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# APPLICATION OF HOARE'S THEOREM TO SYMMETRIES OF RIEMANN SURFACES

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**Abstract.** Let $X$ be a compact Riemann surface of genus $g > 1$. A symmetry $S$ of $X$ is an anticonformal involution. We write $|S|$ for the number of connected components of the fixed points set of $S$. Suppose that $X$ admits two distinct symmetries $S_1$ and $S_2$; then we find a bound for $|S_1| + |S_2|$ in terms of the genus of $X$ and the order of $S_1S_2$. We discuss circumstances in which the bound is attained, showing that this occurs only for hyperelliptic surfaces. In this way we generalize a theorem of S.M. Natanzon.

## 1. Introduction

Let $X$ be a compact Riemann surface of genus $g > 1$. A *symmetry* $S$ of $X$ is an anticonformal involution $S\colon X \to X$ and a Riemann surface that admits a symmetry is called *symmetric*. By Harnack's theorem [1, 5, 9], the fixed-point set of $S$ is either empty or consists of $k \le g + 1$ disjoint simple closed curves or, as we shall call them, *mirrors*. We write $|S|$ for the number of mirrors of $S$, so that $|S|$ is the number of components of the fixed-point set of $S$.

Suppose that $X$ admits two distinct symmetries $S_1$, $S_2$. If $|S_1| = |S_2| = g+1$ then by a theorem of Natanzon [8], $S_1$ and $S_2$ commute and $X$ is hyperelliptic. Usually, however, the total number of mirrors of $S_1$ and $S_2$ is much less than $2g + 2$. In Theorem 3 we find a sharp upper bound for $|S_1| + |S_2|$ in terms of the genus of $X$ and the order of $S_1S_2$. In particular, if $S_1$ and $S_2$ do not commute then $|S_1| + |S_2| \le g + 2$ (Corollary 3).

Theorem 3 follows from a recent theorem of Hoare [3] on subgroups of NEC groups. Using this theorem we find in Section 5 a graphical technique that allows us to determine $|S_1|$ and $|S_2|$ in terms of the signature of a certain NEC group. We then find, in Theorem 2, a formula for $|S_1| + |S_2|$ involving this signature.

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Finally, in Section 7, we discuss circumstances in which the bound of Theorem 3 is attained. We show that this is the case only for hyperelliptic surfaces, a result which extends Natanzon's original theorem.

## 2. NEC groups

A non-Euclidean crystallographic (NEC) group is a discrete group of isometries of the hyperbolic plane $\mathbf{H}$. As usual, we can choose for our model of $\mathbf{H}$, the upper half-plane with the Poincaré metric. Then every isometry is given by a Möbius or anti-Möbius transformation (the latter being a Möbius transformation composed with $z \to -\bar{z}$). We shall assume that an NEC group has compact quotient space. If $\Delta$ is such a group then its algebraic and geometric structure is determined by its *signature*

$$\sigma(\Delta) = \big(h; \pm; [m_1, \ldots, m_r]; \big\{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\big\}\big). \tag{1}$$

The quotient space $\mathbf{H}/\Delta$ is then a surface, possibly with boundary, and in the signature $h$ is the genus of $\mathbf{H}/\Delta$, $k$ is the number of its boundary components and $+$ or $-$ is used according to whether the surface is orientable or not. The integers $m_1, \ldots, m_r$ are the *proper periods* of $\Delta$ and represent the branching over interior points of $\mathbf{H}/\Delta$ in the natural projection $p\colon \mathbf{H} \to \mathbf{H}/\Delta$. The $k$ brackets $(n_{i1}, \ldots, n_{is_i})$ are the period cycles and represent the branching over the $i^{\text{th}}$ hole. The integers $n_{ij}$ are the *link periods*. The maximal finite subgroups of $\Delta$ are either cyclic of order $m_i$ $(i = 1, \ldots, r)$ or dihedral of order $2n_{ij}$ $(i = 1, \ldots, k$, $j = 1, \ldots, s_i)$ and each period represents a conjugacy class of such subgroups.

Associated to the signature (1) we have a presentation of the group $\Delta$ and a formula for the area of a fundamental domain for $\Delta$. If $\sigma(\Delta)$ has a $+$ sign then $\Delta$ has generators

$$\begin{array}{rl} x_1, \ldots, x_r & \text{(elliptic elements)} \\ c_{10}, \ldots, c_{1s_1}, \ldots, c_{k0}, \ldots, c_{ks_k} & \text{(reflections)} \\ e_1, \ldots, e_k & \text{(orientation preserving elements)} \\ a_1, b_1, \ldots, a_h, b_h & \text{(hyperbolic elements)} \end{array}$$

and relations

$$\begin{aligned} &x_i^{m_i} = 1 \qquad (i = 1, \ldots, r) \\ &c_{i,j-1}^2 = c_{ij}^2 = (c_{i,j-1}c_{ij})^{n_{ij}} = 1, \qquad i = 1, \ldots, k,\ j = 1, \ldots, s_i \\ &e_i c_{i0} e_i^{-1} = c_{is_i} \qquad (i = 1, \ldots, k) \\ &x_1 x_2 \cdots x_r e_1 e_2 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_h b_h a_h^{-1} b_h^{-1} = 1. \end{aligned}$$

If $\sigma(\Delta)$ has a minus sign then we just replace the hyperbolic generators $a_i$, $b_i$ by glide reflection generators $a_1, \ldots, a_h$ and replace the last relation by

$$x_1 x_2 \cdots x_r e_1 e_2 \cdots e_k a_1^2 a_2^2 \cdots a_h^2 = 1.$$

The hyperbolic area of a fundamental domain for $\Delta$ is given by

$$(2)\qquad \mu(\Delta) = 2\pi\Big(\varepsilon h - 2 + \sum_{i=1}^{r}\Big(1 - \frac{1}{m_i}\Big) + k + \tfrac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\Big(1 - \frac{1}{n_{ij}}\Big)\Big),$$

(see [11]) where $\varepsilon = 2$ if there is a $+$ sign and $\varepsilon = 1$ if there is a $-$ sign. If $\Lambda < \Delta$ is a subgroup of finite index then

$$(3)\qquad |\Delta : \Lambda| = \mu(\Lambda)/\mu(\Delta).$$

For more details about signatures see ([1, 6, 11]).

Now let $X$ be a compact Riemann surface of genus $g > 1$. Then there is a Fuchsian surface group $\Gamma$ (i.e. an NEC group of signature $(g; +; [\,]\{\,\})$ such that $X = \mathbf{H}/\Gamma$. Let $G$ be a group of automorphisms of $X$ containing a symmetry $S$. Let $\Delta$ be the group generated by all the liftings to $\mathbf{H}$ of the elements of $G$, then $\Delta$ is an NEC group and there is a smooth homomorphism $\theta$: $\Delta \to G$ whose kernel is $\Gamma$. ($\theta$ is smooth means that $\theta$ maps finite subgroups of $\Delta$ isomorphically into $G$.)

Let $L = \langle S\rangle$, (the group generated by $S$) and $\Lambda = \theta^{-1}(L)$. We then have

$$(4)\qquad X/L = \mathbf{H}/\Gamma/\Lambda/\Gamma = \mathbf{H}/\Lambda.$$

For any NEC group $\Phi$, we let $\Phi^+$ (the *canonical Fuchsian group* of $\Phi$) denote the subgroup of index one or two consisting of the orientation-preserving elements of $\Phi$. As $\Gamma$ contains only orientation-preserving transformations and is torsion-free, $\Lambda$ contains orientation-reversing elements. As $|\Lambda : \Gamma| = 2$, $\Lambda^+ = \Gamma$ so that $\Lambda$ has no elliptic elements. Therefore $\Lambda$ has a signature of the form

$$(5)\qquad (h_0; \pm; [\,]\{(\,)^k\})$$

where the notation signifies that there are $k$ empty period cycles. Thus $k$ is the number of period cycles of $\Lambda$ which is the number of mirrors of $S$. Conversely it is clear that if there is an NEC group $\Lambda$ with signature (5) and containing $\Gamma$ as subgroup of index two then $\Lambda$ defines a symmetry on $X$ with exactly $k$ mirrors. This shows that the number of mirrors of a symmetry is an algebraic invariant of an NEC group.

## 3. Dihedral group actions

We now suppose that $X$ admits two distinct symmetries $S_1$, $S_2$. We let $G = \langle S_1, S_2\rangle$ so that $G$ is isomorphic to the dihedral group $D_n$ of order $2n$. We also let $\Delta$ be the lift of $G$ to $\mathbf{H}$, $L_i = \langle S_i\rangle$ and $\Lambda_i = \theta^{-1}(L_i)$, ($i = 1, 2$), where $\theta$: $\Delta \to G$ is the homomorphism of Section 2. We have a subgroup diagram

$$\begin{tikzcd}[column sep=small] & \Delta \arrow[dl, dash] \arrow[dr, dash] & \\ \Lambda_1 \arrow[dr, dash] & & \Lambda_2 \arrow[dl, dash] \\ & \Gamma & \end{tikzcd}$$

where $\Lambda_i$ has signature $\big(h_i;\pm;[\,]\{(\ )^{k_i}\}\big)$.

Now given the signature of an NEC group, Hoare's theorem [3] gives us a procedure for calculating the signature of a subgroup, given the permutation representation of the group on the cosets. We will use the techniques of this theorem to compute $k_1$, $k_2$ from the signature of $\Delta$. We find that the algebra in Hoare's results gives us fairly precise information about $k_1+k_2=|S_1|+|S_2|$.

We shall find the following simple lemma, also used by Hoare, useful.

**Lemma 1.** *Let $D_n$ act by right multiplication on the $n$ $L_i$-cosets $(i=1,2)$. If $n$ is even then $S_1$ fixes exactly two $L_1$-cosets and no $L_2$-cosets while if $n$ is odd then $S_1$ fixes exactly one $L_1$-coset and one $L_2$-coset.*

*Proof.* Let $Q=S_1S_2$. Then the $n$ $L_1$-cosets are $L_1Q^r$ $(r=0,\dots,n-1)$ and the action of $S_1$ is $L_1Q^r\to L_1Q^rS_1$. If $n$ is even the only fixed cosets correspond to $r=0,n/2$ while if $n$ is odd the only fixed cosets correspond to $r=0$. If in the action on the $L_2$-cosets, $S_1$ fixes $L_2Q^r$ then $Q^rS_1Q^{-r}=S_2$ and thus $Q^{2r+1}=I$. Hence there are no fixed cosets if $n$ is even and one if $n$ is odd.

Another way of stating this lemma is as follows:

**Lemma 1$'$.** *Let $D_n$ act by right multiplication on the $n$ $L_1$-cosets. If $n$ is even then $S_1$ fixes exactly two $L_1$-cosets and $S_2$ fixes no $L_1$-cosets. If $n$ is odd then both $S_1$ and $S_2$ fix exactly one $L_1$-coset.*

We now explain how Hoare's ideas apply. Let $c\in\Delta$ be a reflection so that $\theta$ maps $c$ to a conjugate of $S_1$ or $S_2$ in $G\cong D_n$. The action of $c$ on the $\Lambda_i$-cosets is the same as the action of $\theta(c)$ on the $L_i$-cosets. Suppose that $d$ is another reflection in $\Delta$ and that $cd$ has finite order $\mu$. Then $cd$, in its action on the $\Lambda_i$-cosets, gives a product of disjoint $\mu$-cycles. (Otherwise some power of $cd$, not equal to the identity is an elliptic element of $\Gamma$ which is impossible as $\Gamma$ is a surface group.) From Lemma 1, each of $c$ and $d$ fixes two cosets or no cosets if $\mu$ is even and each fixes 1 coset if $\mu$ is odd. By Theorem 1, II(ii) of [3], (or an extension of Lemma 1) each cycle of $cd$ contains either two or none of these fixed cosets.

Suppose that $c$ fixes a coset $\Lambda_1\alpha$. Then $c_\alpha=\alpha c\alpha^{-1}$ is a reflection in $\Lambda_i$ which is called a reflection induced by $c$ or just an *induced reflection*. Consider a cycle of $cd$ which contains $\alpha$ (we now identify $\alpha$ and $\Lambda_i\alpha$). By the above discussion this cycle contains another fixed point $\beta$ of $c$, if $\mu$ is even, or a fixed point $\beta$ of $d$, if $\mu$ is odd. In the first case we say that $c_\alpha$ and $c_\beta$ are *linked* and in the second case we say that $c_\alpha$ and $d_\beta$ are linked and write $c_\alpha \sim c_\beta$ in the first case and $c_\alpha \sim d_\beta$ in the second case. Once all the links are known we put them together to form *chains* and then each chain (by Hoare's results) gives a period cycle in $\Lambda_i$.

*Note.* In general $c_\alpha \sim c_\beta$ (or $c_\alpha \sim d_\beta$) means that $c_\alpha c_\beta$ (or $c_\alpha \sim d_\beta$) has finite order and so gives a link period in a period cycle of $\Lambda_i$. In our cases the period cycles of the subgroups $\Lambda_i$ are empty so that $c_\alpha \sim c_\beta$ means that $(c_\alpha c_\beta)^1 = 1$ or $c_\alpha = c_\beta$ (or $c_\alpha \sim d_\beta$ in the other case).

Before we consider the general situation it might be helpful to look at some examples.

## 4. Examples

*Example* 1. Let $n = 12$ and $\Delta$ have signature

$$\big(0; +; [\,]; \{(2,4,4,6,3)\}\big),$$

and presentation

$$\big\langle c_i,\ (1 \le i \le 5) \big| c_i^2 = 1\ (1 \le i \le 5),\\ (c_1c_2)^2 = (c_2c_3)^4 = (c_3c_4)^4 = (c_4c_5)^6 = (c_5c_1)^3 = 1\big\rangle.$$

Consider the following homomorphism

$$\theta\colon \Delta \to D_{12} = \big\langle S_1, Q \big| S_1^2 = (S_1Q)^2 = Q^{12} = 1\big\rangle;$$

$$\theta(c_1) = S_1, \quad \theta(c_2) = S_1Q^6, \quad \theta(c_3) = S_1Q^9, \quad \theta(c_4) = S_1Q^6, \quad \theta(c_5) = S_1Q^8.$$

We have the following decomposition of $D_{12}$ as a union of $L_1$-cosets ($L_1 = \langle S_1\rangle$):

$$D_{12} = \bigcup_{i=0}^{11} L_1Q^i$$

and letting $i$ denote the coset $L_1Q^i$ we obtain the following permutation representation of $D_{12}$ on the right $L_1$-cosets,

$$\begin{array}{lllllllll}
c_1 \to & (0) & (6) & (1\ \ 11) & (2\ \ 10) & (3\ \ 9) & (4\ \ 8) & (5\ \ 7) \\
c_2 \to & (3) & (9) & (1\ \ 5) & (2\ \ 4) & (7\ \ 11) & (8\ \ 10) & (0\ \ 6) \\
c_3 \to & (0\ \ 9) & & (1\ \ 8) & (2\ \ 7) & (3\ \ 6) & (4\ \ 5) & (10\ \ 11) \\
c_4 \to & (3) & (9) & (1\ \ 5) & (2\ \ 4) & (7\ \ 11) & (8\ \ 10) & (0\ \ 6) \\
c_5 \to & (4) & (10) & (1\ \ 7) & (2\ \ 6) & (3\ \ 5) & (0\ \ 8) & (11\ \ 9).
\end{array}$$

The induced reflections are

$$c_{1,0}, \quad c_{1,6}, \quad c_{2,3}, \quad c_{2,9}, \quad c_{4,3}, \quad c_{4,9}, \quad c_{5,4}, \quad c_{5,10}.$$

We now form the products

$$\begin{array}{ll}
c_1c_2 \to & (0 \;\; 6)\;(1 \;\; 7)\;(2 \;\; 8)\;(3 \;\; 9)\;(4 \;\; 10)\;(5 \;\; 11) \\
c_2c_3 \to & (0 \;\; 3 \;\; 6 \;\; 9)\;(1 \;\; 4 \;\; 7 \;\; 10)\;(2 \;\; 5 \;\; 8 \;\; 11) \\
c_3c_4 \to & (0 \;\; 9 \;\; 6 \;\; 3)\;(1 \;\; 10 \;\; 7 \;\; 4)\;(2 \;\; 11 \;\; 8 \;\; 5) \\
c_4c_5 \to & (1 \;\; 3 \;\; 5 \;\; 7 \;\; 9 \;\; 11)\;(2 \;\; 4 \;\; 6 \;\; 8 \;\; 10 \;\; 0) \\
c_5c_1 \to & (1 \;\; 5 \;\; 9)\;(2 \;\; 6 \;\; 10)\;(3 \;\; 7 \;\; 11)\;(4 \;\; 8 \;\; 0).
\end{array}$$

As fixed points $0, 6$ of $c_1$ belong to the same cycle of $c_1c_2$ we have a link $c_{1,0} \sim c_{1,6}$. Similarly, we have links $c_{2,3} \sim c_{2,9}$, (from $c_2c_3$), $c_{4,3} \sim c_{4,9}$ (from $c_3c_4$), $c_{4,3} \sim c_{4,9}$ (from $c_4c_5$), $c_{5,4} \sim c_{5,10}$ (from $c_4c_5$), $c_{5,4} \sim c_{1,0}$ (from $c_5c_1$), $c_{5,10} \sim c_{1,6}$ (from $c_5c_1$). We then get the chains

$$\begin{array}{l}
c_{1,0} \sim c_{1,6} \sim c_{5,10} \sim c_{5,4} \sim c_{1,0} \\
c_{2,3} \sim c_{2,9} \sim c_{2,3} \\
c_{4,3} \sim c_{4,9} \sim c_{4,3}.
\end{array}$$

As there are three chains, Hoare's theorem implies that $\Lambda_1$ has three period cycles and hence $|S_1| = 3$. Similarly, we have the action of the generators on the $L_2$-cosets,

$$\begin{array}{ll}
c_1 \to & (0 \;\; 11)\;(1 \;\; 10)\;(2 \;\; 9)\;(3 \;\; 8)\;(4 \;\; 7)\;(5 \;\; 6) \\
c_2 \to & (0 \;\; 5)\;(1 \;\; 4)\;(2 \;\; 3)\;(6 \;\; 11)\;(7 \;\; 10)\;(8 \;\; 9) \\
c_3 \to & (4)\;(10)\;(0 \;\; 8)\;(1 \;\; 7)\;(2 \;\; 6)\;(3 \;\; 5)\;(9 \;\; 11) \\
c_4 \to & (0 \;\; 5)\;(1 \;\; 4)\;(2 \;\; 3)\;(6 \;\; 11)\;(7 \;\; 10)\;(8 \;\; 9) \\
c_5 \to & (0 \;\; 7)\;(1 \;\; 6)\;(2 \;\; 5)\;(3 \;\; 4)\;(8 \;\; 11)\;(9 \;\; 10),
\end{array}$$

and notice how Lemma 1 applies: e.g. $c_1$ fixes 2 $L_1$-cosets and 0 $L_2$-cosets, $c_3$ fixes 0 $L_1$-cosets and 2 $L_1$-cosets. The induced reflections are $c_{3,4}$ and $c_{3,10}$ and from the products $c_3c_4$ and $c_4c_5$ we find the single chain $c_{3,4} \sim c_{3,10} \sim c_{3,4}$. Thus $|S_2| = 1$. This example suggests that Lemma 1 should restrict the size of $|S_1| + |S_2|$. As we shall see in Theorem 1 this is indeed the case, but before then we consider another example which illustrates other aspects of Hoare's techniques.

*Example* 2. Let $n = 3$ and $\Delta$ have signature

$$\big(0; +; [3]; \{(3, 3, 3)\}\big),$$

and presentation

$$\big\langle e, x, c_0, c_1, c_2, c_3 \big| c_i^2 = 1 \;(0 \leq i \leq 3),$$
$$x^3 = ex = (c_0c_1)^3 = (c_1c_2)^3 = (c_2c_3)^3 = ec_0e^{-1}c_3 = 1\big\rangle.$$

Consider the following homomorphism

$$\theta\colon \Delta \to D_3 = \langle S_1, Q \,|\, S_1^2 = (S_1Q)^2 = Q^3 = 1\rangle;$$

$$\theta(c_0) = S_1, \quad \theta(c_1) = S_1Q, \quad \theta(c_2) = S_1, \quad \theta(c_3) = S_1Q, \quad \theta(e) = Q.$$

With a similar notation to Example 1,

$$D_3 = \bigcup_{i=0}^{2} L_1 Q^i$$

and we have the following permutation representation on the cosets

$$\begin{array}{rccc}
c_0 \to & (0) & (1 & 2) \\
c_1 \to & (0 & 1) & (2) \\
c_2 \to & (0) & (1 & 2) \\
c_3 \to & (0 & 1) & (2) \\
e \to & (0 & 1 & 2)
\end{array}$$

giving induced reflections $c_{0,0}$, $c_{1,2}$, $c_{2,0}$, $c_{3,2}$ on $\Lambda_1$. As

$$\begin{array}{rccc}
c_0c_1 \to & (0 & 1 & 2), \\
c_1c_2 \to & (0 & 2 & 1), \\
c_2c_3 \to & (0 & 1 & 2), \\
(ec_0e^{-1})c_3 \to & (0) & (1) & (2),
\end{array}$$

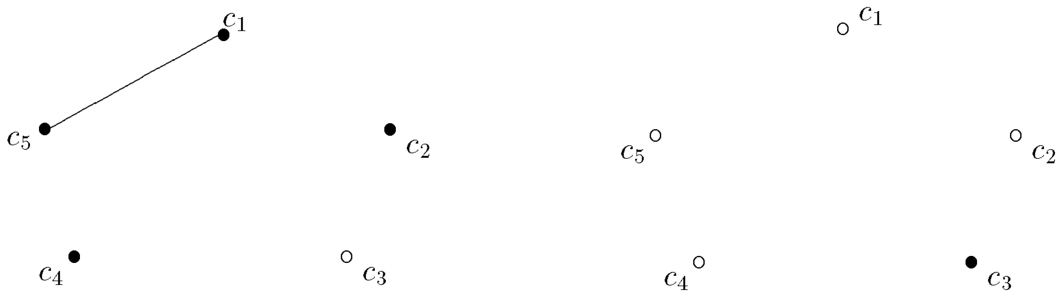
we get links $c_{0,0} \sim c_{1,2}$, $c_{1,2} \sim c_{2,0}$, $c_{2,0} \sim c_{3,2}$, from the first three products. In the final product $c_0$ fixes $0$, $ec_0e^{-1}$ fixes $2$ and $c_3$ also fixes $2$. By Theorem 1 (II(ii)) and Example 1 of [3], we have a link $c_{3,2} \sim c_{0,0}$. This gives the single chain $c_{0,0} \sim c_{1,2} \sim c_{2,0} \sim c_{3,2} \sim c_{0,0}$ and thus $|S_1| = 1$. As $S_2$ is conjugate to $S_1$ in $D_3$, $|S_2| = 1$ as well. As we shall see, this case is typical of the cases when $n$, and hence all the periods of $\Delta$ are odd.

## 5. A graphical technique

We now describe a method that enables us to perform the computations in Section 4 fairly automatically. The examples there show that the number of mirrors depend on the parity of $n$, the parities of the link periods and the parities of the integers $i$ in $S_1Q^i$. We suppose that $n$ is even. If $\theta(c_j) = S_1Q^i$ with $i$ even then $c_j$ gives a permutation of the cosets with two fixed points $\alpha$, $\beta$; if $i$ is odd then there are no fixed points. If the link period $n_j$ is even (recall the relation $(c_jc_{j+1})^{n_j} = 1$) then $\alpha$, $\beta$ lie in the same cycle of the (permutation induced) by $c_jc_{j+1}$, giving a link $c_{j,\alpha} \sim c_{j,\beta}$. (All suffices are modulo $s$ the length of the period cycle.) If $n_{j-1}$ is even then we get a link $c_{j,\beta} \sim c_{j,\alpha}$ giving a chain $c_{j,\alpha} \sim c_{j,\beta} \sim c_{j,\alpha}$.

If $n_j$ is odd then the cycle $c_j c_{j+1}$ containing $\alpha$ also contains a fixed point $\gamma$ of $c_{j+1}$ and the cycle containing $\beta$ contains a fixed point $\delta$ of $c_{j+1}$. We then get links $c_{j,\alpha} \sim c_{j+1,\delta}$ and $c_{j,\beta} \sim c_{j+1,\delta}$. If $n_{n+1}$ is odd we get a link $c_{j+1,\delta} \sim c_{j+2,\varepsilon}$ and so on (and this argument shows that $|S_1| = 1$ if all the link periods of the period cycle are odd). If $n_{j+1}$ is even we get $c_{j+1,\delta} \sim c_{j+1,\gamma} \sim c_{j,\alpha} \sim \cdots$. This process can be illustrated graphically as follows. Suppose that the period cycle has length $s$. Represent each generator $c_j$ by a vertex of an $s$-gon labelled $c_j$. If $\theta(c_j) = S_1 Q^i$, colour the vertex black or white according as to whether $i$ is even or odd respectively. If $c_j$ and $c_{j+1}$ are two black vertices we join them by a black edge if and only if $n_j$ is odd. Then $|S_1|$ is the number of black components. To find $|S_2|$ we just change the colour of the vertices (as $S_2 Q^i = S_1 Q^{i+1}$, changing the parity of the index) and still join black vertices by a black edge if $n_j$ is odd.

*Example.* From Example 1 of Section 4 we obtain the following pictures.




$|S_1| = 3$ $\qquad\qquad$ $|S_2| = 1$

From these graphical ideas we get a simple numerical result.

*Notation.* We let $t = |S_1| + |S_2|$ the total number of fixed curves of $S_1$ and $S_2$.

**Theorem 1.** *With the previous notation, let $n$ be even and suppose that $\Delta$ has a single period cycle and at least one even period. Then $t$ is equal to the number of even link periods in the period cycle.*

*Proof.* Suppose that the period cycle has length $s$. In terms of the above graphs we need to show that if the total number of black edges in one graph is $u$ (this is the number of odd link periods) then the number of black components in both graphs is $s - u$. This is by induction on $u$. If $u = 0$ then the total number of black components is the number of black vertices in both graphs which is the number of vertices in one graph, namely $s$. Suppose the result is true if there are $u$ black edges. Now increase the number of black edges by one. A black edge cannot join vertices of different colours for if $\theta(c_j) = SQ^{i_1}$ and $\theta(c_{j+1}) = SQ^{i_2}$

then $\theta(c_j c_{j+1}) = Q^{i_2 - i_1}$. As $n$ is even and $c_j c_{j+1}$ has odd period, $i_2 \equiv i_1 \bmod 2$. Thus if we introduce another black edge it joins two black vertices in one of the graphs so that the total number of black components is $(s-u)-1 = s-(u+1)$, and the proof follows.

We now extend this result to the case when $\Delta$ is a general NEC group. Let us define a non-empty period cycle to be *odd* if it only contains odd link periods. Then, as we saw near the beginning of Section 5, an odd period cycle induces just one mirror on $U/\Lambda_1$ and one on $U/\Lambda_2$ and so contributes 2 to $t = |S_1| + |S_2|$. Now consider an empty period cycle. The corresponding generators are $c_i$ and $e_i$. These obey the relation $e_i c_i e_i^{-1} = c_i$. With the previous notation let $\theta(c_i) = S_1$. As the centralizer of $S_1$ in $G$ is $\{1\}$ if $n$ is odd and $\{1, Q^{n/2}\}$ if $n$ is even, $\theta(e_i) = 1$ if $n$ is odd while $\theta(e_i) = 1$ or $Q^{n/2}$ if $n$ is even. In the case when $n$ is even the permutation on the $L_1$-cosets induced by $c_i$ is

$$(0)\left(\frac{n}{2}\right)(1\ n-1)(2\ n-2)\dots$$

and the permutation induced by $e_i$ is either

$$(0)(1)\dots(n-1) \qquad \text{if } \theta(e_i) = 1$$

or

$$\left(0\ \frac{n}{2}\right)\left(1\ \frac{n}{2}+1\right)\dots \qquad \text{if } \theta(e_i) = Q^{n/2}.$$

The induced reflections are $c_{i,0}$ and $c_{i,n/2}$. If $\theta(e_2) = 1$ we get two chains $c_{i,0} \sim c_{i,0}$ and $c_{i,n/2} \sim c_{i,n/2}$ and if $\theta(e_i) = Q^{n/2}$ we get a single chain $c_{i,0} \sim c_{i,n/2} \sim c_{i,0}$. (Compare Example 2 in Section 4.) Thus if $e_i \in \Gamma$ then every empty period cycle gives rise to two mirrors of $S_1$ (and none of $S_2$ by Lemma 1) and so contributes 2 to $t$, whilst if $e \in \Gamma$ then every empty period cycle gives rise to one mirror of $S_1$, and none of $S_2$, contributing 1 to $t$.

*Notation.* We let $\alpha$ be the number of even link periods of $\Delta$, $\beta$ the number of odd period cycles of $\Delta$, $\gamma$ the number of empty period cycles of $\Delta$ and $\delta$ the number of $e_i$ generators associated to the empty period cycles which do not belong to $\Gamma$.

We proved the following result.

**Theorem 2.** *With the above notation:*

(i) *if $n$ is even then $t = \alpha + 2\beta + 2\gamma - \delta$,*
(ii) *if $n$ is odd then $t = 2\beta + 2\gamma$.*

*Note.* If $n$ is even then we usually let $\theta(e_i) = 1$ so that $\delta = 0$. If $n$ is odd then $|S_1| = |S_2| = \beta + \gamma$.

## 6. The main application

**Theorem 3.** *Let $S_1S_2$ have order $n$.*

(i) *If $n$ is even then $t \le (4g/n)+2$.*
(ii) *If $n$ is odd then $t \le \big((2g-2)/n\big)+4$.*

*Proof.* (i) Let $\Delta$ have signature (1). Then by [11], $\Delta^+$ is a Fuchsian group of signature

$$\big(h^+;+;[m_1,m_1,\dots,m_r,m_r,n_{11},\dots,n_{1s_1},\dots,n_{ks_k}]\{\ \}\big)$$

where $h^+=\varepsilon h+k-1$, $\varepsilon$ being defined in (2). As there is an epimorphism from $\Delta^+$ onto $C_n$ we know by Harvey's theorem [2] that

(i) If $m$ is the least common multiple (lcm) of the periods of $\Delta^+$ then $m|n$ and also these periods obey the *lcm condition* which says that the lcm of all the periods with one period deleted is also equal to $m$.
(ii) If $h^+=0$, then $m=n$.
(iii) The number of periods of $\Delta^+$ cannot equal 1 and if $h^+=0$ their number if $\ge 3$.
(iv) If $2|m$ then the number of periods divisible by the maximum power of 2 dividing $m$ is even.

Besides the notation $\alpha$, $\beta$, $\gamma$ introduced earlier we also let $\eta$ equal the number of period cycles with at least one even link period. Then $k=\beta+\gamma+\eta$. Our proof is just to apply the area and index formulae (2) and (3) but we need to break up the calculation into several cases.

(a) $r\ge 1$. From (2) and (3)

$$\begin{aligned}2g-2 &\ge 2n\Big(-2+k+\sum_{i=1}^{n}\Big(1-\frac{1}{m_1}\Big)+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\Big(1-\frac{1}{n_{ij}}\Big)\Big)\\ &\ge 2n\Big(-2+\beta+\gamma+\eta+\frac{1}{2}+\frac{\alpha}{4}\Big).\end{aligned}$$

If $\alpha\ne 0$, then $\eta\ge 1$ so that

$$2g-2\ge 2n\Big(\beta+\gamma+\frac{\alpha}{4}-\frac{1}{2}\Big)\ge 2n\Big(\frac{t}{4}-\frac{1}{2}\Big).$$

Hence $t\le 4(g-1)/n+2<(4g/n)+2$. If $\alpha=0$ then $\eta=0$ and now

$$2g-2\ge 2n\Big(\beta+\gamma-\frac{3}{2}\Big)\ge 2n\Big(\frac{t-3}{2}\Big).$$

Therefore

$$t\le\frac{2(g-1)}{n}+3<\frac{4g}{n}+2,$$

the final inequality being true unless $n \geq 2g+2$; but in this case it follows from the proof of Theorem 4 of [12] that $\Delta^+$ is a triangle group or has signature $\big(0;+;[2,2,m,m]\{\ \}\big)$ with $m$ odd. As $\Delta^+$ has at least one proper period, $\Delta$ has at most two even link periods so that $t \leq 2$, by Theorem 2.

We can now assume that $r = 0$.

(b) $\alpha = 0$. We first assume that $\Delta$ has no link periods and so has signature $\big(h;\pm;[\ ]\{(\ )^k\}\big)$ with $\varepsilon h - 2 + k \geq 1$ as $\mu(\Delta) > 0$. Then $t = 2k$ by Theorem 2. Also $2g-2 = 2n(\varepsilon h - 2 + k) \geq 2n$ giving $n \leq g-1$. Hence $2g-2 \geq 2n\big((t/2)-2\big)$ giving

$$t \leq \frac{2(g-1)}{n} + 4 < \frac{4g}{n} + 2$$

as $n < g+1$.

If $\Delta$ has link periods then they are all odd. If there are $\beta_0$ odd link periods then $\beta_0 \geq 2$ by (i) of Harvey's theorem, and clearly $\beta_0 \geq \beta$. Also, by (iii) of Harvey's theorem $h^+ > 0$ as $m$ is odd and $n$ is even. Therefore $\mu(\Delta^+) \geq 2h^+ - 2 + \frac{2}{3}\beta_0 \geq \frac{4}{3}$. As $2g-2 = n\eta(\Delta^+)$, $n < \frac{3}{2}(g-1)$. Now

$$\begin{aligned} 2g-2 &= 2n\bigg(\varepsilon h - 2 + k + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\Big(1 - \frac{1}{n_{ij}}\Big)\bigg) \\ &\geq 2n\Big(k - 2 + \frac{2}{3}\Big) = 2n\Big(\frac{t}{2} - \frac{4}{3}\Big). \end{aligned}$$

Therefore $t \leq \big(2(g-1)/n\big) + 8/3 < (4g/n) + 2$ as $n < \frac{3}{2}(g+1)$.

Thus from now on we can assume that $r = 0$, $\alpha \geq 1$ and thus $\eta \geq 1$.

(c) We now assume that $h^+ > 0$. Then

$$\begin{aligned} 2g-2 &\geq 2n\Big(\varepsilon h + \beta + \gamma + \eta - 2 + \frac{\alpha}{4}\Big) \\ &= 2n\Big(\varepsilon h + \frac{t}{4} + \frac{\beta+\gamma}{2} + \eta - 2\Big). \end{aligned}$$

If $\beta = \gamma = 0$, then $h^+ = \varepsilon h + \eta - 1 \geq 1$ so that $2g-2 \geq 2nt/4$ and $t \leq (4g-4)/n$. If $\beta + \gamma > 0$, then $\varepsilon h + \eta + (\beta+\gamma)/2 \geq \frac{3}{2}$, and $2g-2 \geq 2n\big((t/4) - \frac{1}{2}\big)$ giving

$$t \leq \frac{4(g-1)}{n} + 2 < \frac{4g}{n} + 2.$$

(d) We now assume that $h^+ = 0$ so that $k = 1$. If the unique period cycle is odd or empty then $t = 2$ so that we can assume that it contains even link periods.

Thus $\Delta^+$ has signature $(0;+;[n_1,\dots,n_t,\nu_1,\dots,\nu_\tau])$ where the $n_i$ are even and the $\nu_i$ are odd. Then

$$\begin{aligned}\frac{2g-2}{n} &= -2+\sum_{i=1}^{t}\Big(1-\frac{1}{n_i}\Big)+\sum_{i=1}^{\tau}\Big(1-\frac{1}{\nu_i}\Big)\\ &\geq -2+\frac{t}{2}+\sum_{i=1}^{\tau}\Big(1-\frac{1}{\nu_i}\Big).\end{aligned}$$

Now if $\tau>0$ then $\tau\geq 2$ by Harvey's theorem, part (i). Therefore

$$\frac{2g-2}{n}\geq -2+\frac{t}{2}+\frac{4}{3}$$

and thus

$$t\leq \frac{4g-4}{n}+\frac{4}{3}<\frac{4g}{n}+2.$$

Thus we can assume that $\tau=0$, i.e. that all the link periods in the period cycle are even. We assume that $u$ of them are equal to $2$ and $v$ are greater than $2$. We denote the latter by $n_1,\dots,n_v$. Then

$$\begin{aligned}2g-2 &= n\bigg(-2+\frac{u}{2}+\sum_{i=1}^{v}\Big(1-\frac{1}{n_i}\Big)\bigg)\geq n\Big(-2+\frac{u}{2}+\frac{3v}{4}\Big)\\ &= n\Big(-2+\frac{u+v}{2}+\frac{v}{4}\Big)\geq n\Big(-1+\frac{t}{2}\Big)\end{aligned}$$

if $v\geq 4$, by Theorem 2. Now $t\leq \big(4(g-1)/n\big)+2<(4g/n)+2$. If $v=3$, then $\{n_1,n_2,n_3\}\neq\{4,4,4\}$, $\{4,4,6\}$ or $\{4,6,6\}$ by conditions (i) and (iv) of Harvey's theorem and thus $\sum_1^3 1/n_i\leq\frac{1}{2}$, so that

$$2g-2=n\bigg(1+\frac{u}{2}-\sum_1^3\frac{1}{n_i}\bigg)\geq n\Big(\frac{u+1}{2}\Big).$$

Thus

$$t=u+3\leq\frac{4(g-1)}{n}+2<\frac{4g}{n}+2.$$

If $v=2$ then by the lcm condition $\Delta^+$ has signature $\big(0;+;[2,2,\dots,n,n];\{\ \}\big)$ and as $n$ is even $t=(4g/n)+2$. We cannot have $v=1$ by the lcm condition and if $v=0$ then $n=2$ and $t=2g+2=(4g/2)+2$.

(ii) This is much easier! We have

$$2g-2\geq 2n(-2+k)=2n(-2+\beta+\gamma)=2n\Big(\frac{t}{2}-2\Big)$$

by Theorem 2. Thus $t\leq\big(2(g-1)/n\big)+4$.

**Notes.** The bound in Theorem 3 (i) is only obtained if and only if $\Delta$ has signature $\big(0;+;[\,];\{2,\dots,2,n,n\}\big)$ with $n$ even. A similar bound, but with topological hypotheses, has been found by Natanzon in [9].

## 7. Attainment of the bounds

We show that the bounds in Theorem 3 are attained and then study some of the Riemann surfaces which attain these bounds. First we prove the following lemma which is of independent interest.

**Lemma 2.** *Let $\Lambda$ be a Fuchsian group of signature $\big(0;+;[m_1,\dots,m_r];\{\ \}\big)$ and suppose that there exists a homomorphism $\varphi\colon \Lambda \to C_n$. If $\Lambda^*$ is an NEC group of signature $\big(0;+;[\,]\{(m_1,\dots,m_r);\}\big)$ with $(\Lambda^*)^+ = \Lambda$ then $\varphi$ extends to a homomorphism $\varphi^*\colon \Lambda^* \to D_n$. This extension is essentially unique.*

*Proof.* Suppose that $\Lambda$ has presentation $\big\langle x_1,\dots,x_r \,|\, x_1^{m_1} = \cdots = x_r^{m_r} = x_1x_2\cdots x_r = 1\big\rangle$ and suppose that $C_n$ is generated by $q$ and that $\varphi(x_i) = q^{\sigma_i}$. Then $n \mid \sum_{i=1}^r \sigma_i$. Let

$$D_n = \big\langle y,q \,|\, y^2 = q^n = (yq)^2 = 1\big\rangle$$

and

$$\Lambda^* = \big\langle c_1,\dots,c_r \,|\, c_1^2 = \cdots = c_r^2 = (c_1c_2)^{m_1} = \cdots = (c_{r-1}c_r)^{m_{r-1}} = (c_rc_1)^{m_r} = 1\big\rangle,$$

with $c_ic_{i+1} = x_i$.

If we let $\varphi^*(c_1) = y$ then we must have $\varphi^*(c_2) = yq^{\sigma_1}$, $\varphi^*(c_3) = yq^{\sigma_1+\sigma_2}$, $\dots$, $\varphi^*(c_r) = yq^{\sigma_1+\cdots+\sigma_{r-1}}$. $\varphi^*\colon \Lambda^* \to D_n$ is a homomorphism that extends $\varphi$, and it is unique once the image of $c_1$ is known. As the image of $c_1$ must be a reflection any two extensions differ by an automorphism of $D_n$.

**Theorem 4.** (i) *The bound in Theorem* 3 (i) *is attained for every even $n$ and every $g > 1$ such that $n|4g$.*

(ii) *The bound in Theorem* 3 (ii) *is attained for every odd $n$ and every $g > 1$ such that $n|g-1$.*

*Proof.* (i) Let $\Delta$ be an NEC group of signature $\big(0;+;[\ ]\{(2^{(4g/n)},n,n);\}\big)$ ($4g/n$ periods equal to 2). Then $\Delta^+$ has signature $\big(0;+;[2^{(4g/n)},n,n];\{\ \}\big)$ and as $n$ is even there is a homomorphism $\theta\colon \Delta^+ \to C_n = \langle Q\rangle$. This takes the involution generators of $\Delta^+$ to $Q^{n/2}$ and the generators of order $n$ to $Q$ and $Q^{-1}$. By Lemma 2, $\theta$ extends to a homomorphism $\theta^*\colon \Delta \to D_n$. If $\Gamma$ is the kernel of $\theta^*$ then $X = \mathbf{H}/\Gamma$ is the surface we require.

(ii) We let $\Delta_1$ have signature $\big(0;+;[\,]\{(\ )\big((g-1)/n\big)+2\}\big)$. There is clearly a smooth homomorphism $\theta_1\colon \Delta_1 \to D_n$ and if $\Gamma_1$ is the kernel of $\theta_1$ then $X_1 = \mathbf{H}/\Gamma_1$ as the required surface.

**Theorem 5.** *Let $X$ be a Riemann surface for which the bound of Theorem* 3 (i) *is attained. Then $X$ is hyperelliptic with $(S_1S_2)^{n/2}$ being the hyperelliptic involution.*

*Proof.* The proof of Theorem 3 (i) shows that if $X = \mathbf{H}/\Gamma$ is a surface for which the bound is attained then $\Gamma$ must be a normal subgroup of an NEC group $\Delta$ of signature $\{0;+;[\ ]\{(2^{(4g/n)}n,n)\}\}$ and Theorem 4 shows that $\Gamma$ must be the kernel of $\theta^+\colon \Delta^+ \to C_n$. By [10] we calculate that $(\theta^+)^{-1}(\langle(S_1S_2)^{n/2}\rangle)$ has signature $(0;+;[2^{(2g+2)}];\{\ \})$ so by [7], $(S_1S_2)^{n/2}$ is the hyperelliptic involution.

Finally, we investigate the number of mirrors of each class of symmetries of the surfaces for which the bound of Theorem 3 (i) is attained. If $\Delta$, $\Delta^+$ are as in the proof of Theorem 4 (i), then with the notation of the proof of Lemma 2, the homomorphism $\theta^+\colon \Delta^+ \to C_n$ is given by $x_1 \to Q^{n/2}, \ldots, x_{4g/n} \to Q^{n/2}$, $x_{(4g/n)+1} \to Q$, $x_{(4g/n)+2} \to Q^{-1}$, (this is unique up to automorphisms of $C_n$) and then the homomorphism $\theta\colon \Delta \to D_n$ is given, in the case $4g/n$ even, by $c_1 \to S_1$, $c_2 \to S_1Q^{n/2}$, $c_3 \to S_1$, $\ldots$, $c_{4g/n} \to S_1/Q^{n/2}$, $c_{(4g/n)+1} \to S_1$, $c_{(4g/n)+2} \to S_1Q$ and in the case $4g/n$ odd by

$$c_1 \to S_1, \quad c_2 \to S_1Q^{n/2}, \ldots, c_{4g/n} \to S_1,$$
$$c_{(4g/n)+1} \to S_1Q^{n/2}, \quad c_{(4g/n)+2} \to S_1Q^{n/2+1}.$$

As there are no odd link periods, the graphs of Section 5 contain no black edges and the number of fixed curves of $S_1$ is the number of black vertices, i.e. the number of even exponents of $Q$. We immediately obtain

**Theorem 6.** *If $4|n$ then the surfaces $X$ for which the bounds of Theorem 3 (i) are attained admit two classes of symmetries, one with $(4g/n)+1$ mirrors and the other with $1$ mirror. If $n \equiv 2 \bmod 4$ then both classes of symmetries have $(2g/n)+1$ mirrors.*

We consider some corollaries concerning "$M$-symmetries" [8]. These are symmetries with $g+1$ mirrors, the maximum possible number by Harnack's bound, and they are the symmetries to which most attention has been paid in the literature. From Theorems 3 (i) and 5 we obtain Natanzon's theorem [8].

**Corollary 1.** *If a Riemann surface $X$ admits two $M$-symmetries $S_1$, $S_2$ then $S_1S_2$ has order $2$. Also, $X$ is hyperelliptic and $S_1S_2$ is the hyperelliptic involution.*

Also from Theorem 3 we deduce

**Corollary 2.** *If $X$ admits an $M$-symmetry $S_1$ and another symmetry $S_2$ then $S_1S_2$ has order $2$ or $4$.*

**Corollary 3.** *If $S_1$ and $S_2$ do not commute then $|S_1| + |S_2| \le g+2$ and this bound is attained for every $g \ge 2$.*

*Proof.* The order of $S_1S_2$ is not equal to $2$. We see from Theorem 3 that $|S_1| + |S_2| \leq g + 2$ if $n$ is even and, by Theorem 4, this bound is attained for every $g$ with $n = 4$. If $n$ is odd then $|S_1| + |S_2| \leq \frac{2}{3}(g-1) + 4$ and this bound is attained only if $3|g-1$. Thus for $n$ odd, $|S_1| + |S_2| \leq g + 2$ with equality only for $g = 4$, $n = 3$.

Another important question about a symmetry $S$ of a Riemann surface $X$ is to decide whether its mirrors separate $X$, (e.g. see [1]). This is equivalent to whether $X/\langle S \rangle$ is orientable or not, or the NEC group $\Lambda$ of (5) has orientable quotient space or not. The techniques of [4] allow us to do this. We note that the mirrors of an $M$-symmetry always separate. We do not go into the details here but it can be shown that in Theorem 6 that when $n > 4$, the mirrors of all symmetries do not separate. If $n = 4$, then one class of symmetries consists of $M$-symmetries and by the above remark the mirrors of each such symmetry do separate. The other class of symmetries have only one mirror which can be shown to be non-separating.

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