A first chapter is provided which contains sufficient concepts from abstract fuzzy topology to make the book self-contained.

"ABELIAN VARIETIES" (Second Edition)

By D. Mumford


This book is a systematic account of the basic results about abelian varieties. It includes an exposition, on the one hand, of the analytic methods and results applicable when the ground field $k$ is the complex field $\mathbb{C}$ and, on the other hand, of the scheme-theoretic methods and results used to deal with inseparable isogenies when the ground field $k$ has characteristic $p$.

The revised second edition contains, in addition, appendices on "The Theorem of Tate" and the "Mordell-Weil Theorem".

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**PROBLEM PAGE**

First item this time is one of those intriguing problems which is often solved more easily by non-mathematicians! I heard it from Richard Dumby who traces it back to John Conway.

1. Find the next entry in the following sequence:

$$1, 11, 21, 1211, 111221, 3112211, \ldots.$$  

Here is another problem with a simple solution which is not so simple to discover.

2. Find an infinite family of pairs of distinct integers $m,n$ such that:

- $m,n$ have the same prime factors, and
- $m-1$, $n-1$ have the same prime factors.

Now for the solutions to some earlier problems, from March 1986.

1. How long is the recurring block of digits in $(0.001)^2$?

I first heard this problem from David Fowler, who uses it as an example to show that simple arithmetic can be surprisingly tricky.

Many people's first guess at the answer is 6 digits or 9 digits, but in fact the recurring block has 2997 digits! To be precise:

$$(0.001)^2 = 0.000001002 \ldots 996997999.$$  

In case you think that there is a misprint here, the string 998 is indeed absent.
Once the pattern in this recurring block has been noticed it is not hard to show that

$$\frac{(0.\overline{00 \ldots 0})^2}{n} = \frac{1}{(10^n-1)^2}$$

has a recurring block of $n(10^n-1)$ digits. To verify that the decimal expansion has the form

$$\frac{(0.\overline{00 \ldots 0})^2}{n} = 0.\overline{00 \ldots 000 \ldots 1 \ldots 99 \ldots 9}$$

with the string $99 \ldots 9$ missing, one can apply the identity

$$\frac{10^m(10^n-1) + 1}{(10^n-1)^2} = \frac{m + (m+1)(10^n-1) + 1}{(10^n-1)^2}$$

with $m = 0, 1, \ldots, 10^n-2$, in the long division $1/(10^n-1)^2$.

2. Prove that at least one of the numbers

$$\pi + e, \pi e$$

is transcendental.

Thanks to Des MacHale for supplying this problem and its solution.

We use the facts that if $x$ and $y$ are both algebraic numbers then so are $xy$ and $x/y$ (see Herstein's Topics in Algebra, page 172), and also $\sqrt{x}$. Thus if both $\pi + e$ and $\pi e$ are algebraic we deduce that

$$\pi = \frac{1}{2}((\pi + e)^2 - 4\pi e) + (\pi + e)$$

is algebraic, which is clearly false.

The argument clearly holds for any pair of transcendental numbers $\alpha, \beta$ and Des points out that there are generalisations to more than two numbers, involving the symmetric functions.

3. Suppose that $a_n \neq 0$, for $n = 1, 2, \ldots$. How large can

$$\lim_{n \to \infty} \frac{a_n}{e^{a_1+a_2+\ldots+a_n}}$$

be?

Tom Carroll (a postgraduate at the OU) recently encountered a series of this form while constructing a certain sub harmonic function.

In fact the series is convergent with sum less than 1. One can see this by noting that

$$\frac{a_n}{e^{a_1+a_2+\ldots+a_n}} < \frac{e^{a_n-1}}{e^{a_1+a_2+\ldots+a_n-1}}$$

since $e^x \geq 1+x$. Thus, by telescoping cancellation, the nth partial sum of (*) is at most

$$1 - \frac{1}{e^{a_1+a_2+\ldots+a_n}} < 1.$$ 

To see that the number 1 is best possible here, consider

$$\lim_{n \to \infty} \frac{a_n}{e^{a_1+a_2+\ldots+a_n}} = 1, \quad a > 0,$$

and notice that

$$\lim_{a \to 0} \frac{a}{e^a - 1} = 1.$$