifold of $M$ is an ordinary translation.

In the following $V \cdot W$ denotes the inner product of the vectors $V$ and $W$ in $\mathbb{R}^n$ and $|V|$ is the length of $V$.

**Lemma 4.2.** Let $\mathbb{R}^n$ be of cohomogeneity one under the action of a closed Lie subgroup $G \subset \text{Iso}(\mathbb{R}^n)$ and let $\varphi = (A, b) \in G$, $A \neq \text{Id}$. Then there is a normal geodesic $\gamma$ on $\mathbb{R}^n$ such that the function $F(t) = d_{\varphi}^2 \circ \gamma(t)$ is a strictly convex function with the minimum point $t_0 \in \mathbb{R}$.

**Lemma 4.3.** If $\mathbb{R}^n$ is of cohomogeneity one under the action of a closed Lie subgroup $G$ of $\text{Iso}(\mathbb{R}^n)$ and if all the orbits are regular and one orbit is isometric to $\mathbb{R}^{n-1}$, then other orbits are isometric to $\mathbb{R}^{n-1}$.

**Theorem 4.4.** If $M$ is a flat cohomogeneity one Riemannian manifold under the action of a closed Lie group $G \subset \text{Iso}(M)$ and $M$ is not toruslike, then

a) Either each orbit $D$ of $M$ is isometric to $\mathbb{R}^k \times \mathbb{T}^m$ for some $m, k, m + k = n - 1$, or there is a singular orbit $B$ in $M$;

b) If there is a unique singular orbit $B$ which is non-exceptional, then $B$ is isometric to $\mathbb{R}^k \times \mathbb{T}^m$ for some $m, k$ and $\pi_1(M) = \mathbb{Z}^m$.

**Proof:** Let $\tilde{M} = \mathbb{R}^n$ be the universal covering manifold of $M$ and let $\tilde{G}$ be the corresponding covering Lie group of $G$, which acts on $\tilde{M} = \mathbb{R}^n$ by cohomogeneity one.

(a): Since $M$ is not toruslike there is a deck transformation $\varphi$ such that $\varphi = (A, b), A \neq \text{Id}$. By Lemma 4.2 there is a normal geodesic $\gamma$ in $\tilde{M}$ such that the function $F(t) = d_{\varphi}^2 \circ \gamma(t)$ is a strictly convex function with a minimum point $t_0$. Since $d_{\varphi}^2$ is constant along orbits we get that the orbit $\tilde{D}_0 = \tilde{G}\gamma(t_0)$ is the
minimum point set of $d^2_\varphi$. Thus by 2.4(a) it is totally geodesic in $\tilde{M} = \mathbb{R}^n$, so it is flat and therefore isometric to $\mathbb{R}^r$ for some $r$. Now let there is not any singular orbit in $M$. So $\tilde{M} = \mathbb{R}^n$ does not have any singular orbit, therefore $r = n - 1$ and $\tilde{D}_0$ is isometric to $\mathbb{R}^{n-1}$, so by Lemma 4.3 we get that each orbit $\tilde{D}$ of $\tilde{M}$ is isometric to $\mathbb{R}^{n-1}$, therefore each orbit $D (= \pi(\tilde{D}))$ of $M$ is flat, and since it is homogeneous we get by Theorem 2.5 that $D$ is isometric to $\mathbb{R}^k \times \mathbb{T}^m$, for some $m, k$, $m + k = n - 1$. This proves the part (a).

(b): Let $B$ be the unique non-exceptional singular orbit of $M$ and $\tilde{B} = \pi^{-1}(B)$ and let $F(t)$ be the function obtained in the proof of part (a) with the minimum point $t_0$. For each $t \in R$ we have $\tilde{G}_\gamma(t) = g^{-1}(F(t))$, where $g = d^2_\varphi$. If $c$ and $b$ are regular values of $g$ then $g^{-1}(c)$ and $g^{-1}(b)$ are diffeomorphic (see [3], p. 10, Corollary 3.11), from these facts we get that $\tilde{B} = g^{-1}(F(t_0))$ (because if not, then $\tilde{B} = g^{-1}(b)$ where $b$ is a regular value of $g$, and so $B$ must be diffeomorphic to principal orbits which is a contradiction). So $\tilde{B}$ is the minimum point set of $g$ and therefore by 2.4(a) it is totally geodesic in $M$ and is flat, thus $B$ is flat. Since it is homogeneous we get by 2.5 that $B$ is diffeomorphic to $\mathbb{R}^k \times \mathbb{T}^m$ and by 2.3(b) we have $\pi_1(M) = \pi_1(B) = \mathbb{Z}^m$. □

References


