Polynomials and Holomorphic Functions
on Infinite Dimensional Spaces
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TRACE-ZERO MATRICES AND
POLYNOMIAL COMMUTATORS

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Let $\mathbb{F}$ denote a field and $M_n(\mathbb{F})$ the algebra of $n \times n$ matrices over the field $\mathbb{F}$. If $X \in M_n(\mathbb{F})$, $\text{tr}(X)$ will denote the trace of the matrix $X$. A well known result of Albert and Muckenhoupt [1] states that if $\text{tr}(X) = 0$ then there exist matrices $A, B \in M_n(\mathbb{F})$ such that $X$ is the commutator of $A$ and $B$,

$$X = [A, B] = AB - BA.$$

Let $p$ denote a polynomial in $\mathbb{F}[x]$ of degree greater than or equal to one. The Polynomial Commutator of $A$ and $B$ relative to $p$ is defined to be

$$p[A, B] = p(AB) - p(BA).$$

It is easy to check, by examining the eigenvalues, that $\text{tr}(p[A, B])$ is always zero. The Albert-Muckenhoupt result states that if $X \in M_n(\mathbb{F})$ with $\text{tr}(X) = 0$ then, for $p(x) = x$,

$$X = p[A, B],$$

for some $A, B \in M_n(\mathbb{F})$. We show that, if the field $\mathbb{F}$ has characteristic zero the Albert-Muckenhoupt result may be extended to general polynomials of degree greater than, or equal to, one.

**Theorem.** Let $\mathbb{F}$ be a field of characteristic zero and let $p \in \mathbb{F}[x]$ have degree greater than or equal to one. If $X \in M_n(\mathbb{F})$ is of trace zero then there exist matrices $A, B \in M_n(\mathbb{F})$ such that

$$X = p[A, B].$$

First we prove the following elementary
Lemma. If $F$ is a field of characteristic zero and $X \in M_n(F)$ is of trace zero then we can choose a basis of $F^n$ such that, relative to this basis, $X$ has zeros on its main diagonal.

Proof: Since $\text{tr}(X) = 0$ and $F$ is of characteristic zero, $X$ is not a scalar matrix. Thus there exists a vector $v \in F^n$ such that $v$ and $Xv$ are linearly independent.

Set $v_1 = v$, $v_2 = Xv$ and extend to a basis $v_1, v_2, \ldots, v_n$ of $F^n$. Relative to this basis

$$X = [x_{ij}]_{n \times n} \quad \text{with} \quad x_{11} = 0.$$ Further the matrix

$$Y = [x_{ij}]_{(n-1) \times (n-1)} \quad (2 \leq i, j \leq n)$$

has trace zero and the proof may be completed by induction.

Proof of Theorem: Since $\text{tr}(X) = 0$ we may take

$$X = [x_{ij}]_{n \times n} \quad \text{with} \quad x_{ii} = 0 \quad (1 \leq i \leq n).$$

Now

$$X = L - U,$$

where $L$ is a lower triangular matrix, $U$ is an upper triangular matrix and both have zeros on the main diagonal.

Let $D$ be the diagonal matrix

$$D = \text{diag}(d_1, \ldots, d_n),$$

then $p(D)$ is the diagonal matrix

$$p(D) = \text{diag}(p(d_1), \ldots, p(d_n)),$$

and since $F$ is an infinite field and the degree of $p$ is greater than, or equal to, one, we may choose the $d_i$ so that the $p(d_i)$ are distinct ($1 \leq i \leq n$). Then

$$X = (L + p(D)) - (U + p(D))$$

$$= L_1 - U_1$$

where $L_1 = L + p(D)$ is lower triangular and $U_1 = U + p(D)$ is upper triangular. The diagonal entries of $L_1$ and $U_1$ are $p(d_i), (1 \leq i \leq n)$, and since these have been chosen distinct, the matrices $L_1, U_1$ and $p(D)$ are all similar. Thus there exist invertible $S, T \in M_n(F)$ so that

$$X = S^{-1}p(D)S - T^{-1}p(D)T,$$

$$= p(S^{-1}DS) - p(T^{-1}DT).$$

Taking $A = S^{-1}T$ and $B = T^{-1}DS$ gives

$$X = p(AB) - p(BA) = p[A, B]$$

which completes the proof.

Remarks

1. The result does not remain true if the restriction that $F$ is of characteristic zero be dropped.
2. It would be interesting to investigate the latitude in equation (*) for fixed $X$ and $p$, in the possible choices of $A$ and $B$.

Reference


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