Editor: Phil Rippon

My first problem this time is a remarkable result about spherical triangles, which was apparently first proved by a computer!

26.1 Prove that if the area of a spherical triangle is one quarter of the area of the sphere, then the midpoints of its sides form an equilateral spherical triangle with angles of 90°.

A discussion of the algebraic verification of theorems in geometry and a BASIC program to prove this result can be found in the article *A new method of automated theorem proving* by Yang Lu ("The mathematical revolution inspired by computing" edited by J. H. Johnson and M. J. Loomes, Oxford University Press, 1991). It might be argued that a computer program cannot tell you why the result holds, in the way that a conventional proof should do.

Next is a problem that I heard recently from my school mathematics teacher, Mr Harold Taylor. It was inspired, he says, by a discussion of the relative sizes of bifurcating blood vessels, given on a television science programme.

26.2 A pipe from A is split into two smaller pipes at P to supply B and C. Given that the pipe AP costs $k$ times as much per unit length as do PB and PC, determine the position of P so that the total cost is a minimum.

Now, here is some recent news about one of my older problems. Problem 11.2 asked you to prove that the sequence

$$a_{n+2} = |a_{n+1}| - a_n, \quad n = 0, 1, 2, \ldots,$$

where $a_0, a_1 \in \mathbb{R}$, is always periodic with period 9. Just before last Christmas, Alan Beardon noticed a connection between this problem and the theory of Hecke groups (certain discrete groups of Möbius transformations). This insight has led to a number of extensions and related results, now being written up by Alan, Shaun Bullett and myself; for example, the sequence

$$a_{n+2} = 2 \cos(\pi/p)|a_{n+1}| - a_n, \quad n = 0, 1, 2, \ldots,$$

where $p \in \{2, 3, \ldots\}$ and $a_0, a_1 \in \mathbb{R}$ is always periodic with period $p^2$. For $p = 3$, we obtain the sequence (1).

Finally, here is a solution to problem 23.2 which appeared in issue 23.

23.2 Let $s(n)$ denote the number of triples $(a, b, c)$, where $a, b, c$ are positive integers with

$$a + b + c = n, \quad a \leq b \leq c \quad \text{and} \quad a + b > c.$$

Determine a simple formula for $s(n)$.

The motivation behind this counting problem is that each such triple $(a, b, c)$ determines an integer-sided triangle, which is unique up to congruence. We denote the set of all such triples by $S_n = \{(a, b, c) : a, b, c \in \mathbb{N}, \ a + b + c = n, \ a \leq b \leq c, \ a + b > c\}$, and record below the elements of $S_n$, for $0 \leq n \leq 10$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$S_n$</th>
<th>$s(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>(1,1,1)</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>(1,2,2)</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>(2,2,2)</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>(2,2,3),(1,3,3)</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>(2,3,3)</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>(3,3,3),(2,3,4),(1,4,4)</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>(3,3,4),(2,4,4)</td>
<td>2</td>
</tr>
</tbody>
</table>
On the basis of this table, it is clear that \( s(n) \) is somewhat irregular, but it appears that \( s(n+3) = s(n) \) if \( n \) is odd. Indeed, it is clear that if \( (a, b, c) \in S_n \), then \( (a+1, b+1, c+1) \in S_{n+3} \) and the reverse implication holds also if \( n \) is odd (because if \( a+b+c \) is odd, then \( a+b-c \) is odd, so that

\[
(a+1)+(b+1) > c+1 \implies a+b > c-1
\]

\[
\quad \implies a+b > c.
\]

Thus

\[
s(2m+1) = s(2m+4), \quad m = 0, 1, 2, \ldots,
\]

and so the problem reduces to the evaluation of \( s(2m) \), \( m = 0, 1, 2, \ldots \). To do this, we first prove that

\[
s(2m+3) = s(2m) + \left\lfloor \frac{1}{2}(m+2) \right\rfloor,
\]

where \( \lfloor x \rfloor \) denotes the integer part of \( x \). For, if \( (a+1, b+1, c+1) \in S_{2m+3} \) but \( (a, b, c) \notin S_{2m} \), then

\[
a+1+b+1 > c+1 \quad \text{and} \quad a+b \leq c,
\]

so that \( a+b = c \). Hence

\[
2m = a+b+c \iff a+b = m \iff (a+1)+(b+1) = m+2.
\]

Now, there are \( \left\lfloor \frac{1}{2}(m+2) \right\rfloor \) pairs \( (a+1, b+1) \) with \( a+1 \leq b+1 \) and \( (a+1)+(b+1) = m+2 \), so that (3) follows.

Combining (2) and (3) gives, for \( m = 0, 1, 2, \ldots \),

\[
s(2m+6) = s(2m) + \left\lfloor \frac{1}{2}(m+2) \right\rfloor
\]

and hence

\[
s(2m+12) = s(2m) + \left\lfloor \frac{1}{2}(m+2) \right\rfloor + \left\lfloor \frac{1}{2}(m+5) \right\rfloor
\]

\[
= s(2m) + m + 3.
\]

Applying this recurrence relation repeatedly, we find that if \( 2m = 12k+2i \), where \( i = 0, 1, 2, 3, 4, 5 \) and \( k = 0, 1, 2, \ldots \), then

\[
s(2m) = s(2i) + (i+3) + (i+9) + \cdots + (i+6(k-1)+3)
\]

\[
= s(2i) + 6(k-1)k/2 + k(i+3)
\]

\[
= s(2i) + k(3k+i)
\]

\[
= s(2i) + (m^2-i^2)/12,
\]

since \( k = (m-i)/6 \). Thus, in this case,

\[
s(2m) - m^2/12 = s(2i) - i^2/12.
\]

On examining the table above, we find that, for \( i = 0, 1, 2, 3, 4, 5 \),

\[
s(2i) \text{ is the nearest integer to } i^2/12.
\]

Hence, for \( m = 0, 1, 2, \ldots \),

\[
s(2m) \text{ is the nearest integer to } m^2/12,
\]

so that, by (2),

\[
s(2m+1) \text{ is the nearest integer to } (m+2)^2/12.
\]

To get some feeling for this formula, it is a nice exercise to find the first value of \( n \) for which \( s(n) > n \).

Phil Rippon,
Faculty of Mathematics,
The Open University,
Milton Keynes MK7 6AA,
UK.