Note on the Diophantine Equation

\[ x^x y^y = z^z \]

James J. Ward.

In a letter to the Editor of the Irish Times, Dr. Des McHale issued the challenge of finding any solution \((x, y, z)\), with none of \(x, y, z = 1\), of the Diophantine equation

\[ x^x y^y = z^z. \]

This had appeared as a problem in the first Irish Universities Mathematical Olympiad and apparently none of the contestants found a non-trivial solution. The purpose of this note is to indicate a method for generating solutions to this equation.

**Lemma:** Suppose \(X, Y, Z, \varphi\) are natural numbers such that

(i) \(X + Y - Z = 1\) and

(ii) \(\varphi \geq 2\) and

(iii) \(\varphi = 2^\varphi/(X^X Y^Y)\);

then \(x = \varphi X, y = \varphi Y, z = \varphi Z\) have the property that

\[ x^x y^y = z^z. \]

**Proof:** Consider \(x^x y^y\): this equals

\[ (\varphi X)^\varphi X (\varphi Y)^\varphi Y = \varphi^{\varphi (X+Y)} (X^X Y^Y)^\varphi, \]

On the other hand \(z^z\) equals

\[ \varphi^\varphi (2^Z)^\varphi \]
So \( x^\varphi y^\varphi = z^\varphi \) if and only if
\[
\varphi^{(X+Y)}(X^X Y^Y)^\varphi = \varphi^{(Z^Z)}(Z^Z)^\varphi \\
\iff \varphi^{(X+Y-Z)}(X^X Y^Y)^\varphi = (Z^Z)^\varphi \\
\iff \varphi^{(X^X Y^Y)^\varphi} = (Z^Z)^\varphi \quad \text{since} \quad X + Y - Z = 1 \\
\iff \varphi^{X^X Y^Y} = Z^Z \quad \text{which follows from (iii)}.
\]

Now suppose \( X = 2^{\alpha} \) and \( Y = p^\beta \) where \( p \) is odd and \( \alpha, \beta \geq 1 \). Consider \((2^\alpha - p^\beta)^2\). This is
\[
2^{2\alpha} + p^{2\beta} - 2^{\alpha+1} p^\beta = X + Y - Z
\]
say for \( Z = 2^{\alpha+1} p^\beta \). In this case one has \( X + Y - Z = 1 \) if and only if
\[
(2^\alpha - p^\beta) = \pm 1. \tag{*}
\]
Subject to this we want to ensure that \( Z^Z / X^X Y^Y \) is an integer \( \geq 2 \). Now \( \varphi := Z^Z / X^X Y^Y \) in this case can be written as
\[
\varphi = \frac{2^{(\alpha+1)(2^{\alpha+1}p^\beta)} \cdot p^{(2^{\alpha+1}p^\beta)}}{2^{2\alpha+1} \cdot p^{2\beta p^\beta}}.
\]
The power of 2 in \( \varphi \) equals
\[
(\alpha + 1)(2^{\alpha+1} \cdot p^\beta) - \alpha 2^{2\alpha+1} \tag{1}
\]
The power of \( p \) in \( \varphi \) equals
\[
\beta 2^{\alpha+1} p^\beta - 2\beta p^\beta \tag{2}
\]
Equation (2) is \( \geq 0 \) \( \iff \) \( 2^\alpha - p^\beta \geq 0 \) (on dividing (2) by \( 2\beta p^\beta \)). Therefore in (\( * \)) we shall require \( 2^\alpha - p^\beta = +1 \). Inserting this condition into (1) we get
\[
(\alpha + 1)(2^{\alpha+1}(2^\alpha - 1)) - \alpha 2^{2\alpha+1} \tag{3}
\]
Dividing by \( 2^{\alpha+1} \), for (1) to be non-negative we require
\[
(\alpha + 1)(2^\alpha - 1) - \alpha 2^\alpha \geq 0 \\
\iff 2^\alpha - 1 \geq \alpha.
\]

However this holds for all \( \alpha \geq 1 \). Using \( 2^\alpha - p^\beta = 1 \), (2) becomes \( 2\beta p^\beta \) and (3) simplifies to \( 2^{\alpha+1}(p^\beta - \alpha) \). From this, it is apparent that \( \varphi \geq 2 \).

Since \( 2^1 - p = 1 \) implies \( \varphi = 1 \) we shall now assume \( \alpha \geq 2, \beta \geq 1 \).

**Examples:**
(i) Choose \( \alpha = 2 \), then \( 2^2 - p^\beta = 1 \) gives \( p = 3, \beta = 1 \).

Then \( X = 2^{2^2} = 16, Y = 3^{2^1} = 9 \) and \( Z = 2^{2^{2^2}+1} p^\beta = 2^{2^2} \cdot 3^2 = 24 \). Note that \( X + Y - Z = 1 \).

Letting \( \varphi = Z^Z / X^X Y^Y \), the power of 2 in \( \varphi \) equals \( 2^{2^{2^2}+1} (p^\beta - 2) \) which in this example is \( 8 \). The power of \( p \) in \( \varphi \) equals \( 2\beta p^\beta \) which equals \( 2 \cdot 1 \cdot 3 = 6 \), so
\[
\varphi = 2^8 3^6.
\]
Hence
\[
x = 2^{12} \cdot 3^6, y = 2^8 \cdot 3^6 \quad \text{and} \quad z = 2^{11} \cdot 3^7
\]
is a solution of the Diophantine equation
\[
x^\varphi y^\varphi = z^\varphi.
\]

(ii) Choose any power of 2, say \( 2^k \) where \( k \geq 2 \). Then \( p = 2^k - 1 \) is always odd and clearly \( 2^k - p = 1 \). So we can take
\[
X = 2^{2^k}, Y = p^2 \quad \text{and} \quad Z = 2^{2^k+1} p
\]
and compute \( \varphi \) as before. For instance if we take \( 2^4 \) then \( p = 15 \) and we get
\[
X = 2^{2^4}, Y = 225 \quad \text{and} \quad Z = 480 \quad \varphi = \frac{2^{322}(15)^{30}}{etc.}
\]

Department of Mathematics
University College
Galway