CONFORMAL MAPPINGS AND SPECIAL NETWORKS
OF WEYL SPACES

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Abstract. In this paper, we show that a totally umbilical hypersurface of a recurrent Weyl space is conformally recurrent. Also, while a totally umbilical hypersurface of a recurrent Weyl space is conharmonically recurrent or conharmonically Ricci-recurrent, theorems concerning some special nets are proved.

1. Introduction

A differentiable manifold of dimension n having conformal metric tensor g and symmetric connection \( \nabla \) satisfying the compatibility condition

\[
\nabla g = 2(TXg)
\]

where \( T \) is a 1-form (complementary covector field) is called a Weyl space which is denoted by \( W_n(g, T) \). After renormalization of the metric tensor \( g \)

\[
\tilde{g} = \lambda^2 g
\]

the vector field \( T \) is transformed [1] into

\[
\tilde{T} = T + d\ln \lambda
\]

An object \( A \) defined on \( W_n(g, T) \) is called a satellite of \( g \) of weight \{p\} if it admits a transformation of the form \( \tilde{A} = \lambda^p A \) under the renormalization of \( g \). Suppose that the metrics of \( W_n \) and \( W_{n+1} \) are elliptic and that they are given, respectively, by \( g_{ij} \, du^i \, du^j \) and \( g_{ab} \, dx^a \, dx^b \) which are connected by the relations

\[
g_{ij} = g_{ab} x_i^a x_j^b \quad i, j = 1, 2, \ldots, n, \quad a, b = 1, 2, \ldots, n + 1
\]
where $x^a_i$ denotes the covariant derivative of $x^a$ with respect to $u^i$. The prolonged derivative and the prolonged covariant derivative in the direction of vector $x$ of the satellite $A$ of $g$ of weight $\{p\}$ are defined by the laws, respectively,

$$\ddot{A} = \partial A - p(TX A), \quad \nabla A = \nabla A - p(TX A) \quad (1)$$

where $\partial h$ is the partial derivative of $A$ [2–4]. By $\bar{g} = \lambda^2 g$ and second equality in (1) it follows that for every $z$, $\nabla z g = 0$. It is easy to see that prolonged covariant derivative preserve weights of the satellites.

The prolonged covariant derivative of $A$, relative to $W_n$ and $W_{n+1}$, are related by

$$\nabla_k A = x^a_k \nabla_a A. \quad (2)$$

Let $n^a$ be the contravariant components of the vector field in $W_{n+1}$ normal to $W_n$, and let it be normalized by the condition $g_{ab}n^a n^b = 1$. The moving frame $\{x^i_a, n_a\}$ in $W_n$, reciprocal to the moving frame $\{x^a_i, n^a\}$ is defined by the relations [4]

$$n^a_a = 1, \quad n_a x^a = 0, \quad n^a x_i^a = 0, \quad x^a_i x^a_j = \delta_i^j. \quad (3)$$

Differentiating covariantly with respect to $u^k$ both sides of the last equality (3) and remembering that

$$\nabla_k x^a_i = \nabla_k x^a_i = w_{ik} n^a \quad (4)$$

we find that $\nabla_k x^a_i$, regarded as a function of $x$’s, is a vector of $W_n$, and so it can be expressed in the form [5]

$$\nabla_k x^a_i = \nabla_k x^a_i = \Omega^a_k n_a. \quad (5)$$

Let $v^i (r = 1, 2, \ldots, n)$ be the contravariant components of the $n$ independent vector fields $v$ in $W_n$ which are normalized by the condition $g_{ij} v^i v^j = 1$. Following [1], we define the covector fields $\tilde{v}^i$ satisfying the equalities

$$v^i \tilde{v}^j = \delta^i_j, \quad v^i \tilde{v}^p = \delta^p_r \quad r, p = 1, 2, \ldots, n. \quad (6)$$

Let $v^a$ and $v^i$ be, respectively, the contravariant components of the vector fields $v$ in $W_n$ relative to $W_{n+1}$ and $W_n$. Then, we have

$$v^a = x^a_i v^i. \quad (7)$$
The generalised Gauss equation is obtained, in the following form [6]

$$R_{hijk} = w_{hj}w_{ik} - w_{hk}w_{ij} + \bar{R}_{bced}x^b_i x^c_j x^d_k$$  \hspace{1cm} (8)

where $\bar{R}_{bced}$ is the covariant curvature tensor of $W_{n+1}$.

A hypersurface of a Weyl space is called **totally umbilical** if the following expression holds

$$w_{ij} = \mu g_{ij}$$  \hspace{1cm} (9)

where $\mu$ is a satellite of $g_{ij}$ with weight $\{-1\}$. From this definition, it follows that $\mu = \frac{M}{n}$ where $M$ is the mean curvature of the hypersurface, defined by $M = w_{ij}g^{ij}$. A hypersurface of a Weyl space is totally geodesic if

$$w_{ij} = 0.$$  \hspace{1cm} (10)

We will use the following relations [7]

$$B_{hi...jk}^{ab...cd} = x^a_i x^b_j \cdots x^d_k.$$  \hspace{1cm} (11)

If $\bar{a}^a_{\rho}$ and $a^i_{\rho}$, respectively, the components of the Chebyshev vector fields of the first kind with respect to $W_{n+1}$ and $W_n$, then the following relations hold (see [5] and [8])

$$\bar{a}^a_{\rho} = \kappa n^a_{\rho} + a^i_{\rho} x^a_i, \hspace{1cm} r \neq p$$

$$a^i_{\rho} = \frac{v^k}{p} \bar{\nabla}_k v^i_{\rho}, \hspace{1cm} r \neq p$$

$$\kappa = w_{ik} v^i_{\rho} v^k_{\rho}.$$  \hspace{1cm} (11)

Let any net $(v_1, v_2, \ldots, v_n)$ in $W_n$ be a Chebyshev net of the first kind with respect to $W_{n+1}$, in this case, the following condition holds [9]

$$\bar{a}^a_{\rho} = 0.$$  \hspace{1cm} (12)

If $\bar{b}^a_{\rho}$ and $b^i_{\rho}$ are, respectively, the components of the Chebyshev vector fields of the second kind with respect to $W_{n+1}$ and $W_n$, then the following relations hold [5, 8]

$$\bar{b}^a_{\rho} = (-\Omega^i_k v^j_{\rho} v^k_{\rho}) n_a + b^i_{\rho} x^a_i b_i = \frac{v^k}{r_k} \bar{\nabla}_k v^i_{\rho} \Omega^i_k = w_{km} g^{mi}$$  \hspace{1cm} (13)

(no summation over $r$).
Let any net \((v, v', \ldots, v_n)\) in \(W_n\) be a Chebyshev net of the second kind with respect to \(W_{n+1}\), in this case, the following condition holds [9]

\[
\frac{r}{b_n} = 0. \tag{14}
\]

If \(\bar{c}_a^r\) and \(c_i^r\) are, respectively, the components of the geodesic vector fields of the net \((v, v', \ldots, v_n)\) with respect to \(W_{n+1}\) and \(W_n\), then they are connected by the relations [5, 8]

\[
\bar{c}_a^r = \kappa \iota^a r + c_i^r x_i^a c_i^r = v^k \hat{\nabla}_k v^r \kappa = w_{ik} v^i v^k. \tag{15}
\]

Let any net \((v, v', \ldots, v_n)\) in \(W_n\) be a geodesic net with respect to \(W_{n+1}\), in this case the following condition holds [9]

\[
\bar{c}_a^r = 0.
\]

If \(W_n\) admits of a tensor field \(T_{\ldots}\) such that

\[
\hat{\nabla}_k T_{\ldots} = \lambda_k T_{\ldots} \tag{16}
\]

where \(\lambda_k\) is non-zero vector field of \(W_n\), then \(W_n\) is called a \textit{T-recurrent Weyl space}. We note that since the prolonged covariant derivative preserves the weight, \(\lambda_s\) is a satellite of \(g_{ij}\) with weight \(\{0\}\).

Let \(W_n\) be a hypersurface of recurrent Weyl space \(W_{n+1}\) with recurrence vector \(\lambda_a\) which is not orthogonal to the hypersurface \(W_n\). If we denote the tangential component of \(\phi_a\) by \(\phi_r\), then we have

\[
\phi_k = \phi_a x_a^r.
\]

Since \(W_{n+1}\) is recurrent Weyl space, we can write

\[
\hat{\nabla}_r \bar{R}_{abcd} = \phi_r \bar{R}_{abcd}. \tag{17}
\]

According to [6], we have

\[
\hat{\nabla}_r R_{hi,jk} = \hat{\nabla}_r \Omega_{hi,jk} + \phi_c \bar{R}_{abcd} B_{hi,jkr}^a + \bar{R}_{abcd} B_{ijkl} w_{hr} n^a
\]

\[
+ \bar{R}_{abcd} B_{hi,jk} w_{r} n^b + \bar{R}_{abcd} B_{hak} w_{jr} n^c + \bar{R}_{abcd} B_{hi,j} w_{kr} n^d.
\]
2. Conformal Mappings and Special Nets of Weyl Spaces

Let $\tau$ be a conformal mapping of $W_n(g, T)$ onto $W^*_n(g^*, T^*)$. In this case, we have

$$g^* = g.$$  \hspace{1cm} (18)

The covariant vector $P_k$ is defined by

$$P = T - T^*$$  \hspace{1cm} (19)

is called the vector of the conformal mapping. Clearly, $P$ has zero weight.

Let $C$ be a smooth curve in $W_n(g, T)$ and let $C^*$ be its image under the conformal mapping $\tau$. Denote the parameters of $C$ and $C^*$ by $S$ and $S^*$, respectively. Denote the coordinates of a current point $P$ on $C$ by $x^i$ and those of the corresponding point $P^*$ by $x^*_i$. Then for the tangent vectors $v^i$ and $v^*_i$ at corresponding points, we have

$$v^*_i = v^i.$$  \hspace{1cm} (20)

Let $\nabla$ and $\nabla^*$ be the Weyl connections of $W_n(g, T)$ and $W^*_n(g^*, T^*)$ and let the connection coefficients be denoted by $\Gamma^*_{jk}$ and $\Gamma^i_{jk}$, respectively, then the tensor $T^i_{jk}$ is called the affine deformation tensor, where

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^*_i_{jk}.$$  \hspace{1cm} (21)

Another expression for affine deformation tensor can be written in [10] as follows

$$T^i_{jk} = P_j \delta^i_k + P_k \delta^i_j - P_m g^{im} g_{jk}.$$  \hspace{1cm} (22)

In this case, from the conformal transformation which is given by (1), (2), (3) and (4), the covariant curvature tensor $R_{hi,jk}$ transforms $R^*_{hi,jk}$ as in the following expression, [11]

$$R_{hi,jk} = R_{hi,jk} + g_{hk} P_{ij} + g_{ij} P_{hk} - g_{ik} P_{hj} - g_{hj} P_{ik} + 2 g_{ih} \nabla[k] P_{lj}$$  \hspace{1cm} (22)

where we have put

$$P_{ij} = \nabla_i P_j - P_i P_j + \frac{1}{2} g^{kl} g_{ij} P_k P_l$$

and

$$R^* = R + 2(n - 1) P_m^m.$$  \hspace{1cm} (23)
From this transformation, using (5) and (6), we can easily obtain that the conformal curvature tensor of $W_n$ Weyl space is in the following form, [12]

$$
C_{ijk}^h = R_{ijk}^h - \frac{1}{n-2} (\delta_k^h R_{ij} - \delta_j^h R_{ik} + g_{ij} g^{hm} R_{mk} - g_{ik} g^{hm} R_{mj}) \\
+ \frac{2}{n(n-2)} (\delta_k^h R_{[ij]} - \delta_j^h R_{[ik]} + g_{ij} g^{hm} R_{[mk]} - g_{ik} g^{hm} R_{[mj]}) - (n-2) \delta_i^h R_{[jk]} + \frac{R}{(n-1)(n-2)} (\delta_k^h g_{ij} - \delta_j^h g_{ik}) .
$$

(24)

Let us suppose that the conformal transformation (1) be a conharmonic one, we obtain from the above expression, [11]

$$
P_h^h = g^{hk} \nabla_h P_k + \frac{1}{2} (n-2) P^h P_h = 0 .
$$

(25)

In this case, the conharmonic curvature tensor of Weyl space is in the following form, [13]

$$
K_{ijk}^h = R_{ijk}^h - \frac{1}{n} (\delta_k^h R_{ij} - \delta_j^h R_{ik} + g_{ij} g^{hm} R_{mk} - g_{ik} g^{hm} R_{mj}) + 2 \delta_i^h R_{[jk]} \\
- \frac{1}{n-2} (\delta_k^h R_{(ij)} - \delta_j^h R_{(ik)} + g_{ij} g^{hm} R_{(mk)} - g_{ik} g^{hm} R_{(mj)})
$$

(26)

where $R_{[ij]} = \frac{1}{2} (R_{ij} - R_{ji})$ and $R_{(ij)} = \frac{1}{2} (R_{ij} + R_{ji})$. From (9), the conharmonic Ricci tensor of a Weyl space can be easily obtained in the form

$$
K_{ij} = \frac{R}{2-n} g_{ij} , \quad n \neq 2 .
$$

Now, we prove the following theorems about the conformally recurrent and conharmonically Ricci-recurrent Weyl spaces.

**Theorem 1.** If $W_n$ is a totally umbilical hypersurface of a recurrent Weyl space $W_{n+1}$ then $W_n$ is also conformally recurrent.

**Proof:** If we consider that $W_n$ is a totally umbilical hypersurface of a recurrent Weyl space $W_{n+1}$ then we have, [6]

$$
\hat{\nabla}^r R_{hijk} = \phi_r R_{hijk} + \frac{M}{n^2} [ (\hat{\nabla}^r M) G_{hivk} + (\hat{\nabla}^k M) G_{hijr} + (\hat{\nabla}^r M) G_{kfrh} \\
+ (\hat{\nabla}^h M) G_{kjir} ] + \frac{2M}{n^2} (\hat{\nabla}^r M) G_{hjik} - \frac{M^2}{n^2} \phi_r G_{hijk}
$$

(27)

where $G_{hijk} = g_{hi} g_{jk} - g_{hk} g_{ij}$. 

If we consider the form \( \hat{\nabla}_r C_{hijk} - \phi_r C_{hijk} \), taking the prolonged covariant derivative of the conformal curvature tensor, then we obtain from (7)

\[
\hat{\nabla}_r C_{hijk} = \phi_r C_{hijk} + (\hat{\nabla}_r R_{hijk} - \phi_r R_{hijk}) - \frac{M}{n^2} ((\hat{\nabla}_j M) G_{hijk} + (\hat{\nabla}_k M) G_{hijr} + (\hat{\nabla}_i M) G_{kjrh} + (\hat{\nabla}_h M) G_{kjir} + (2(\hat{\nabla}_r M) - M\phi_r) G_{hijk}).
\] (28)

From (10) and (11), we can obtain

\[ \hat{\nabla}_r C_{hijk} = \phi_r C_{hijk} \]

which is the required result. □

**Theorem 2.** Let a totally umbilical hypersurface \( W_n \) of recurrent Weyl space \( W_{n+1} \) be conharmonically Ricci-recurrent \((n > 2)\). If any net \((v_1, v_2, \ldots, v_n)\) in \( W_n \) is a Chebyshev net of the first kind with respect to \( W_{n+1} \), it is also a Chebyshev net of the first kind with respect to \( W_n \) and the converse is also true.

**Proof:** Let a totally umbilical hypersurface \( W_n \) of recurrent Weyl space \( W_{n+1} \) be conharmonically Ricci recurrent \((n > 2)\). According to [14], we say that \( W_n \) is also recurrent.

If a totally umbilical hypersurface of a recurrent Weyl space is recurrent then we have, [15]

\[ M = 0, \quad \lambda_r \neq 0, \quad n > 2. \] (29)

With the help of (9), (12) and (12), we get

\[ \bar{a}_r = a_r^i x_i^a, \quad r \neq p. \] (30)

From (12), (13) and (30) the proof is clear. □

**Theorem 3.** Let a totally umbilical hypersurface \( W_n \) of recurrent Weyl space \( W_{n+1} \) be conharmonically Ricci-recurrent \((n > 2)\). If any net \((v_1, v_2, \ldots, v_n)\) in \( W_n \) is a Chebyshev net of the second kind with respect to \( W_{n+1} \), it is also a Chebyshev net of the second kind with respect to \( W_n \), and the converse is also true.

**Proof:** Let a totally umbilical hypersurface \( W_n \) of recurrent Weyl space \( W_{n+1} \) be conharmonically Ricci-recurrent \((n > 2)\). Then, \( M = 0 \). From (9) and (14), we get

\[ \bar{b}_a = b_i x_i^a. \] (31)
Using (14), (15) and (31) the proof is completed. □

**Theorem 4.** Let a totally umbilical hypersurface $W_n$ of a recurrent Weyl space $W_{n+1}$ be conharmonically Ricci-recurrent ($n > 2$). If any net $(v_1, v_2, \ldots, v_n)$ in $W_n$ is a geodesic net with respect to $W_{n+1}$, it is also a geodesic net with respect to $W_n$ and conversely.

**Proof:** Let a totally umbilical hypersurface $W_n$ of recurrent Weyl space $W_{n+1}$ be conharmonically Ricci-recurrent ($n > 2$). Then, $M = 0$. In this case, using (9) and (16), we get

$$\bar{c}^a_r = c^i r_i^a. \quad (32)$$

With the help of the equations (16) and (32) and the expression $\bar{c}^a_r = 0$, the result is easily obtained. □

**Remark 1.** Conharmonically recurrent Weyl space is also conharmonically Ricci-recurrent, [13].

**Corollary 1.** Let a totally umbilical hypersurface $W_n$ of recurrent Weyl space $W_{n+1}$ be conharmonically recurrent ($n > 2$). If any net $(v_1, v_2, \ldots, v_n)$ in $W_n$ is a Chebyshev net of the first kind with respect to $W_{n+1}$, it is also a Chebyshev net of the first kind with respect to $W_n$ and the converse is also true.

**Corollary 2.** Let a totally umbilical hypersurface $W_n$ of recurrent Weyl space $W_{n+1}$ be conharmonically recurrent ($n > 2$). If any net $(v_1, v_2, \ldots, v_n)$ in $W_n$ is a Chebyshev net of the second kind with respect to $W_{n+1}$, it is also a Chebyshev net of the second kind with respect to $W_n$ and conversely.

**Corollary 3.** Let a totally umbilical hypersurface $W_n$ of recurrent Weyl space $W_{n+1}$ be conharmonically recurrent ($n > 2$). If any net $(v_1, v_2, \ldots, v_n)$ in $W_n$ is a geodesic net with respect to $W_{n+1}$, it is also a geodesic net with respect to $W_n$ and conversely.

**References**


