leads to the same linearized value, here \( \lambda, \mu \) are ordinary numbers.
Case (ii): \( A, B \) anti-commute; that is, \( AB = -BA \), which will be the case when \( A \) and \( B \) are fermion operators. In that case, consistency demands that \( \lambda \) and \( \mu \) anti-commute with one another, and also with the operators \( A, B \), then \( \lambda, \mu \) may be taken as Grassmann or Clifford numbers.
Thus a general Hamiltonian, after linearization by this method, will look naturally like an element of a superalgebra, with \( A_1 \)-type elements multiplied by Grassmann (or Clifford) numbers, just as in the simple example above. This approach has recently been used to give a superalgebraic model of superconductivity [6].

References


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Integrals of Subharmonic Functions

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This article reviews a problem concerning potential theory in \( \mathbb{R}^n \) which has its roots in classical complex analysis. One of the interesting features of the problem is the way in which the solution has gradually emerged, sometimes in a surprising fashion. The article is based on a lecture given at the First September Meeting of the Society, held at Trinity College, Dublin.

1 Background in C

Let \( N(f, r) \) denote the maximum modulus of an analytic function \( f \) on the circle \( \{ z \in \mathbb{C} : |z| = r \} \). The starting point for our discussion is provided by the following facts from elementary complex analysis.

Hadamard’s Three Circles Theorem. If \( f \) is analytic on \( \{|z| < R\} \) and \( f \neq 0 \), then \( \log N(f, r) \) is convex as a function of \( \log r \).

Principle of Removable Singularities. If \( f \) is analytic on \( \{ 0 < |z| < R \} \) and \( rN(f, r) \to 0 \) as \( r \to 0^+ \), then \( f \) has an analytic continuation to \( \{ |z| < R \} \).

The latter result is saying that either \( N(f, r) \) behaves badly near 0 or else 0 is a removable singularity for \( f \), in which case \( N(f, r) \) is continuous at 0. The Three Circles Theorem has the following analogue for supraeme over lines. (See [14, p.180] for an important application of this result in the proof of the M. Riesz convexity theorem.)

Three Lines Theorem. Let \( f \) be bounded and analytic on \( \mathbb{R} \times (0, 1) \), continuous on \( \mathbb{R} \times [0, 1] \), and let \( f \neq 0 \). Then

\[
y \mapsto \sup \{ \log|f(x + iy)| : x \in \mathbb{R} \}
\]

defines a convex function on \([0, 1] \).
We will be concerned with analogues of the above results for integrals of subharmonic functions. We recall that a function \( s \) defined on a connected open subset \( \omega \) of \( \mathbb{R}^n \) (\( n \geq 1 \)) and taking values in \( [-\infty, +\infty) \) is called subharmonic if \( s \) is upper semicontinuous (u.s.c.), that is, \( \limsup_{y \to x} s(y) = s(x) \) for all \( x \in \omega \); 

(i) \( s \) is upper semicontinuous (u.s.c.), i.e. \( \limsup_{y \to x} s(y) = s(x) \) for all \( x \in \omega \); 

(ii) the mean of \( s \) over the boundary of any closed ball in \( \omega \) is greater than or equal to its value at the centre.

**Notes.** (I) A function \( h \) is harmonic (i.e. \( h \) satisfies Laplace’s equation) if and only if both \( h \) and \( -h \) are subharmonic.  

(II) If \( f \) is analytic on \( C \) and \( f \neq 0 \), then \( \log |f| \) is subharmonic. (Here we are identifying \( C \) with \( \mathbb{R}^2 \) in the usual way).

(III) Condition (ii) above can be replaced by (ii'): for any open set \( W \) with compact closure in \( \omega \), and for any continuous function \( h \) on \( \overline{W} \) which is harmonic on \( W \) and satisfies \( h \geq s \) on \( \partial W \), we have \( h \geq s \) on \( W \).

(IV) Although it is usual to work with subharmonic functions on open subsets of \( \mathbb{R}^n \), where \( n \geq 2 \), the definition also makes sense for \( n = 1 \). We discuss this further at the end of Section 3.

### 2 Convexity Theorems

If \( s \) is a non-negative subharmonic function on \( \mathbb{R}^{n-1} \times (0, 1) \), put

\[
M(z_n) = \int_{\mathbb{R}^{n-1}} s(z_1, \ldots, z_n) \, dz_1 \ldots dz_{n-1} \quad (0 < z_n < 1).
\]

The following analogue of the Three Lines Theorem is essentially due to Hardy, Ingham and Pólya [8] in the case \( n = 2 \). (See also [13, 9]).

**Theorem 1** If \( M(\cdot) \) is locally bounded on \( (0, 1) \), then it is convex.

**Proof** (\( n = 2 \)). Let \( 0 < \alpha < \beta < 1 \), and choose \( a, b \) such that \( ay + b = M(y) \) for \( y = \alpha, \beta \). Now define

\[
h_t(z, y) = ay + b + \varepsilon \cosh(\pi x) \sin(\pi y)
\]

(a harmonic function), and

\[
u_t(z, y) = \int_{-\ell}^{\ell} s(z + t, y) \, dt,
\]

which is subharmonic because it is finite valued, u.s.c. (by Fatou’s Lemma) and submeanvalued (by Tonelli’s Theorem). Also \( u_t \leq h_t \) on \( \mathbb{R} \times \{\alpha, \beta\} \) and

\[
u_t(z, y) - h_t(z, y) \to -\infty \quad (|z| \to \infty, \alpha \leq y \leq \beta),
\]

so (cf. (ii') above) \( u_t \leq h_t \) on \( \mathbb{R} \times [\alpha, \beta] \). Letting \( \varepsilon \to 0^+ \) and \( \ell \to \infty \), we get \( M(y) \leq ay + b \) for \( y \in [\alpha, \beta] \), proving convexity.

**Question.** Is local boundedness the “right” condition?

The hypothesis cannot be dispensed with entirely. To give some idea of possible behaviour we give below a few simple examples when \( n = 2 \).

**Examples**

(i) \( s(x, y) \equiv 1; \quad M(y) \equiv +\infty \).

(ii) \( s(x, y) = e^{2xy} |\sin \pi y|; \quad M(y) = \begin{cases} 0 & \text{if } y \in \{0, \frac{1}{2}, 1\} \\ +\infty & \text{otherwise.} \end{cases} \)

(iii) \( s(x, y) = \frac{e^y}{x^2 + (y + 1)^2}; \quad M(y) = \frac{xe^y}{y + 1} \).

(iv)
Thus $M(\cdot)$ may be everywhere infinite, or everywhere finite, or neither. Even if $M(\cdot)$ is always finite, it need not be convex.

**Theorem 2.** If $M(\cdot)$ is locally integrable on $(0, 1)$, then it is finite and convex.

This result, due to Kuran [10], shows that convexity holds provided we restrict the type of discontinuity that is allowed to occur. It was substantially improved when Rippon [12] applied a result of Domar to obtain the following.

**Theorem 3.** If $\log^+ M(\cdot)$ is locally integrable on $(0, 1)$, then $M(\cdot)$ is finite and convex.

It was also shown in [12] that the hypothesis here is best possible, so the convexity property of $M(\cdot)$ is now satisfactorily described. However, we will mention a recent generalization [7] which shows what happens when integration of $s$ is carried out with respect to fewer of the co-ordinates.

3  A Generalization

A subset $E$ of $\omega$ is called polar if there is a subharmonic function on $\omega$ which takes the value $-\infty$ on $E$. A function $s$ is said to be quasi-subharmonic if the function $\bar{s}(X) = \lim \sup_{Y \to X} s(Y)$ is subharmonic, and $\bar{s}$ equals $s$ except on a polar set.

Let $X = (X', X'') \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, $(2 \leq m \leq n - 1)$, and put

$$P(s, X'') = \int_{\mathbb{R}^{n-m}} s(X', X'') \, dX',$$

$$P_\infty(s, X'') = \sup\{s(X', X'') : X' \in \mathbb{R}^{n-m}\}.$$

**Theorem 4.** Let $s$ be subharmonic on $\mathbb{R}^{n-m} \times (0, 1)$.  

(i) If $\{\log^+ P(s^+, \cdot)^{m+1}\}^{m+1}$ is locally integrable on $(0, 1)^m$, then $P(s, \cdot)$ is either subharmonic on $(0, 1)^m$ or identically valued $-\infty$.

(ii) If $\{\log^+ P_\infty(s^+, \cdot)^{m+1}\}^{m+1}$ is locally integrable on $(0, 1)^m$, then $P_\infty(s, \cdot)$ is quasi-subharmonic on $(0, 1)^m$.

Notes. The hypotheses can be weakened slightly [7]. A version of (i) with stronger hypotheses was proved independently by Aikawa [1].

4  Growth Theorems

We now consider analogues for $M(\cdot)$ of the Principle of Removable Singularities. In what follows, we assume that $s$ is a non-negative subharmonic function on the half-space $\mathbb{R}^{n-1} \times (0, +\infty)$, and that $M(\cdot)$ is finite and convex on $(0, +\infty)$. We also note that, if $M(\cdot)$ is bounded on $(a, +\infty)$ for some $a > 0$, then $M(\cdot)$ is decreasing (wide sense).

The following is due to Flett [6].

**Theorem 5.** If $M(y) = O(y^{n-1})$ as $y \to +\infty$, then $M(\cdot)$ is decreasing.

**Proof:** Let $B(X, r)$ denote the open ball of centre $X$ and radius $r$, and let $\nu$ denote the volume of $B(O, 1)$. By hypothesis there exists $c > 0$ such that $M(y) \leq cy^{n-1}$ for all $y \geq \frac{1}{2}$. If $x_n \geq 1$, then

$$s(X) \leq \int_{B(X, x_n/2)} s(Y) \, dY$$

$$\leq c \int_{x_n/2}^{x_n} s(Y) \, dy$$

$$= \frac{1}{\nu(x_n/2)^{n-1}} \int_{x_n/2}^{3x_n/2} M(y) \, dy$$
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Corollary. If \( \lim \inf_{y \to +\infty} y^{-n} \varphi(y) < +\infty \) then \( \varphi(\cdot) \) is decreasing.

Open Question. Can the hypothesis about the harmonic majorant be removed from Theorem 5?

This question appears to be difficult. If the answer is "yes", then Rippon's condition (*) is best possible [12].

5 An Extension

We mention now a recent result [3] which shows what can be said about the growth of \( \varphi = \varphi(s, \cdot) \) when we drop the requirement that \( s \) be non-negative. Again, \( s \) denotes a subharmonic function on \( \mathbb{R}^{n-1} \times (0, +\infty) \).

Theorem 9. If \( \log^+ M(s^+, y) = o(y) \) and

\[
\int_{1}^{\infty} y^{-n-1} M(s, y) \, dy < +\infty,
\]

then \( \varphi(s, \cdot) \) and \( \varphi(s^+, \cdot) \) are decreasing, and \( \varphi(s^-, y) = o(y) \).

The proof of Theorem 9 begins by estimating the distributional Laplacian of \( s \) on strips and using this to show that \( s \) has a harmonic majorant on \( \mathbb{R}^{n-1} \times (0, +\infty) \). With regard to the sharpness of the result we mention the following. (i) If \( \log^+ M(s^+, y) = o(y) \), then all three conclusions fail. (ii) If we replace \( y^{-n-1} \) by \( y^{-n-1-o} \), the counterexample of §4 (involving Legendre functions) applies. (iii) The conclusion about \( \varphi(s^+, \cdot) \) is best possible in that, if \( \phi(y) \) decreases to 0 as \( y \to +\infty \), then there is a negative subharmonic function \( s \) such that \( \varphi(s^-, y) \geq y\phi(y) \).

6 Other Results

A number of papers have dealt with \( M(\Phi \circ s, \cdot) \), where \( \Phi \) is an increasing, convex function (whence \( \Phi \circ s \) is subharmonic). We mention here only the case \( \Phi(x) = x^p \), where \( p > 1 \). The following is a refinement of a result of Brawn [4] in the light of Theorem 3.

Theorem 10. If \( s \) is non-negative and subharmonic on \( \mathbb{R}^{n-1} \times (0, 1) \) and \( \log^+ M(s^p, \cdot) \) is locally integrable on \( (0, 1) \), then \( \{M(s^p, \cdot)\}^{1/p} \) is finite and convex.
The convexity property here is replaced by subharmonicity if we integrate only over $\mathbb{R}^{n-m}$ as in §3, (see [6]). With regard to growth theorems, we mention the following result of Armitage [2].

**Theorem 11.** If $s$ is non-negative and subharmonic on $\mathbb{R}^{n-1} \times (0, +\infty)$ and $M(s^*, y) = O(y^{n+p-1})$ as $y \to +\infty$, then $M(s^*, y)$ decreases to 0 as $y \to +\infty$.

Thus, with $s$ replaced by the “strongly subharmonic” function $s^*$, we can weaken the hypotheses of Theorem 7 and strengthen the conclusion.

**Acknowledgement.** I am grateful to David Armitage for his assistance in compiling this account.

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