HARMONIC FORMS ON COMPACT SYMPLECTIC 2-STEP NILMANIFOLDS

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Abstract. In this paper we study harmonic forms on compact symplectic nilmanifolds. We consider harmonic cohomology groups of dimension 3 and of codimension 2 for 2-step nilmanifolds and give examples of compact 2-step symplectic nilmanifolds $G/I$ such that the dimension of harmonic cohomology groups varies.

1. Introduction

Let $(M, G)$ be a Poisson manifold with a Poisson structure $G$, that is, a skew-symmetric contravariant 2-tensor $G$ on $M$ satisfying $[G, G] = 0$, where $[,]$ denotes the Schouten-Nijenhuis bracket. For a Poisson manifold $(M, G)$, Koszul [5] introduced a differential operator $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$ by $d^* = [d, i(G)]$, where $\Omega^k(M)$ denotes the space of all $k$-forms on $M$. The operator $d^*$ is called the Koszul differential. For a symplectic manifold $(M^{2m}, \omega)$, let $G$ be the skew-symmetric bivector field dual to $\omega$. Then $G$ is a Poisson structure on $M$. Brylinski [1] defined the star operator $*: \Omega^k(M) \to \Omega^{2m-k}(M)$ for the symplectic structure $\omega$ as an analogue of the star operator for an oriented Riemannian manifold and proved that the Koszul differential $d^*$ satisfies $d^* = (-1)^k * d*$ on $\Omega^k(M)$ and the identity $*^2 = \text{id}$. A form $\alpha$ on $M$ is called harmonic form if it satisfies $d\alpha = d^*\alpha = 0$. Let $\mathcal{H}^k_\omega(M) = \mathcal{H}^k(M)$ denote the space of all harmonic $k$-form on $M$. Brylinski [1] defined symplectic harmonic $k$-cohomology group $H^k_{\omega-hr}(M) = \mathcal{H}^k(M)$
by $\mathcal{H}_k^b(M)/(\mathcal{B}_k^b(M) \cap \mathcal{H}_k^b(M))$, as a subspace of de Rham cohomology group $H^k_{DR}(M)$. We denote by $L_\omega: \Omega^k(M) \to \Omega^{k+2}(M)$ the linear operator defined by $\omega$ and the induced homomorphism in de Rham cohomology groups by $L_{[\omega]}: H^k_{DR}(M) \to H^{k+2}_{DR}(M)$.

Mathieu [6] proved the following theorem.

**Theorem 1.** (Mathieu) Let $(M^{2m}, \omega)$ be a symplectic manifold of dimension $2m$. Then the following two assertions are equivalent:

a) For any $k \leq m$, the homomorphism $L_{[\omega]}^k: H^k_{DR}(M) \to H^{k+2}_{DR}(M)$ is surjective.

b) For any $k$, $H^k_{DR}(M) = H^k_{\omega_{hr}}(M)$.

Mathieu’s theorem is a generalization of Hard Lefschetz Theorem for compact Kähler manifolds. Mathieu [6] proved also that, for $i = 0, 1, 2$, $H^i_{hr}(M) = H^i_{DR}(M)$. Yan [10] gave a simpler, elegant proof of Mathieu’s Theorem by using a special type of infinite dimensional $\mathfrak{sl}(2)$-representation theory.

In connection with the study of harmonic forms, we are interested in the following question raised by Khesin and McDuff (see Yan [10]).

Question: On which compact manifold $M$, there exists a family $\omega_t$ of symplectic forms such that the dimension of $H^k_{\omega_t hr}(M)$ varies for some $k$?

For 6-dimensional compact nilmanifolds, the above question is considered independently by one of the present authors [9] and Ibáñez et al [3]. Actually Ibáñez et al [3] have proved that there exist at least five 6-dimensional nilmanifolds $M$ with a family $\omega_t$ of symplectic forms such that the dimension of $H^k_{\omega_t hr}(M)$ varies by computing $H^4_{\omega_t hr}(M)$ and $H^5_{\omega_t hr}(M)$. Note that, in [9], it is proved that the dimension of $H^{2m-1}_{\omega hr}(M)$ for compact 2-step nilmanifold $M^{2m}$ is independent of symplectic forms $\omega$ (cf. Theorem 5).

In this paper we study symplectic harmonic cohomology groups $H^3_{\omega hr}(M)$ of dimension 3 and $H^{2m-2}_{\omega hr}(M)$ of codimension 2 for compact nilmanifolds and give examples of higher dimensional compact 2-step symplectic nilmanifolds $G/\Gamma$ such that the dimension of harmonic cohomology group varies.

### 2. Harmonic Cohomology Groups of Nimanifolds

For a $2m$-dimensional symplectic manifold $(M, \omega)$ let $\mathbf{G}$ be the skew-symmetric bivector field dual to $\omega$. Then $\mathbf{G}$ is a Poisson structure on $M$. By the Darboux’s theorem, going to the canonical coordinates $\{p_1, q_1, \ldots, p_m, q_m\}$, we can write symplectic structure $\omega$ as $\omega = dp_1 \wedge dq_1 + \cdots + dp_m \wedge dq_m$ and respectively the Poisson structure $\mathbf{G}$ as $\mathbf{G} = \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + \cdots + \frac{\partial}{\partial q_m} \wedge \frac{\partial}{\partial p_m}$. 
Brylinski [1] defined the star operator \( \star : \Omega^k(M) \to \Omega^{2m-k}(M) \) by requiring
\[
\alpha \wedge \star \beta = (\wedge^k(G)) (\alpha, \beta) v_M
\]
for \( k \)-forms \( \alpha, \beta \), where \( v_M = \omega^m/m! \). The star operator \( \star \) satisfies the identities
\[
\star^2 = \text{id}, \quad d^* = (-1)^k \star d
\]
and consequently, the Koszul differential \( d^* \) is a symplectic codifferential of the exterior differential \( d \) with respect to the star operator \( \star \). We denote by \( L_\omega = L : \Omega^k(M) \to \Omega^{k+2}(M) \) the operator defined by \( L(\alpha) = \alpha \wedge \omega \).

The following Propositions are due to Yan [10]:

**Proposition 1.** (Duality on forms) The linear mapping \( L^k : \Omega^{m-k}(M) \to \Omega^{m+k}(M) \) is an isomorphism for any \( k \).

**Proposition 2.** (Duality on harmonic forms) The linear mapping \( L^k : \mathcal{H}^{m-k}(M) \to \mathcal{H}^{m+k}(M) \) is an isomorphism for any \( k \). In particular, we have \( H^i_{hr}(M) = \text{Im} \{ L^k : H^i_{hr}(M) \to H^i_{DR}(M) \} \).

Note also that we have \( H^i_{hr}(M) = H^i_{DR}(M) \) for \( i = 0, 1, 2 \). Thus we have the following corollary from Proposition 2.

**Corollary 1.** We have
\[
H^i_{hr}(M) = \text{Im} \{ L^{i-1} : H^1_{DR}(M) \to H^i_{DR}(M) \}
\]
and
\[
H^i_{hr}(M) = \text{Im} \{ L^{i-1} : H^2_{DR}(M) \to H^i_{DR}(M) \}.
\]

Let \( \mathfrak{g} \) be a Lie algebra and put \( \mathfrak{g}^{(0)} = \mathfrak{g} \) and let \( \mathfrak{g}^{(i+1)} = [\mathfrak{g}, \mathfrak{g}^{(i)}] \) for \( i \geq 0 \). A Lie algebra \( \mathfrak{g} \) is said to be \((r+1)\)-step nilpotent if \( \mathfrak{g}^{(r)} \neq (0) \) and \( \mathfrak{g}^{(r+1)} = (0) \) and a Lie group \( G \) is said to be \((r+1)\)-step nilpotent if the Lie algebra \( \mathfrak{g} \) is \((r+1)\)-step nilpotent. If \( G \) is a simply-connected \((r+1)\)-step nilpotent Lie group and \( \Gamma \) is a lattice of \( G \), that is, a discrete subgroup of \( G \) such that \( G/\Gamma \) is compact, then \( G/\Gamma \) is called an \((r+1)\)-step compact nilmanifold. We denote by \( \wedge^k \mathfrak{g}^* \) the space of all left \( G \)-invariant \( k \)-forms on \( G \) and regard it as a subspace of \( \Omega^k(G/\Gamma) \). Then we have a subcomplex \( (\wedge^* \mathfrak{g}^*, d) \) of the de Rham complex \( (\Omega^*(G/\Gamma), d) \) of compact nilmanifold \( G/\Gamma \) and denote by \( H^k(\mathfrak{g}) \) the \( k \)-th cohomology groups of the complex \( (\wedge^* \mathfrak{g}^*, d) \).

**Theorem 2.** (Nomizu) For a compact nilmanifold \( G/\Gamma \), the inclusion \( \iota : (\wedge^* \mathfrak{g}^*, d) \to (\Omega^*(G/\Gamma), d) \) induces an isomorphism on cohomology groups
\[
\iota^* : H^k(\mathfrak{g}) \cong H^k_{DR}(G/\Gamma).
\]
For a symplectic form $\omega$ on a compact nilmanifold $G/\Gamma$, there exists a $G$-invariant closed 2-form $\omega_0$ on $G/\Gamma$ such that $\omega - \omega_0 = d\gamma$. Note that $\omega_0$ is also a symplectic form on $G/\Gamma$. For a $G$-invariant symplectic form $\omega_0$, we denote by $\mathcal{H}^k(g)$ the space of all $G$-invariant harmonic $k$-forms on $G/\Gamma$ and by $H^k_{\omega_0hr}(g) = \mathcal{H}^k(g)/(\mathcal{B}^k(g) \cap \mathcal{H}^k(g))$ a subspace of Lie algebra cohomology group $H^k(g)$.

**Proposition 3.** Let $(G/\Gamma, \omega)$ be a compact symplectic nilmanifold and let $\omega_0$ be a $G$-invariant symplectic form such that $\omega - \omega_0 = d\gamma$ as above. Then we have

$$H^k_{\omega_0hr}(G/\Gamma) = H^k_{\omega_0hr}(G/\Gamma) = H^k_{\omega hr}(g)$$

for any $k$.

**Proposition 4.** Let $(G/\Gamma, \omega)$ be a $2m$-dimensional compact symplectic nilmanifold with a $G$-invariant symplectic form $\omega \in \Lambda^2(g^*)$. Then the linear mapping

$$L^k : \mathcal{H}_{\omega hr}^{m-k}(g) \rightarrow \mathcal{H}_{\omega hr}^{m+k}(g)$$

is an isomorphism for any $k$.

Now we may assume that symplectic structures on $G/\Gamma$ are $G$-invariant in order to study harmonic cohomology groups on compact nilmanifolds $M$. A nilpotent Lie algebra $g$ with a non-degenerate invariant closed 2-form is called a symplectic nilpotent Lie algebra. For an $(r+1)$-step nilpotent Lie algebra $g$, let $a^{(i)}$ denote a complementary vector subspace of $g^{(i+1)}$ in $g^{(i)}$: $g^{(i)} = g^{(i+1)} + a^{(i)}$ for $i = 0, 1, \ldots, r$. For simplicity, put $\Lambda^{i_0, \ldots, i_r} = \Lambda^{i_0, 0\ldots, 0}_{i_1, \ldots, i_r}$ and $\Lambda^{i_0, \ldots, i_r} = \Lambda_{i_0 + \ldots + i_r}$. Then we have $\Lambda^* g^* = \sum_{i_0 + \ldots + i_r = a} \Lambda^{i_0, \ldots, i_r}$.

The following lemma is due to Benson and Gordon [2].

**Lemma 1.** Each closed 2-form $\theta \in \Lambda^2 g^*$ belongs to $\Lambda^{1,0,\ldots,0,1} + \sum \Lambda^{i_0,\ldots,i_r} \theta$.

Let $\{\lambda_1, \ldots, \lambda_n\}$ be a basis of $\Lambda^{0,\ldots,0,1}$. By Lemma 1, a $G$-invariant symplectic form $\omega$ can be written in the form

$$\omega = \beta_1 \wedge \lambda_1 + \cdots + \beta_n \wedge \lambda_n + \sum_{i_0, \ldots, i_r = 0} \Lambda^{i_0, \ldots, i_r} \theta$$

modulo $\Lambda^{1,0,\ldots,0}$. Note that $\beta_1, \ldots, \beta_n$ are linearly independent, since $\omega$ is non-degenerate. We extend these elements to a basis $\{\beta_1, \ldots, \beta_n, \ldots, \beta_{n_0}\}$ for $\Lambda^{1,0,\ldots,0}$. Put $a^{(0)}_1 = \text{span}\{\beta_1, \ldots, \beta_n\}$ and $a^{(0)}_2 = \text{span}\{\beta_{n+1}, \ldots, \beta_{n_0}\}$. Then $a^{(0)*} = a^{(0)*}_1 + a^{(0)*}_2$. Put $n_1 = \dim a^{(0)*}_1 = n_r$ and $n_2 = \dim a^{(0)*}_2 = n_0 - n_r$. For simplicity, we also put $\Lambda^{(i_0,i_1),i_2,\ldots,i_r} = \Lambda^{i_0,\ldots,i_r}$. Then $\omega = a^{(0)*}_1 + a^{(0)*}_2 + \sum \Lambda^{i_0,\ldots,i_r} \theta$.
\[ \wedge^{i_0} a_1^{(0)*} \wedge \wedge^{i_0} a_2^{(0)*} \wedge \wedge^{i_1} a^{(1)*} \wedge \cdots \wedge \wedge^{i_r} a^{(r)*}. \]
Moreover, let \( \{ \beta_1^{(k)}, \ldots, \beta_{n_k}^{(k)} \} \) be a basis of \( a^{(k)*} \) and put
\[
\{ \omega_1, \ldots, \omega_{2m} \} = \{ \beta_1, \ldots, \beta_{n_0}, \ldots, \beta_1^{(k)}, \ldots, \beta_{n_{n_k}}^{(k)}, \ldots, \lambda_1, \ldots, \lambda_r \}.
\]
Let \( \{ X_1, \ldots, X_{2m} \} \) be a basis of \( \mathfrak{g} \) which is dual to the basis \( \{ \omega_1, \ldots, \omega_{2m} \} \).
If we write the symplectic form \( \omega \) as
\[
\omega = \sum a_{ij} \omega_i \wedge \omega_j \quad a_{ij} = -a_{ji} \in \mathbb{R}
\]
then it is easy to see that the Poisson structure \( \mathbf{G} \) which is dual to \( \omega \) is given by
\[
\mathbf{G} = -\sum c_{ij} X_i \wedge X_j \quad (3)
\]
where \( c_{ij} \) is the inverse of the transposed matrix of \( (a_{ij}) \).

**Lemma 2.** With respect to the basis \( \{ X_1, \ldots, X_{2m} \} \) above, \( \mathbf{G} \) is given in the form
\[
(c_{ij}) = \begin{pmatrix}
0_{n_r,n_r} & 0_{n_r,n_0-n_r} & 0_{n_r,2m-n_0-n_r} & E_{n_r} \\
0_{n_0-n_r,n_r} & * & * & * \\
0_{2m-n_0-n_r,n_r} & * & * & * \\
-E_{n_r} & * & * & * \\
\end{pmatrix}. \quad (4)
\]

**Proof:** Note that \( (c_{ij}) \) is an alternating matrix. We have \( a_{j n_0 + \cdots + n_{r-1} + i} = \delta_{ji} \) for \( i, j = 1, \ldots, n_r \) and \( a_{jk} = 0 \) for \( j = 1, \ldots, n_r; k = 1, \ldots, n_0 + \cdots + n_{r-1} \) from (2). Then we have
\[
c_{ik} = c_{ik} a_{i n_0 + \cdots + n_{r-1} + i} = \sum_{l=1}^{n_0 + \cdots + n_r} c_{ik} a_{l n_0 + \cdots + n_{r-1} + i} = \delta_{k n_0 + \cdots + n_{r-1} + i}
\]
for \( i = 1, \ldots, n_r \) and \( k = 1, \ldots, n_0 + \cdots + n_r \). \( \Box \)

**Lemma 3.** Let \( G/\Gamma \) be a 2-step compact nilmanifold with a \( G \)-invariant symplectic form \( \omega = \beta_1 \wedge \lambda_1 + \cdots + \beta_{n_1} \wedge \lambda_{n_1} \) modulo \( \wedge^{2,0} \). Then we have
\[
d^{\lambda_k} = \sum_{i<j \leq n_1} b_{ij}^{k} \beta_i \wedge \beta_j + \sum_{i \leq n_1, j < n_1} b_{ij}^{k} \beta_i \wedge \beta_j \in \wedge^{(2,0),0} + \wedge^{(1,1),0}
\]
for \( k = 1, \ldots, n_1 \).
Proof: Put $d\lambda_k = \sum_{i,j \leq n_1} b_{ij}^k \beta_i \wedge \beta_j + \sum_{i \leq n_1 < j} b_{ij}^k \beta_i \wedge \beta_j + \sum_{n_1 < i < j} b_{ij}^k \beta_i \wedge \beta_j$. Since $\omega$ is closed, we have

$$\beta_1 \wedge \sum_{n_1 \leq i < j} b_{ij}^1 \beta_i \wedge \beta_j + \cdots + \beta_{n_1} \wedge \sum_{n_1 \leq i < j} b_{ij}^{n_1} \beta_i \wedge \beta_j = 0.$$ 

Therefore, since $\beta_1 \wedge \beta_{i_1} \wedge \beta_{j_1}, \ldots, \beta_{n_1} \wedge \beta_{i_n} \wedge \beta_{j_n}$ ($n_1 < i_k < j_k$) are linearly independent, we conclude that $b_{ij}^k = 0$ for $n_1 < i < j$. □

Note that, from Lemma 2, we have $i(G)(\wedge^{(2,0)}) = i(G)(\wedge^{(1,1)}) = 0$.

Theorem 3. Let $G/\Gamma$ be a 2-step compact nilmanifold with a $G$-invariant symplectic form $\omega$. Then we have $B^3(g) \subset H^3(g)$.

Proof: Since $d\wedge^{2,0} = 0$ and $d\wedge^{1,1} \subset \wedge^{3,0}$, we have to consider only the case of $\wedge^{0,2}$. From Lemma 3, we end with

$$\wedge^{0,2} \xrightarrow{d} \wedge^{(2,0), 1} + \wedge^{(1,1), 1}.$$ 

Thus, from (4) in Lemma 2, we get

$$i(G)(d\wedge^{0,2}) \subset \wedge^{1,0} \xrightarrow{d} 0,$$

which implies that $d\wedge^{0,2} \subset H^3(g)$. □

By a straightforward computation, we have also that

$$\ast (\omega_{i_1} \wedge \cdots \wedge \omega_{i_s}) = \sum_{j_1 < \cdots < j_s} (-1)^a \det (c_{kh})_{h = i_{j_1}, \ldots, i_{j_s}} \omega_{j_1} \wedge \cdots \wedge \omega_{j_s} \wedge \cdots \wedge \omega_{2m} \quad (5)$$

where $\omega^m/m! = a \cdot (\omega_{j_1} \wedge \cdots \wedge \omega_{j_s}) \wedge (\omega_{1} \wedge \cdots \wedge \omega_{j_1} \wedge \cdots \wedge \omega_{j_s} \wedge \cdots \wedge \omega_{2m})$.

In fact, let

$$\ast (\omega_{i_1} \wedge \cdots \wedge \omega_{i_s}) = \sum_{j_1 < \cdots < j_s} a_{j_1 \cdots j_s} \omega_{j_1} \wedge \cdots \wedge \omega_{j_s} \wedge \cdots \wedge \omega_{2m}.$$ 

Then, we get that

$$(\omega_{j_1} \wedge \cdots \wedge \omega_{j_s}) \wedge \ast (\omega_{i_1} \wedge \cdots \wedge \omega_{i_s})$$

$$= (\wedge^s(G))(\omega_{j_1} \wedge \cdots \wedge \omega_{j_s}, \omega_{i_1} \wedge \cdots \wedge \omega_{i_s}) \omega^m/m!$$

$$= \det(i(G)(\omega_k, \omega_h))_{h = i_{j_1}, \ldots, i_{j_s}} \omega^m/m! = (-1)^a \det(c_{kh})_{h = i_{j_1}, \ldots, i_{j_s}} \omega^m/m!.$$
Thus we have finally
\[ a^{j_1\ldots j_s}_{i_1\ldots i_s} = (-1)^s a \cdot \det(c_{k,l})_{k=i_1,\ldots,i_s, l=j_1,\ldots,j_s}. \]

**Theorem 4.** Let \( \mathfrak{g} \) is an \( (r+1) \)-step symplectic nilpotent Lie algebra. Then, for \( q = 0, \ldots, n_r \), we have
\[ \bigwedge^{n_0,\ldots,n_r-1, n_r-q} \subset \mathcal{H}^{2m-q}(\mathfrak{g}). \]

**Proof:** Note that the star operator \( * : \mathcal{H}^{m-k}(\mathfrak{g}) \to \mathcal{H}^{m+k}(\mathfrak{g}) \) is an isomorphism for each \( k \) and we have \( \bigwedge^{(q,0),0,\ldots,0} \subset \mathcal{H}^{q}(\mathfrak{g}) \) from (4) in Lemma 2. Now, from (5) and (4) in Lemma 2, we see that the star operator
\[ *(\bigwedge^{(q,0),0,\ldots,0})^{n_0,\ldots,n_r-1, n_r-q} \]
is an isomorphism. Thus we have \( \bigwedge^{n_0,\ldots,n_r-1, n_r-q} \subset \mathcal{H}^{2m-q}(\mathfrak{g}) \) for \( q = 0, \ldots, n_r \). \( \square \)

**Corollary 2.** Let \( G/\Gamma \) be a 2-step compact nilmanifold with a \( G \)-invariant symplectic form \( \omega \). Then we have
\[ \bigwedge^{n_0,n_1-q} \subset \mathcal{H}^{2m-q}(\mathfrak{g}). \]
In particular, we have that
\[ \dim H^2_{\text{DR}}(G/\Gamma) - \dim H^{2m-2}_{h}\bigwedge^{n_0,3,n_1, n_0-3,n_1} = \dim(d \bigwedge^{n_0-2,n_1-1} \cap \mathcal{H}(\mathfrak{g})) + \dim d \bigwedge^{n_0-2,n_1-1} - n_1. \]

**Proof:** Since \( d \bigwedge^{n_0-2,n_1-1} \subset \bigwedge^{n_0,n_1-2}, d \bigwedge^{n_0-2,n_1-1} \) is a subspace of \( \mathcal{H}^{2m-2}(\mathfrak{g}) \). Thus we have that
\[ \dim H^2(g) - \dim H^{2m-2}_{h}\bigwedge^{n_0-3,n_1} = \dim \mathcal{H}^2(g) - \dim (B^2(g) \cap \mathcal{H}^2(g)) - \dim \mathcal{H}^{2m-2}(g) \]
\[ + \dim (B^{2m-2}(g) \cap \mathcal{H}^{2m-2}(g)) \]
\[ = \dim (B^{2m-2}(g) \cap \mathcal{H}^{2m-2}(g)) - \dim B^2(g) \]
\[ = \dim(d \bigwedge^{n_0-3,n_1} \cap \mathcal{H}^{2m-2}(g)) + \dim d \bigwedge^{n_0-3,n_1} - \dim B^2(g) \]
\[ = \dim(d \bigwedge^{n_0-3,n_1} \cap \mathcal{H}^{2m-2}(g)) + \dim d \bigwedge^{n_0-2,n_1-1} - n_1. \]
\( \square \)
Theorem 5. Let $G/\Gamma$ be a 2-step compact nilmanifold with a $G$-invariant symplectic form $\omega$. Then we have

$$\bigwedge^{(n_0^1, n_0^2-p), n_1-q} \subset \mathcal{H}^{2m-p-q}(\mathfrak{g}).$$

(7)

In particular,

$$\bigwedge^{(n_0^1, n_0^2-1), n_1} + \bigwedge^{(n_0^1, n_0^2), n_1-1} = \mathcal{H}^{2m-1}(\mathfrak{g})$$

which implies $\dim H^{2m-1}_{hr}(\mathfrak{g}) = n_0^2 = \dim \mathfrak{g} - 2 \dim[\mathfrak{g}, \mathfrak{g}]$.

Proof: From Lemma 3, it is obvious that $\bigwedge^{(n_0^1, n_0^2-p), n_1-q} \subset Z^{2m-p-q}(\mathfrak{g})$. Since $d^* = (-1)^k \ast d \ast$ on $\bigwedge^k (\mathfrak{g}^*)$, it is enough to prove that

$$\ast : \bigwedge^{(n_0^1, n_0^2-p), n_1-q} \rightarrow \bigwedge^{p+q,0}.$$

Note also that

$$\ast (\omega_{i_1} \wedge \cdots \wedge \omega_{i_s})$$

$$= \sum_{j_1 < \cdots < j_s} (-1)^s a \det (c_{kh})_{(k, h = 1, \ldots, l_s)} \omega_{i_1} \wedge \cdots \wedge \hat{\omega}_{j_1} \wedge \cdots \wedge \hat{\omega}_{j_s} \wedge \cdots \wedge \omega_{2m} \ (8)$$

where $s = n_0^1 + n_0^2 + n_1 - p - q$. Thus if $\{j_1, \ldots, j_s\} \not\subset n_0 + j$, then we get that $\det (c_{kh})_{(k, h = 1, \ldots, l_s)} = 0$ from Lemma 2. In fact, noting that $n_1 = n_1$, we have

$$\det (c_{kh})_{(k, h = 1, \ldots, l_s)} = \begin{vmatrix} c_{j_1}^{i_1} & \cdots & c_{j_1}^{i_{n_0^1}} & \cdots & c_{j_1}^{i_{l_s}} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ c_{j_r}^{i_1} & \cdots & c_{j_r}^{i_{n_0^1}} & \cdots & c_{j_r}^{i_{l_s}} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ c_{j_s}^{i_1} & \cdots & 0 & \cdots & c_{j_s}^{i_{n_0^1}} & \cdots & c_{j_s}^{i_{l_s}} \\
\end{vmatrix} = 0. \square$$

Thus we get that if $\det (c_{kh})_{(k, h = 1, \ldots, l_s)} \neq 0$, $\{j_1, \ldots, j_s\} \supset \{n_0 + 1, \ldots, n_0 + n_1\}$. Therefore we have $\ast (\bigwedge^{(n_0^1, n_0^2-p), n_1-q}) \subset \bigwedge^{p+q,0}$. \square

In particular, we have $\bigwedge^{(n_0^1, n_0^2-1), n_1} + \bigwedge^{(n_0^1, n_0^2), n_1-1} \subset \mathcal{H}^{2m-1}(\mathfrak{g})$. Since $L^{m-1} : \mathcal{H}^1(\mathfrak{g}) \rightarrow \mathcal{H}^{2m-1}(\mathfrak{g})$ is an isomorphism by Proposition 2, we obtain that $\dim \mathcal{H}^{2m-1}(\mathfrak{g}) = n_0$. On the other hand, we have that $\dim(\bigwedge^{(n_0^1, n_0^2-1), n_1} + \bigwedge^{(n_0^1, n_0^2), n_1-1}) = (n_0 - n_1) + n_1 = \dim \mathcal{H}^{2m-1}(\mathfrak{g})$. Thus we have proved our second claim. The last claim follows from the fact that $B^{2m-1}(\mathfrak{g}) = \bigwedge^{(n_0^1, n_0^2), n_1-1}$ which is due to Benson and Gordon [2].
3. Examples

Example 1. We consider a 2-step nilpotent Lie algebra of dimension $2m$ given by $\mathfrak{g} = \text{span}\{X_1, \ldots, X_m, Y_1, \ldots, Y_m\}$ with $[X_i, X_{i+1}] = Y_i$, $i = 1, \ldots, m$, where we set $X_{m+1} = X_1$. Then a simply connected nilpotent Lie group $G$ with the Lie algebra $\mathfrak{g}$ has a lattice $\Gamma$. Let $\{\beta_1, \ldots, \beta_m, \lambda_1, \ldots, \lambda_m\}$ be the dual basis of $\{X_1, \ldots, X_m, Y_1, \ldots, Y_m\}$. Note that $d \Lambda^{k,0} = (0)$ and $d \Lambda^{k,j} \subset \Lambda^{k+2j-1}$.

It is easy to see that $d: \Lambda^{0,j} \to \Lambda^{2j-1}$ is injective.

The space of invariant closed 2-forms $\Lambda^2(\mathfrak{g})$ is given by

$$\Lambda^2(\mathfrak{g}) = \left\{ \sum_{i=1}^{m} a_{i,i+1}(\beta_i \wedge \lambda_{i+1} - \beta_{i+2} \wedge \lambda_i) + \sum_{i=1}^{m} a_{i,i} \beta_i \wedge \lambda_i \right\} + \Lambda^2$$

where we have put $\beta_{m+1} = \beta_1$, $\beta_{m+2} = \beta_2$, $\lambda_{m+1} = \lambda_1$.

Now we consider a symplectic form $\omega$ given by an element of $\Lambda^{1,1}$. Then the Poisson structure $G$ is of the form

$$G = - \sum_{i,j=1}^{m} c_{ij} X_i \wedge Y_j$$

with respect to the basis $\{X_1, \ldots, X_m, Y_1, \ldots, Y_m\}$. We see that

$$i(G): \Lambda^3 \to (0), \quad i(G): \Lambda^1 \to \Lambda^0, \quad i(G): \Lambda^2 \to \Lambda^1$$

and the space of harmonic 3-forms $\mathcal{H}^3(\mathfrak{g})$ is given by

$$\mathcal{H}^3(\mathfrak{g}) = \Lambda^3 + \Lambda^2 \mathcal{H}^3(\mathfrak{g}) \cap \Lambda$$

For $m \geq 6$, we see that

$$\Lambda^3 \cap \Lambda = \left\{ \sum_{j=1}^{m} b_j \beta_{j+1} \wedge \lambda_j \wedge \lambda_{j+1}; b_j \in \mathbb{R}, j = 1, \ldots, m \right\}$$

where we have put $\beta_{m+1} = \beta_1$, $\lambda_{m+1} = \lambda_1$.

Case 1. The case when $\omega = \sum_{j=1}^{m} a_{jj} \beta_j \wedge \lambda_j$ where $a_{jj} \neq 0$.

Note that $G = - \sum_{j=1}^{m} \frac{1}{a_{jj}} X_j \wedge Y_j$ and hence we have $\mathcal{H}^3(\mathfrak{g}) \cap \Lambda = (0)$ for $m \geq 6$.

Case 2. The case when $\omega = \sum_{j=1}^{m} a_{jj+1}(\beta_j \wedge \lambda_{j+1} - \beta_{j+2} \wedge \lambda_j)$. 

...
Note that for the above basis \( \omega \) can be written in the form \( \left( \begin{smallmatrix} 0 & A \\ \lambda & \lambda^{-1} \end{smallmatrix} \right) \), where
\[
A = \begin{pmatrix}
0 & a_{12} & 0 & 0 & 0 & \ldots & 0 & -a_{m-1,m} & 0 \\
0 & 0 & a_{23} & 0 & 0 & \ldots & 0 & 0 & 0 \\
-1 & 0 & 0 & a_{34} & 0 & \ldots & 0 & 0 & 0 \\
0 & -a_{23} & 0 & 0 & a_{45} & \ldots & 0 & 0 & 0 \\
0 & 0 & -a_{34} & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & -a_{45} & 0 & \ldots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \ldots & a_{m-3,m-2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & a_{m-2,m-1} & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & a_{m-1,m} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \ldots & -a_{m-2,m-1} & 0 & 0 \\
\end{pmatrix}.
\]

We can prove also that, for \( m \geq 3 \), the matrix \( A \) is non-degenerate if and only if \( m = 3\ell \) for \( \ell \in \mathbb{N} \). Put \( A^{-1} = (c_{ij}) \) for \( m = 3\ell \). Then we have also that the components of the matrix \( A^{-1} = (c_{ij}) \) satisfy the conditions
\[
c_{jj} = 0 \quad \text{for} \quad j = 1, \ldots, m, \quad c_{j+1,j} = 0 \quad \text{for} \quad j = 1, \ldots, m - 1 \quad \text{and} \quad c_{1m} = 0.
\]

Since \( G \) is given by \( \left( \begin{smallmatrix} 0 & A^{-1} \\ \lambda A & 0 \end{smallmatrix} \right) \), we see that for \( \alpha = \sum_{\ell=1}^{m} b_{\ell} \beta_{\ell+1} \wedge \lambda_{\ell} \wedge \lambda_{j+1} = i(G)\alpha = 0 \) and \( \alpha \) therefore is harmonic. This implies that \( H^3(G) \cap \Lambda^{1,2} = Z^3(g) \cap \Lambda^{1,2} \) and \( \dim(H^3(G) \cap \Lambda^{1,2}) = 3\ell \) for \( \ell \geq 2 \). Thus, from Theorem 3, we see that compact 2-step nilmanifolds \( G/\Gamma \) admit such symplectic structures that the dimension of harmonic cohomology group \( H^3_{\omega,br}(G/\Gamma) \) varies.

**Example 2.** For \( p \geq 2 \) let \( \mathfrak{h}(1,p) \) be a 2-step nilpotent Lie algebra of dimension \( 2p + 1 \) spanned by \( \{X_1, \ldots, X_{2p+1}\} \) with
\[
[X_1, X_i] = X_{p+i} \quad i = 2, \ldots, p + 1
\]

We consider the Lie algebra \( \mathfrak{g} = \mathfrak{h}(1,p) \oplus \mathbb{R} \) of dimension \( 2p + 2 \). Then a simply connected nilpotent Lie group \( G \) with the Lie algebra \( \mathfrak{g} \) has a lattice \( \Gamma \). Let \( X_{2p+2} \) denote a generator of the Lie algebra \( \mathbb{R} \) and let \( \{\omega_1, \ldots, \omega_{p+1}, \omega_{p+2}, \ldots, \omega_{2p+1}, \omega_{2p+2}\} \) be the dual basis of the basis \( \{X_1, \ldots, X_{p+1}, X_{p+2}, \ldots, X_{2p+1}, X_{2p+2}\} \). Then we have
\[
\bigwedge^{(0,1),0} \supset \text{span} \{\omega_{2p+2}\}, \quad \bigwedge^{(0,0),1} = \text{span} \{\omega_{p+2}, \ldots, \omega_{2p+1}\}.
\]

Consider a \( G \)-invariant symplectic structure \( \omega \) on \( G/\Gamma \). We write the Poisson structure \( \mathbf{G} \) dual to \( \omega \) as \( \mathbf{G} = -\sum c_{ij} X_i \wedge X_j \) with respect to the basis \( \{X_1, \ldots, X_{2p+1}, X_{2p+2}\} \) above.
We prove that
\[
\dim H^2(\mathfrak{g}) - \dim H^2_{\omega, hr}(\mathfrak{g}) = \begin{cases} pC_2, & \text{if } c_{12p+2} \neq 0 \\ pC_2 + (p - 1) & \text{if } c_{12p+2} = 0. \end{cases}
\]

In this example we use the following notations. For \( i < j \), we put
\[
\hat{\omega}_i \hat{\omega}_j = \omega_1 \wedge \cdots \wedge \hat{\omega}_i \wedge \cdots \wedge \hat{\omega}_j \wedge \cdots \wedge \omega_{2p+2}.
\]
Similarly, for \( i_1 < \cdots < i_k \) we put
\[
\hat{\omega}_{i_1} \cdots \hat{\omega}_{i_k} = \omega_1 \wedge \cdots \wedge \hat{\omega}_{i_1} \wedge \cdots \wedge \hat{\omega}_{i_k} \wedge \cdots \wedge \omega_{2p+2}.
\]
For \( 2 \leq i < j \leq p + 1 \), we put
\[
\alpha_{ij} = \omega_2 \wedge \cdots \wedge \hat{\omega}_i \wedge \cdots \wedge \hat{\omega}_j \wedge \cdots \wedge \omega_{p+1} \wedge \omega_{2p+2} \wedge d(\omega_{p+2} \wedge \cdots \wedge \omega_{2p+1}).
\]
Then we have that
\[
\begin{align*}
\text{for } 2 \leq i < j < k, & \quad d(\hat{\omega}_i \hat{\omega}_j \hat{\omega}_k) = 0 \\
\text{for } 2 \leq i < j \leq p + 1, & \quad d(\hat{\omega}_1 \hat{\omega}_i \hat{\omega}_j) = -\alpha_{ij} \\
\text{for } 2 \leq i \leq p + 1, & \quad d(\hat{\omega}_1 \hat{\omega}_i \hat{\omega}_{2p+2}) = (-1)^{p} \hat{\omega}_{p+1} \hat{\omega}_{2p+2} \\
\text{for } 2 \leq i, j \leq p + 1, & \quad d(\hat{\omega}_1 \hat{\omega}_i \hat{\omega}_{p+1}) = \begin{cases} \pm \hat{\omega}_{p+1} \hat{\omega}_{p+1} & \text{for } i < j \\ \pm \hat{\omega}_{p+1} \hat{\omega}_{p+1} & \text{for } j < i \\ 0 & \text{for } i = j \end{cases} \\
\text{for } 2 \leq i < j \leq p + 1, & \quad d(\hat{\omega}_1 \hat{\omega}_{p+1} \hat{\omega}_{p+1}) = 0 \\
\text{for } 2 \leq i \leq p + 1, & \quad d(\hat{\omega}_1 \hat{\omega}_{p+1} \hat{\omega}_{2p+2}) = 0.
\end{align*}
\]
From (6) of Corollary 2, we have
\[
\dim H^2(\mathfrak{g}) - \dim H^2_{\omega, hr}(\mathfrak{g}) = \dim(d \bigwedge^{p-1,p} \mathcal{H}^2_{\omega}(\mathfrak{g})) + \dim d \bigwedge^{p-1,p} = pC_2. \quad (10)
\]
Note that \( d(\hat{\omega}_1 \hat{\omega}_{p+1} \hat{\omega}_{2p+2}) = (-1)^{p} \hat{\omega}_{p+1} \hat{\omega}_{2p+2} \in \bigwedge^{(p+1),p-1} \) for \( 2 \leq i \leq p + 1 \). Thus, from Theorem 5, we see that \( d(\hat{\omega}_1 \hat{\omega}_{p+1} \hat{\omega}_{2p+2}) \in \mathcal{H}^2_{\omega}(\mathfrak{g}) \). From (9), we see that \( \dim d \bigwedge^{p-1,p} = pC_2 \). We put
\[
V = \text{span}\{\alpha_{ij}; 2 \leq i < j \leq p + 1\}.
\]
Then, from (10), we have
\[
\dim H^2(\mathfrak{g}) - \dim H^2_{\omega, hr}(\mathfrak{g}) = \dim(V \cap \mathcal{H}^2_{\omega}(\mathfrak{g})) + pC_2. \quad (11)
\]
Note that \( a^{(0)} = \text{span}\{X_1, \ldots, X_{p+1}, X_{2p+2}\} \) and \( a^{(1)} = \text{span}\{X_{p+2}, \ldots, X_{2p+1}\} \). Put \( \Lambda_{a, \omega} = \Lambda^{a^{(0)}} \Lambda^{a^{(1)}} \) and write the Poisson structure \( \mathbf{G} \) dual to \( \omega \) as
\[
\mathbf{G} = \mathbf{G}_{2,0} + \mathbf{G}_{1,1} + \mathbf{G}_{0,2}, \quad \mathbf{G}_{i,j} \in \bigwedge_{i,j}.
\]
Note that

\[
di(G_{2,0})(\bigwedge_{p+1,p-1} \bigwedge_{p-1,p-1} \bigwedge_{p+1,p-2}) = d \bigwedge_{p+1,p-1} \bigwedge_{p-1,p-1} \bigwedge_{p+1,p-2}
\]

\[
di(G_{1,1})(\bigwedge_{p+1,p-1} \bigwedge_{p-2,p-2} \bigwedge_{p+2,p-3}) = d \bigwedge_{p+1,p-1} \bigwedge_{p-2,p-2} \bigwedge_{p+2,p-3}
\]

\[
di(G_{0,2})(\bigwedge_{p+1,p-1} \bigwedge_{p+1,p-3} = d \bigwedge_{p+1,p-1} = (0).
\]

Note that, from Lemma 1, the \(G\)-invariant symplectic structure \(\omega\) on \(G/\Gamma\) is of the form

\[
\omega = \sum_{i<j \leq p+1} a_{ij} \omega_i \wedge \omega_j + \sum_{j \leq 2p+1} a_{j 2p+2} \omega_j \wedge \omega_{2p+2} + \sum_{i \leq p+1<j \leq 2p+1} a_{ij} \omega_i \wedge \omega_j.
\]

Since \(d\omega_{p+i} = -\omega_1 \wedge \omega_i\), we see that \(a_{p+i 2p+2} = 0\) for \(2 \leq i \leq p+1\). Thus the matrix form of \(G\) with respect to the basis \(\{X_1, \ldots, X_{2p+1}, X_{2p+2}\}\) is given by

\[
\begin{pmatrix}
0 & c_{12p+2} & \cdots & c_{12p+1} & c_{12p+2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-c_{1p+2} & \cdots & -c_{p+1p+2} & c_{p+1p+2} & c_{p+1p+2} \\
-c_{p+1p+2} & \cdots & -c_{12p+1} & -c_{12p+2} & \cdots & -c_{p+1p+2}
\end{pmatrix}
\]

Thus \(G_{2,0} = -c_{12p+2}X_1 \wedge X_{2p+2}\). Moreover, we have

\[
di(G_{2,0})(\alpha_{ij}) = -di(G_{2,0})(\omega_1 \wedge \cdots \wedge \omega_i \wedge \cdots \wedge \omega_{p+1} \wedge \omega_{2p+2} \\
\wedge \omega_{p+2} \wedge \cdots \wedge \omega_{p+j} \wedge \cdots \wedge \omega_{2p+1})
\]

\[
+ di(G_{2,0})(\omega_1 \wedge \cdots \wedge \omega_j \wedge \cdots \wedge \omega_{p+1} \wedge \omega_{2p+2} \\
\wedge \omega_{p+2} \wedge \cdots \wedge \omega_{p+i} \wedge \cdots \wedge \omega_{2p+1})
\]

\[
= -2c_{12p+2}(\omega_2 \wedge \cdots \wedge \omega_{p+1} \wedge \omega_{p+2} \wedge \cdots \\
\wedge \omega_{p+i} \wedge \cdots \wedge \omega_{p+j} \wedge \cdots \wedge \omega_{2p+1}).
\]

i) The case of \(c_{12p+2} \neq 0\).

From (12) and (13), we see that \(V \cap H^{2p}(g) = (0)\), and hence we get

\[
\dim H^2(g) - \dim H^{2p}_{\omega-hr}(g) = p C_2.
\]

ii) The case of \(c_{12p+2} = 0\).
Since $G$ is non-degenerate, $c_{1p+k} \neq 0$ for some $k \in \{2, \ldots, p + 1\}$. For simplicity, we may assume that $c_{1p+2} \neq 0$.

Now we put

$$W = \text{span}\{\hat{\omega}_{p+i} \hat{\omega}_{p+j} \hat{\omega}_{p+k}; 2 \leq i < j < k \leq p + 1\}.$$  

Note that $\dim V = pC_2$ and $\dim W = pC_3$.

We consider the linear mapping $\text{di}(G_{1,1}) : V \rightarrow W$. We claim that

$$\dim \text{di}(G_{1,1})(V) = pC_2.$$  

Then $\dim \ker(\text{di}(G_{1,1})) = pC_2 - pC_2 = p - 1$ and hence

$$\dim H^2(g) - \dim H^2_{\omega, \text{hr}}(g) = pC_2 + p - 1.$$  

By a straightforward computation, we see

$$\text{di}(G_{1,1})(\alpha_{ij}) = 2 \sum_{k<i} (-1)^{p+k} c_{1p+k} \tilde{\omega}_{p+i} \tilde{\omega}_{p+j} \tilde{\omega}_{p+k}$$

$$- 2 \sum_{i<k<j} (-1)^{p+k} c_{1p+k} \tilde{\omega}_{p+i} \tilde{\omega}_{p+j} \tilde{\omega}_{p+k}$$  

$$+ 2 \sum_{j<k} (-1)^{p+k} c_{1p+k} \tilde{\omega}_{p+i} \tilde{\omega}_{p+j} \tilde{\omega}_{p+k}.$$  

(14)

Consider the basis

$$\{\alpha_{34}, \ldots, \alpha_{3p+1}, \alpha_{45}, \ldots, \alpha_{p+1}, \alpha_{23}, \ldots, \alpha_{2p+1}\}$$

of $V$ and the basis

$$\{\tilde{\omega}_{p+2} \tilde{\omega}_{p+3} \tilde{\omega}_{p+4}, \tilde{\omega}_{p+2} \tilde{\omega}_{p+3} \tilde{\omega}_{p+5}, \ldots, \tilde{\omega}_{p+2} \tilde{\omega}_{p+3} \tilde{\omega}_{p+4} \tilde{\omega}_{p+5}, \ldots, \tilde{\omega}_{2p-1} \tilde{\omega}_{2p} \tilde{\omega}_{2p+1}\}$$

of $W$. Then, from (14), we see that, with respect to the bases above, the matrix form of $\text{di}(G_{1,1})$ is of the form

$$\text{di}(G_{1,1}) = \begin{pmatrix}
2(-1)^{p} c_{1p+2} & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 2(-1)^{p} c_{1p+2}
\end{pmatrix}\begin{pmatrix} C \end{pmatrix}$$

where $D$ is a matrix of $pC_3 \times pC_2$ and $C$ is a matrix of $pC_2 \times (p - 1)$. Then we see that the rank$(\text{di}(G_{1,1})) \geq pC_2$ and hence $\dim \ker(\text{di}(G_{1,1})) \leq pC_2 - pC_2 = p - 1$.  

For $j = 3, \ldots, p + 1$, we put
\[
\gamma_j = (-1)^{p+2} c_1 p + 2 \alpha_{2j} + \sum_{2 \leq \ell < j} (-1)^{p+\ell} c_1 p + \ell \alpha_{\ell j} + \sum_{j \leq \ell \leq p+1} (-1)^{p+\ell} c_1 p + \ell \alpha_{j\ell}.
\]
From (14), it is easy to conclude that, for $j = 3, \ldots, p + 1$,
\[
d_i(G_{1,1})(\gamma_j) = 0.
\]
Since $\{\gamma_j ; j = 3, \ldots, p + 1\}$ are linearly independent, we have proved our claim.

References