A trivial application of Zorn's lemma shows that every filter on $I$ can be extended to an ultrafilter on $I$.

The following topological property of ultrafilters forms the basis of the definition of the Banach space ultraproduct.

**Theorem 1.1.** Let $K$ be a compact Hausdorff topological space; let $I$ be a non-empty set and $U$ be an ultrafilter on $I$. Then, for each family $(x_i)_{i \in I}$ in $K$, there exists a unique point $x \in K$ such that, for every neighbourhood $V$ of $x$,

$$
(i \in I : x_i \in V) \in U.
$$

The point $x$ is called the limit of $(x_i)_{i \in I}$ with respect to $U$, and is denoted by $\lim_{U} x_i$.

2. **ULTRAPRODUCTS OF BANACH SPACES**

Let $(E_i, \| \cdot \|) : i \in I$ be a family of Banach spaces over $C$ (or $\mathbb{R}$) indexed by the set $I$. $U$ is an ultrafilter on $I$.

Define $\Pi_U$ and $\Pi_{\Pi}$ as follows:

$$
\Pi_U := \{ (x_i)_{i \in I} : x_i \in E_i, \sup_{i \in I} \| x_i \| < \infty \}
$$

$$
\Pi_{\Pi} := \{ (x_i)_{i \in I} : (x_i)_{i \in I} \in \Pi_U, \lim_{U} \| x_i \| = 0 \}.
$$

Note that, for $(x_i)_{i \in I} \in \Pi_U$, $\lim_{U} \| x_i \|$ exists and is unique by theorem 1.1.

Let $\| \cdot \|$ be the supremum norm on $\Pi_U$:

$$
\| (x_i)_{i \in I} \| := \sup_{i \in I} \| x_i \|.
$$

Then $l^\infty((E_i)_{i \in I})$ is the Banach space $(\Pi_U, \| \cdot \|)$ over $C$. It is easy to check that $\Pi_U$ is a closed subspace of $l^\infty((E_i)_{i \in I})$. 

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The ultraproduct of the family \( \{ (E_i, \| \cdot \|) \mid i \in I \} \) modulo \( \mathcal{U} \) is the quotient space \( \prod_i (E_i) / \mathcal{U} \) with the canonical norm, and is denoted \( (E_i) / \mathcal{U} \) or \( E_i \mathcal{U} \). \( (E_i) / \mathcal{U} \) is called a Banach space ultraproduct; in the case where \( E_i = E \) for all \( i \in I \), \( E \mathcal{U} \) is also written \( E / \mathcal{U} \) and is termed the Banach space ultrapower of \( E \) modulo \( \mathcal{U} \).

It is convenient and customary to denote elements of \( (E_i) / \mathcal{U} \) by \( \{ x_i \} \) so that

\[
\{ x_i \} = \{ x_i \mid i \in I \} \in E / \mathcal{U}.
\]

Notice that the quotient norm on \( (E_i) / \mathcal{U} \) is given by the equation:

\[
\| \{ x_i \} \| = \inf_{n \in \mathbb{N}} \| \{ x_i \mid i \in I \} + n \| = \lim_{\mathcal{U}} \| x_i \|.
\]

For each ultrapower \( E / \mathcal{U} \) of \( E \), there is a canonical isometric embedding \( \hat{\mathcal{T}} \) of \( E \) into \( E / \mathcal{U} \):

\[
\hat{\mathcal{T}}(x) := \{ x_i \mid i \in I \} \quad \text{where} \quad x_i = x \quad \text{for all} \quad i \in I \quad \text{and} \quad \lim_{\mathcal{U}} \| x_i \| = \lim_{\mathcal{U}} \| x \|.
\]

If \( E \) is finite dimensional, then \( E \) and \( E / \mathcal{U} \) are isometrically isomorphic. The closed balls of \( E \) are compact so that for every bounded family \( \{ x_i \mid i \in I \} \) in \( E \) the limit \( \lim_{\mathcal{U}} x_i \) exists in \( E \) (by theorem 1.1) and \( \| \lim_{\mathcal{U}} x_i \| = \lim_{\mathcal{U}} \| x_i \| \) so that the map \( \{ x_i \mid i \in I \} \mapsto \lim_{\mathcal{U}} x_i \) is a linear surjection with kernel \( \mathcal{U} \), hence induces an isometric isomorphism of \( E / \mathcal{U} \) and \( E \).

The following proposition introduces the theme of the structure-preserving properties of ultraproducts.

**Proposition 2.1.** The following classes of Banach spaces are closed under ultraproducts:

1. Banach algebras;
2. \( C^* \) algebras;
3. \( C(K) \)-spaces;
4. \( L^p \) spaces.

The class of \( JB^* \) triple systems is closed under ultraproducts.

**Proof.** To prove (1) and (ii) define the natural multiplication and involution on \( (E_i) / \mathcal{U} \):

\[
\{ x_i \} \cdot \{ y_i \} := \{ x_i y_i \} \quad \{ x_i \}^* := \{ x_i^* \}.
\]

For (iii) note that \( C(K) \)-spaces are \( C^* \)-algebras and hence ultraproducts of \( C(K) \)-spaces are \( C(K) \)-spaces by the Gel'fand-Naimark theorem; (iv) requires the representation theorem for \( L^p \) spaces.

Finally, if \((E, \| \cdot \|, \phi)\) is a \( JB^* \) triple system (cf. [2]), then there exists \( M > 0 \) such that for all \( x, y, z \in E \)

\[
\| \phi(x, y, z) \| \leq M \| x \| \| y \| \| z \|
\]

so that \( \Phi(\cdot, \cdot) \) is a \( JB^* \) triple system with

\[
\Phi(x_i, y_i) = \{ \phi(x_i, y_i) \mid i \in I \}.
\]

\( N \) is a \( J^* \) ideal in \( \prod_i (E_i) \) by (**) and hence \( (E / \mathcal{U}, \| \cdot \|, \phi) \) is a \( JB^* \) triple system.

3. ULTRAPOWER PRINCIPLES AND SUPER-PROPERTIES

One of the successful typical applications of Banach space ultraproducts is in the local theory of Banach spaces, i.e., the study of the finite dimensional structure of Banach spaces and its relation to global structure. In particular, finite representability - the most important concept of the local theory - has a simple powerful ultrapower characterisation.
Let $E$ and $F$ be Banach spaces. $F$ is finitely representable in $E$ iff

for all $\epsilon > 0$, for every finite dimensional subspace $M$ of $F$, there exists a finite dimensional subspace $N$ of $E$ with $\dim N = \dim M$, and an isomorphism $\phi$ from $M$ onto $N$ such that

$$(1-\epsilon)||x|| \leq ||\phi(x)|| \leq (1+\epsilon)||x||$$

for all $x \in M$.

The isomorphism $\phi$ is termed a $(1+\epsilon)$ isomorphism. For orientation here are two results.

**PROPOSITION 3.1.**

(i) Every Banach space is finitely representable in itself.

(ii) Finite representability is transitive.

(iii) Every Banach space is finitely representative in $l_\infty$, in $c_0$, and in the separable reflexive Banach space $(\ell_1^r, \|\cdot\|_p)$, the $\ell_\infty$-sum of the family $(\ell_1^r : n \in \mathbb{N})$ where $\mathbb{N}$ is $c_0$ with supremum norm.

The easy proof is omitted. Incomparable deeper is:

**THEOREM 3.2 (Dvoretzky).** $l_1$ is finitely representable in every infinite dimensional Banach space.

The advertised characterisation of finite representability is as follows:

**THEOREM 3.3.** $F$ is finitely representable in $E$ iff there exists an ultrafilter $U$ on a set $I$ such that $F$ is isometric to a subspace of $E^I/U$.

**PROOF.** In the format of an expository note there is space just to isolate one characteristic feature of the proof of 3.3 which occurs in the choice of the index set $I$ and the construction of the ultrafilter $U$ on $I$.

Let $I$ be the set of all pairs $(M, \epsilon)$ where $M$ is a finite dimensional subspace of $F$ and $\epsilon > 0$. Partially order $I$ by $< : (M_1, \epsilon_1) < (M_2, \epsilon_2)$ iff $M_1 \subseteq M_2$ and $\epsilon_1 \leq \epsilon_2$. Associate a filter $A$ with $< on $I$:

$I_0 \in A$ iff $I_0 \subseteq I$ and there exists $(M_0, \epsilon_0) \in I$ with

$I = \{(M, \epsilon) \in I : (M_0, \epsilon_0) < (M, \epsilon)\}$.

Extend $A$ to an ultrafilter $U$ on $I$.

Since $F$ is finitely representable in $E$, for each $I = (M_1, \epsilon_1) \in I$, there exists a $(1+\epsilon_1)$ isomorphism $\phi_I$ from $M_1$ onto $N_1 \subseteq E$:

$$(1-\epsilon_1)||x|| \leq ||\phi_I(x)|| \leq (1+\epsilon_1)||x||$$

for all $x \in M_1$.

Define a mapping $J : F^I/U$ by

$$Jx = (x_I)_U$$

where

$$x_I = \begin{cases} \phi_I(x) & \text{if } x \in M_1, \\ 0 & \text{otherwise} \end{cases}$$

$J$ is the required linear isometry.

Note in particular that $E^I/U$ is finitely representable in $E$ for any ultrafilter $U$ on a non-empty set $I$.

3.2 and 3.3 imply that the modulus of convexity of a uniformly convex infinite dimensional Banach space is dominated by the modulus of convexity of $l_1$.

Ultrapower techniques allow one to deduce information on the global structure of $E$ from its local structure. The reformation of local principles results in corresponding ultrapower principles. One of the best examples of this process is:

**THEOREM 3.4 (Ultrapower Principle of local Reflexivity).**

Let $E$ be a Banach space. There exist an ultrafilter $U$ on a set $I$ and a mapping $J$ from $E^{**}$ into $E^I/U$ such that
projections, so from 2.1 one deduces the following recent theorem of S. Dineen [2]:

**Theorem 3.5.** Let \((E, || ||, \Phi)\) be a JB* triple system. Then the bidual \((E^{**}, || ||, \Phi)\) is a JB* triple system.

Intuitively, a local property of Banach spaces is a property P such that if E has P, then every Banach space locally similar to E also has P. Super-properties are the mathematically precise explication of this intuition. Let P be any property of Banach spaces. E has the property super-P iff every Banach space finitely representable in E has the property P. P is called a super-property iff whenever E has P then E has super-P.

Examples of super-properties include: uniform convexity, super-reflexivity, the properties "E is finitely representable in G" and "G is not finitely representable in E" for arbitrary fixed Banach space G.

The super-properties of infinite dimensional Banach spaces can be ordered in a hierarchy: there is a weakest (trivial) super-property \(\Pi\), the first (non-trivial) super-property \(\Theta\), and the strongest super-property \(\Pi\). Their definitions run:

\(\Pi(E) : E\) is a Hilbert space.
\(\Theta(E) : c_0\) is not finitely representable in \(E\).
\(\Pi(E) : E\) is infinite dimensional.

3.1 and 3.2 show that the following implications hold: \(\Pi(E) \Rightarrow Q(E) \Rightarrow C(E) \Rightarrow W(E)\) where Q is any super-property. 3.2 implies too that \(W\) is equivalent to the super-property \(D\);

\(D(E) : l_2\) is finitely representable in \(E\).

There are many characterisations of \(C\). Here is a recent one deriving from results in [3]. Let BD be the property:

\((- 36 -)\)
BD(E) : every bounded domain in the complex Banach space E is biholomorphically equivalent to a finite product of irreducible complex Banach manifolds.

Then C is equivalent to super-BD.

Immediate consequences of the hierarchy of super-properties are:

1. Hilbert spaces possess every super-property;
2. if l is fails to have a given super-property Q, then no infinite dimensional Banach space has Q;
3. if E is infinite dimensional with even one (non-trivial) super-property, then c₀ is not finitely representable in E.

4. CONCLUSION

The Banach space ultraproduct was developed initially in an interaction of functional analysis and mathematical logic. Thus it is not surprising to find Banach space analogues of theorems of first-order model theory: downward Loewenheim-Skolem theorem, Keisler-Shelah theorem ([8], [11]). A simple corollary of these results is a version of the Banach-Mazur theorem:

**COROLLARY 4.1.** Assuming the continuum hypothesis, there exists a Banach space of density character χ₁ which contains (isometrically isomorphic copies of) every Banach space of density character at most χ₁. In fact, there is an ultrapower of c₀ satisfying 4.1.

Recent applications of ultraproduct techniques to nonlinear classification problems can be found in [7].

Finally, the material of Section 2 can be generalized to define ultrapowers of locally convex spaces [5], [6].

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