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UNIVALENT FUNCTIONS OF GIVEN TRANSFINITE DIAMETER: A MAXIMUM MODULUS PROBLEM

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Abstract. A variational method is used to find the maximum of $|f(\zeta)|$ for fixed $\zeta \in \mathbf{D}$ as f ranges over the class of normalized univalent functions which map the unit disk $\mathbf{D}$ onto regions of prescribed transfinite diameter (logarithmic capacity).

0. Introduction

Let S be the usual class of functions $f(z) = z + a_2 z^2 + \cdots$ that are analytic and univalent in the unit disk $\mathbf{D}$. We say $f \in S_R$ if $f \in S$ and the image $f(\mathbf{D})$ of the unit disk under f has transfinite diameter R. We consider the following problem: given a point $\zeta \in \mathbf{D}$ and a transfinite diameter R ($1 < R < \infty$), find the maximum of $\bigl|f(\zeta)\bigr|$ among all functions $f \in S_R$. In the solution to this problem, it turns out that the extremal function has as its range one of five geometrically distinct types. In each case, the boundary of the range lies in a trajectory of a determined quadratic differential. Depending on R and ζ, the extremal values of $\bigl|f(\zeta)\bigr|$ are given either explicitly or parametrically in terms of complete elliptic integrals. For each choice of R and ζ, the extremal function is unique. The proof uses a special variational method devised by Duren and Schiffer [6] in their recent paper on a related problem.

For basic facts about transfinite diameter, see Goluzin [7, Chapter VII], Hille [8, Chapter 16], Pólya–Szegő [10, Part IV, Chapter 2] or Tsuji [14]. Here we note only that if $R > 0$ is the transfinite diameter of a compact set E in the complex plane and if

$$g(w) = g(w, \infty) = \log |w| + \gamma + O\Bigl(\frac{1}{w}\Bigr)$$

is Green's function (with pole at infinity) for the unbounded component of the complement of E, then $R = e^{-\gamma}$, the logarithmic capacity of E. A set E with $R < \infty$ is clearly bounded.

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1. Existence of solution

Given R $(1 < R < \infty)$ and $\zeta \in \mathbf{D}$, we wish to determine the maximum of $|f(\zeta)|$ for $f \in S_R$. Without loss of generality we may assume that $0 < \zeta < 1$, since other values ζ may then be treated via rotations of $\mathbf{D}$. In order to employ a variational method, we must first prove the existence of an extremal function. It is easily seen that S_R is a normal family. Although various pinching constructions suggest that S_R is not compact, Duren and Schiffer [6] have shown that the larger family $\widehat{S}_R = \cup_{Q \leq R} S_Q$ is a compact normal family. This ensures that the continuous functional $|f(\zeta)|$ attains a maximum over the class $\widehat{S}_R$. Since the extremal values for $|f(\zeta)|$ will be shown to be increasing with R, the maximum of $|f(\zeta)|$ over the class $\widehat{S}_R$ is actually attained within the subclass S_R. Thus we are assured of the existence of an extremal function within the class S_R itself.

2. The Duren–Schiffer variation

A special variation devised by Duren and Schiffer to preserve the class S_R will be used to find the extremal functions. First an interior variation will be used to determine $\left(p'(w)\right)^2$, where $p(w)$ is the analytic completion of Green's function for the complement of $\overline{f(\mathbf{D})}$. Consider the perturbation

$$w^* = w + \sum_{j=1}^{3} \varepsilon_j \frac{w}{w - w_j}, \tag{1}$$

where $w_1, w_2, w_3 \notin \overline{f(\mathbf{D})}$ are all distinct and the ε_j are small complex parameters. For $\varepsilon \equiv \max_j |\varepsilon_j|$ sufficiently small, it is easily seen that

$$f^*(z) = f(z) + \sum_{j=1}^{3} \varepsilon_j \frac{f(z)}{f(z) - w_j} = \sum_{n=1}^{\infty} a_n^* z^n \tag{2}$$

is univalent in D. Notice that

$$a_1^* = 1 - \sum_{j=1}^{3} \varepsilon_j w_j^{-1};$$

therefore, in order to ensure that $f^* \in S$, we set

$$\sum_{j=1}^{3} \varepsilon_j w_j^{-1} = 0. \tag{3}$$

Now, using the method of interior variation, one may calculate an expression for Green's function $g^*(w,\infty)$ of the complement of $\overline{f^*(\mathbf{D})}$. Hence one determines the following relation between R and the transfinite diameter R^* of the region $f^*(\mathbf{D})$:

$$R^* = R\left(1 - \operatorname{Re}\left\{\sum_{j=1}^{3} \varepsilon_j w_j p'(w_j)^2\right\}\right) + O(\varepsilon^2).$$

(See [13] for a derivation of this formula.) For the above expression, $p(w) = p(w,\infty)$ is the multiple-valued analytic completion of $g(w,\infty)$, Green's function of the complement of $\overline{f(\mathbf{D})}$. We wish to ensure that $R^* = R$, so we set

$$\sum_{j=1}^{3} \varepsilon_j w_j p'(w_j)^2 = 0 \tag{4}$$

and modify (1) by a term of order ε^2 (*cf.* Duren and Schiffer [6]).

Without affecting (3) and (4) we may replace ε_j in (1) by $\varrho e^{i\theta}\varepsilon_j$, where $\varrho > 0$ and θ is an arbitrary real constant. Modifying the variation (2) accordingly, we obtain $f \in S_R$ and

$$f^*(\zeta) = f(\zeta)\left(1 + \varrho e^{i\theta} \sum_{j=1}^{3} \frac{\varepsilon_j}{f(\zeta) - w_j}\right) + O(\varrho^2).$$

It follows that

$$\log f^*(\zeta) = \log f(\zeta) + \varrho e^{i\theta} \sum_{j=1}^{3} \frac{\varepsilon_j}{f(\zeta) - w_j} + O(\varrho^2).$$

For any extremal function f, we know $\log\left|f^*(\zeta)\right| \le \log\left|f(\zeta)\right|$, which implies

$$\operatorname{Re}\left\{\varrho e^{i\theta} \sum_{j=1}^{3} \frac{\varepsilon_j}{f(\zeta) - w_j} + O(\varrho^2)\right\} \le 0.$$

But this is true for arbitrary θ, so

$$\sum_{j=1}^{3} \frac{\varepsilon_j}{f(\zeta) - w_j} = 0. \tag{5}$$

(Equation (5) may also be obtained merely by comparing $\left|f^*(\zeta)\right|$ with $\left|f(\zeta)\right|$ and making a judicious choice of θ.)

Equations (3), (4) and (5) provide three linear homogeneous equations in ε_1, ε_2 and ε_3 with a nontrivial solution. This implies the vanishing of a determinant:

$$(6) \qquad \begin{vmatrix} w_1^{-1} & w_2^{-1} & w_3^{-1} \\ \big(f(\zeta)-w_1\big)^{-1} & \big(f(\zeta)-w_2\big)^{-1} & \big(f(\zeta)-w_3\big)^{-1} \\ w_1 p'(w_1)^2 & w_2 p'(w_2)^2 & w_3 p'(w_3)^2 \end{vmatrix} = 0.$$

Every extremal function f satisfies this condition, which remains valid for any choice of distinct points $w_j \notin \overline{f(\mathbf{D})}$. Setting $w_1 = w$ and expanding the determinant (6) along the first column, we get

$$\lambda_1 w^{-1} + \lambda_2\big(f(\zeta)-w\big)^{-1} + \lambda_3 w p'(w)^2 = 0.$$

Since $w_2 \neq w_3$, we see that $\lambda_3 \neq 0$, so we have

$$(7) \qquad p'(w)^2 = aw^{-2} + \frac{b}{\big(f(\zeta)-w\big)w}.$$

Now, since $p(w)$ is the analytic completion of Green's function, it has the form

$$p(w) = \log w + c_0 + c_1 w^{-1} + c_2 w^{-2} + \cdots$$

near infinity; therefore,

$$(8) \qquad p'(w)^2 = w^{-2} - 2c_1 w^{-3} + \cdots.$$

After a short calculation comparing (7) and (8), we obtain $a = b+1$ and

$$(9) \qquad p'(w)^2 = \frac{(b+1)B - w}{w^2(B-w)},$$

where $B = f(\zeta)$.

From the fact that $p(w)$ is the analytic completion of Green's function, we know that $\operatorname{Re}\{p(w)\} = 0$ for w in the boundary of $\overline{f(\mathbf{D})}$. As we shall see later, the boundary of $\overline{f(\mathbf{D})}$ consists of analytic arcs so that $p'(w)$ has an extension to the boundary. This implies that $\big[p'(w)^2 w'(t)^2\big] < 0$ on the boundary of $\overline{f(\mathbf{D})}$, which suggests that the "outer boundary" of $f(\mathbf{D})$ is composed of arcs $w = w(t)$ lying on trajectories of the quadratic differential

$$(10) \qquad \frac{(b+1)B - w}{(w-B)w^2}\, dw^2 > 0.$$

As we shall see, the entire boundary of $f(\mathbf{D})$ consists of arcs $w = w(t)$ which satisfy (10). To prove this we shall apply a special boundary variation developed by Duren and Schiffer [6] which preserves the class S_R. This additional variation is necessary partly because $p(w)$ will not detect any internal boundary components of $f(\mathbf{D})$ (e.g., internal spires) since Green's function for the complement of $\overline{f(\mathbf{D})}$ is independent of such boundary components.

Let Γ be the boundary of $f(\mathbf{D})$. Choose an arbitrary $w_0 \in \Gamma$ and two distinct arbitrary points $w_1, w_2 \notin \overline{f(\mathbf{D})}$. We modify the standard boundary variation (see [12] or [15, Chapter 10])

$$w^* = w + \frac{a\varrho^2 w}{w_0(w - w_0)} + O(\varrho^3)$$

by including two additional terms to produce the variation

$$w^* = V_\varrho(w) = w + \frac{a\varrho^2 w}{w_0(w - w_0)} + \frac{\varepsilon_1 w}{w - w_1} + \frac{\varepsilon_2 w}{w - w_2} + O(\varrho^3), \tag{11}$$

where ε_1, ε_2 are of order ϱ^2 and will be specified later. We make the further assumption that $a(\varrho)$ is bounded away from 0; the reason for this will be made evident later. The variation (11) is analytic and univalent off the union of a small subcontinuum of Γ and two small disks about w_1 and w_2 respectively. If we let $f^* \equiv V_\varrho \circ f$, then f^* is analytic and univalent on $\mathbf{D}$ and $f^*(0) = 0$. By requiring that $V_\varrho'(0) = 1$, we ensure that $f^* \in S$. Thus we require that

$$a\varrho^2 w_0^{-2} + \varepsilon_1 w_1^{-1} + \varepsilon_2 w_2^{-1} = 0. \tag{12}$$

Using the method of interior variation to determine the change in Green's function [12], we calculate the transfinite diameter of R^* in terms of R:

$$R^* = R\left[1 - \operatorname{Re}\left\{\varepsilon_1 w_1 p'(w_1)^2 + \varepsilon_2 w_2 p'(w_2)^2\right\}\right] + O(\varrho^3).$$

We also require that $R^* = R$, so we set

$$\varepsilon_1 w_1 p'(w_1)^2 + \varepsilon_2 w_2 p'(w_2)^2 = 0 \tag{13}$$

and modify (11) by a term of order ϱ^3 (see [6]) to produce a variation with the properties $R^* = R$ and $f^* \in S$. Thus we have $f^* \in S_R$.

Combining equations (9), (12) and (13), we arrive at the equation

$$a\varrho^2 w_0^{-2} = b\left[\frac{\varepsilon_1 B}{w_1(B - w_1)} + \frac{\varepsilon_2 B}{w_2(B - w_2)}\right]. \tag{14}$$

Since $w_0 \neq \infty$ and since we have assumed $a \neq 0$, we see from (14) that $b \neq 0$. Also, we see from (11) that

$$\log f^*(\zeta) = \log B + \frac{a\varrho^2}{w_0(B - w_0)} + \frac{\varepsilon_1}{B - w_1} + \frac{\varepsilon_2}{B - w_2} + O(\varrho^3). \tag{15}$$

After some calculation involving (12), (14) and the fact that $b \neq 0$, we infer from (15) that

$$\log f^*(\zeta) = \log B + \frac{a\varrho^2}{b}\left[\frac{(b+1)B - w_0}{w_0^2(B - w_0)}\right] + O(\varrho^3).$$

If f is extremal, then $\operatorname{Re}\{\log f^*(\zeta)\} \leq \operatorname{Re}\{\log B\}$, which implies

$$\operatorname{Re}\left\{\frac{a\varrho^2}{b}\left[\frac{(b+1)B - w_0}{w_0^2(B - w_0)}\right] + O(\varrho^3)\right\} \leq 0.$$

We now invoke Schiffer's theorem (see [5, p. 297]) to conclude that Γ consists of analytic arcs lying on trajectories of the quadratic differential

$$\frac{1}{b}\left[\frac{(b+1)B - w}{w^2(B - w)}\right] dw^2 > 0. \tag{16}$$

We note here that our assumption that a is bounded away from 0 goes beyond the hypothesis of Schiffer's theorem. However, Schiffer's result is actually stronger than the statement of his theorem indicates, because the proof goes through without modification if we assume $a \neq 0$ (see [5, p. 297–302]. Since Γ consists of analytic arcs, the quadratic differentials (10) and (16) must both be positive on the outer boundary of $f(\mathbf{D})$. Thus $b < 0$ for every choice of ζ and R and we have established that the entire boundary of $f(D)$ consists of arcs lying on trajectories of (10).

3. Analysis of the quadratic differential

The quadratic differential (10) has a simple zero at $w = (b+1)B$, a double pole at the origin and infinity, and a simple pole at $w = B$. (Note that $b \neq 0$ implies $(b+1)B \neq B$.)

It is not hard to see that the trajectories of (10) are symmetric in the real axis. First, symmetry in the line $w = Bt$ can be seen by making the substitution $w = B\omega$ and recognizing the resulting symmetry in the real axis of the ω-plane. Next, note that $f'(0) = 1$ and consider the image under f of the curve $z = \zeta t$, $(0 \leq t \leq 1)$. This image curve has positive direction at the origin and contains B. A symmetry argument enables us to conclude that the image curve is in fact the line segment joining the origin with B in the ω-plane. Therefore the line $w = Bt$, whose tangent direction is of course a constant, must lie in the real axis. We now see that B is real and positive and that all trajectories are symmetric in the real axis. We are now able to show the trajectories for (10) in Figures 1, 2, and 3 below. (See Chapter 8 in [11] for the local trajectory structure of quadratic differentials.)

Note that of the trajectories pictured, only those surrounding B and the origin may form boundaries of the images of extremal functions f. All the candidate

Figure 1. Sketch of trajectories of (10) for $-1 < b < 0$.

Figure 2. Sketch of trajectories of (10) for $b = -1$.

solution regions pictured do in fact occur as boundaries of extremal functions for suitable values of R and ζ.

We now parametrize Γ by $w(t) = f(e^{it})$ and use (10) to obtain

$$F(z) = \frac{z^2 f'(z)^2}{f(z)^2} \, \frac{\big[(b+1)B - f(z)\big]}{B - f(z)} \geq 0, \qquad |z| = 1. \tag{18}$$

Since $f \in S$, it is clear that F is analytic in $\mathbf{D}$ except for a removable singularity at $z = 0$ and a simple pole at $z = \zeta$ with residue

$$a_{-1} = \frac{-b\zeta^2 f'(\zeta)}{f(\zeta)}.$$

Figure 3. Sketch of trajectories of (10) for $b < -1$.

In order to arrive at a convenient expression for $F(z)$, one shows that $F(z)$ is analytic and non-vanishing on $|z| = 1$ except for at most one double zero when Γ has an internal spire or a corner. (The proof for this follows just as in [6, Section 3]). Then by the Schwarz reflection principle, $F(z)$ has a meromorphic extension to the Riemann sphere satisfying $F(1/\bar{z}) = \overline{F(z)}$. Therefore $F(z)$ has the form

$$F(z) = \frac{A(z - z_0)(1 - zz_0)}{(z - \zeta)(1 - \zeta z)} \tag{19}$$

where $f(z_0) = (b + 1)B$ when $w = (b + 1)B$ is inside Γ. When $w = (b + 1)B$ lies on Γ, we have $z_0 = 1$. A symmetry argument shows that the point z_0 in (19) is real and negative. Therefore we may conclude from the Schwarz reflection principle that A is real. The point $(b+1)B$ is in $\overline{f(\mathbf{D})}$ for each of the trajectories above which represent possible solutions.

As suggested by the illustrations above, our work breaks into three cases which we handle in the order (i) $b = -1$; (ii) $-1 < b < 0$; (iii) $b < -1$.

4. The case $b = -1$: ellipses

In this case, $(b + 1)B = 0$ and $z_0 = 0$ so that (19) becomes

$$F(z) = \frac{Az}{(z - \zeta)(1 - \zeta z)} \tag{20}$$

and (10) becomes

$$\frac{dw^2}{w(B-w)} \geq 0. \tag{21}$$

The trajectories of (21) are ellipses with foci $w = 0$ and $w = B$ [4, p. 309–310]. A comparison of the equations (18), (20) and (21) establishes the relation

$$\int_0^B \frac{dw}{\sqrt{w(B-w)}} = \sqrt{A}\int_0^\zeta \frac{dz}{\sqrt{z(\zeta - z)(1-\zeta z)}}. \tag{22}$$

An elementary calculation shows that the left-hand integral is equal to π, while the substitution $u = \sqrt{z/\zeta}$ shows that the right-hand side is equal to $2\sqrt{A}K(\zeta)$ where

$$K(k) = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}, \qquad 0 < k < 1,$$

is the normal complete elliptic integral of the first kind. Thus (22) allows us to evaluate $A = \frac{1}{4}\pi^2 \cdot K(\zeta)^{-2}$. On the other hand, a comparison of (18) and (20) gives

$$\lim_{z\to 0} \frac{F(z)}{z} = -\frac{A}{\zeta} = -\frac{1}{B}.$$

Consequently, for the case $b = -1$, we conclude that

$$B = \frac{4\zeta K(\zeta)^2}{\pi^2}. \tag{23}$$

This expression is independent of R because, as we shall see, the ellipse case ($b = -1$) occurs for only one value of R for each choice of ζ.

By calculating $G(w)$, the analytic completion of Green's function for the complement of the solution ellipses, we shall now obtain a relation between R and B which determines for which values of R the ellipse solution may occur.

Let $w_0 = f(-1)$ ($w_0 < 0$ by symmetry considerations), and define

$$G_0(w) \equiv \int_{w_0}^{w} \frac{dw}{\sqrt{w(w-B)}} = \int_0^w \frac{dw}{\sqrt{w(w-B)}} - \int_0^{w_0} \frac{dw}{\sqrt{w(w-B)}} \equiv I_1 - I_2. \tag{24}$$

It is easily seen that

$$I_1 = \log\left(w\sqrt{w(w-B)} + 2w - B\right) - \log(-B)$$

where the chosen branch of the logarithm is unimportant since we will be concerned only with $\operatorname{Re}\{G_0(w)\}$. Note that by (18) and (20)

$$I_2 = \sqrt{A}\int_0^{-1} \frac{dz}{\sqrt{z(z-\zeta)(1-\zeta z)}}$$

and substitute $z=(t+1)/(t-1)$, $\tau=(1+\zeta)/(1-\zeta)$ to conclude

$$I_2=\frac{-2\sqrt{A}}{1+\zeta}K\Big(\frac{1}{\tau}\Big).$$

Thus (24) takes the form

$$G_0(w)=\log 4w-\log(-B)+\frac{2\sqrt{A}}{1+\zeta}K\Big(\frac{1}{\tau}\Big)+O\Big(\frac{1}{w}\Big).$$

Since the boundary of $f(\mathbf{D})$ lies on a trajectory of the quadratic differential (21),

$$g(w)=\operatorname{Re}\big\{G_0(w)\big\}=\log|w|-\log\frac{B}{4}+\frac{2\sqrt{A}}{1+\zeta}K\Big(\frac{1}{\tau}\Big)+O\Big(\frac{1}{w}\Big) \tag{25}$$

is harmonic in the complement of $\overline{f(\mathbf{D})}$ except for a logarithmic singularity at infinity and is constant on the boundary. But $G_0(w_0)=0$, so $g(w)=0$ on the boundary. This shows that g is Green's function of the complement of $\overline{f(\mathbf{D})}$. The equivalence of transfinite diameter and logarithmic capacity (see Section 0) now shows by (25) that

$$R=\frac{B}{4}\exp\left[\frac{2\sqrt{A}}{1+\zeta}K\Big(\frac{1}{\tau}\Big)\right]=\frac{\zeta K(\zeta)^2}{\pi^2}\exp\left[\frac{\pi}{(1+\zeta)K(\zeta)}K\Big(\frac{1}{\tau}\Big)\right]. \tag{26}$$

Thus the ellipse solution may occur only for the single value of R satisfying (26). This solves the case $b=-1$. For information on the explicit form of functions $f(z)$ which map the unit disk conformally onto the interior of an ellipse, see [9, Chapter VII, Section 3].

5. The case $-1<b<0$

In the case $-1<b<0$, Γ is an analytic Jordan curve surrounding $w=0$, $w=B$ and $w=(b+1)B$. In this case we return to (19) in its original form

$$F(z)=\frac{A(z-z_0)(1-zz_0)}{(z-\zeta)(1-\zeta z)} \tag{19}$$

where $f(z_0)=(b+1)B$ and $0<z_0<\zeta$. Now upon considering (18) along with (19) we have four parameters; B, A, b, z_0. To express them in terms of R and ζ we require four relations among the variables.

Relation 1. Equate the expressions (18) and (19) for $F(z)$ and let z approach 0 to get

$$(b+1) = \frac{Az_0}{\zeta}, \qquad (A>0). \tag{27}$$

Relation 2. We find the analytic completion $H(w)$ of Green's function for the complement of $\overline{f(\mathbf{D})}$ to establish a relation among R, b, B and ζ. If $w_1 = f(1)$, then

$$H(w) = G(w) - G(w_1)$$

where

$$G(w) = \int_B^w \sqrt{\frac{w-(b+1)B}{w-B}}\,\frac{dw}{w}.$$

A calculation now reveals that

$$G(w) = \log w + \log\frac{-4}{bB} + \sqrt{b+1}\log\left(\frac{1-\sqrt{b+1}}{1+\sqrt{b+1}}\right) + O\left(\frac{1}{w}\right) \tag{28}$$

and

$$G(w_1) = \int_B^{w_1} \sqrt{\frac{w-(b+1)B}{w-B}}\,\frac{dw}{w} = \int_\zeta^1 \sqrt{\frac{A(z-z_0)(1-z_0z)}{(z-\zeta)(1-\zeta z)}}\,\frac{dz}{z}. \tag{29}$$

We may substitute $t = (z-1)/(z+1)$, $\tau = (1+\zeta)/(1-\zeta)$ and $t_0 = (1+z_0)/(1-z_0)$ in (29) to obtain

$$G(w_1) = \sqrt{A}\,\frac{2(1+z_0)}{1+\zeta}\int_{-1/\tau}^{0} \frac{(t^2-t_0^{-2})\,dt}{(t^2-1)\sqrt{(t^2-t_0^{-2})(t^2-\tau^{-2})}}.$$

Now let $\omega = -\tau t$. This gives

$$G(w_1) = \sqrt{A}\left[\frac{2(1-z_0)t_0^2}{1+\zeta}K\left(\frac{t_0}{\tau}\right) - \frac{8z_0}{(1-z_0)(1+\zeta)}\Pi\left(\frac{1}{\tau^2}, \frac{t_0}{\tau}\right)\right], \tag{30}$$

where

$$\Pi(\alpha^2, k) = \int_0^1 \frac{dt}{(1-\alpha^2)\sqrt{(1-t^2)(1-k^2t^2)}},$$

the normal complete elliptic integral of the third kind. Therefore,

$$H(w) = \log w + \log\frac{-4}{bB} + \sqrt{b+1}\log\left(\frac{1-\sqrt{b+1}}{1+\sqrt{b+1}}\right) - G(w_1) + O\left(\frac{1}{w}\right).$$

An argument similar to that used for the ellipse case shows that $\mathrm{Re}\{H(w)\}$ is Green's function for the complement of $\overline{f(\mathbf{D})}$. Thus if $G(w_1)$ is given by (30), then

$$\log R = \log \frac{-bB}{4} + \sqrt{b+1}\log\left(\frac{1+\sqrt{b+1}}{1-\sqrt{b+1}}\right) + G(w_1). \tag{31}$$

Relation 3. We use the equality

$$\int_{(b+1)B}^{B} \sqrt{\frac{(b+1)B-w}{w-B}}\,\frac{dw}{w} = \int_{z_0}^{\zeta} \sqrt{\frac{A(z-z_0)(1-z_0 z)}{(\zeta-z)(1-\zeta z)}}\,\frac{dz}{z}. \tag{32}$$

The left side of (32) is easily seen to be $\pi(1-\sqrt{b+1}\,)$ while the right side is an elliptic integral. We refer to formulas 253.00, 253.11, 253.12 and 340.01 in [3] to conclude that the right side of (32) is equal to

$$\frac{2\sqrt{A}}{z_0^{-1}-\zeta}\left[\left(\frac{1}{\zeta}-\zeta\right)\Pi(\alpha^2,k)+\left(\zeta-\frac{1}{\zeta}\right)\Pi\left(\frac{\alpha^2}{\zeta^2},k\right)+\left(\frac{1}{z_0}+z_0-\frac{1}{\zeta}-\zeta\right)K(k)\right], \tag{33}$$

where

$$\alpha^2 = \frac{\zeta - z_0}{\zeta^{-1}-z_0}, \qquad k^2 = \alpha^2\cdot\frac{1-\zeta^{-1}z_0}{1-\zeta z_0}.$$

A special addition formula for elliptic integrals of the third kind ([3, 117.01]) allows us to simplify (33) to

$$\frac{2\sqrt{A}}{z_0^{-1}-\zeta}\left[2\left(\frac{1}{\zeta}-\zeta\right)\Pi(\alpha^2,k)+\left(\frac{1}{z_0}+z_0-\frac{2}{\zeta}\right)K(k)\right]-\frac{\pi}{2}\sqrt{\frac{Az_0}{\zeta}}.$$

Now equality in (32) tells us that

$$\sqrt{b+1} = \frac{\pi}{2}\sqrt{\frac{z_0}{\zeta}}\left(\frac{1}{z_0}-\zeta\right)\left[2\left(\frac{1}{\zeta}-\zeta\right)\Pi(\alpha^2,k)+\left(\frac{1}{z_0}+z_0-\frac{2}{\zeta}\right)K(k)\right]^{-1}, \tag{34}$$

where (27) has been used. Equation (34) enables us to express b (and thus A) in terms of z_0 and ζ.

Relation 4. We now have three equations (27), (31) and (34) relating the parameters B, A, b, z_0 to R and ζ. For a final fourth relation, we use the fact that $f'(0)=1$ along with the equality of the integrals,

$$\int_{f(\varepsilon)}^{(b+1)B} \sqrt{\frac{(b+1)B-w}{B-w}}\,\frac{dw}{w} = \sqrt{\frac{Az_0}{\zeta}}\int_{\varepsilon}^{z_0} \sqrt{\frac{(z_0-z)(z_0^{-1}-z)}{(\zeta-z)(\zeta^{-1}-z)}}\,\frac{dz}{z}.$$

For small $\varepsilon > 0$ the left side of this equation is calculated to be

$$\sqrt{b+1}\log\frac{-4(b+1)B}{bf(\varepsilon)}+\log\frac{1-\sqrt{b+1}}{1+\sqrt{b+1}}+O(\varepsilon).$$

The right integral is equal to

$$\sqrt{\frac{Az_0}{\zeta}}\left(J_\varepsilon+\log\frac{z_0}{\varepsilon}\right)+O(\varepsilon),$$

where

$$J_\varepsilon=\int_\varepsilon^{z_0}\left(\sqrt{\frac{(z_0-z)(z_0^{-1}-z)}{(\zeta-z)(\zeta^{-1}-z)}}-1\right)\frac{dz}{z}.$$

Let $\varepsilon\to 0$ in both sides and recall that

$$\lim_{\varepsilon\to 0}\frac{f(\varepsilon)}{\varepsilon}=1,$$

to obtain the final fourth relation,

$$\log B=-\frac{1}{\sqrt{b+1}}\log\frac{1-\sqrt{b+1}}{1+\sqrt{b+1}}+\log\frac{-bz_0}{4(b+1)}+J_0. \tag{35}$$

The elliptic integral J_0 will be left in open form for simplicity.

We now prove that the three relations (31), (34) and (35) give a well-defined relation between R, ζ and B. To do this we show that for fixed ζ, R and B both depend monotonically on z_0 for values of z_0 between 0 and ζ.

First, the relationship between R and B must be monotone as proven in the following lemma.

Lemma. *Suppose $f\in S_R$ and $|f(\zeta)|\ge|g(\zeta)|$ for all $g\in S_R$. Then for each $\varepsilon>0$ there exist δ satisfying $0<\delta<\varepsilon$, and $f^*\in S_{R+\delta}$ such that $|f^*(\zeta)|>|f(\zeta)|$.*

Since ζ is positive we have $B=f(\zeta)>0$. We now choose a particular variation of the type found in (7) to preserve S while increasing R slightly. First select the points w_0, w_1 and w_2 in (7) so that $w_1>B>0$ and $w_0<0$. After choosing $\varepsilon_1=-\varrho^2$ and $\varepsilon_2=0$ in (7), it is easily verified that for arbitrarily small $\varrho>0$, we may choose $a(\varrho)$ such that:

(a) equation (8) is satisfied, thus S is preserved,
(b) the transfinite diameter R^* of the function $f^*=V_\varrho\circ f$ satisfies $R<R^*<R+\varepsilon$;
(c) $\operatorname{Re}\{\log f^*(\zeta)\}>\operatorname{Re}\{\log f(\zeta)\}$.

Note that (c) is satisfied using (11). This proves the lemma.

Next we divide both sides of (32) by $\sqrt{b+1} = \sqrt{Az_0/\zeta}$ and differentiate with respect to z_0 to obtain

$$\frac{\partial b}{\partial z_0} > 0. \tag{36}$$

Equation (35) may now be used along with (36) to show that $\log B$ and thus B has positive derivative with respect to z_0. Therefore both R and B are monotone with respect to z_0.

This monotonicity indicates that the case $-1 < b < 0$ may occur only for values of R and ζ corresponding to $0 < z_0 < \zeta$. Since $z_0 = 0$ corresponds to $R = 1$ and $z_0 = \zeta$ corresponds to the ellipse case ($b = -1$), we see that the case $-1 < b < 0$ may occur only in the sub-region of the R, ζ half-strip ($\{(R,\zeta) : R > 1, 0 < \zeta < 1\}$) lying below the curve (26). Further, the monotonicity of R with respect to z_0 shows that when the extremal function falls only into this case, the extremal function is unique.

6. The case $b < -1$

Here there are two subcases:

Subcase α: Γ consists of a Jordan curve passing through $w = (b+1)B$ along with a slit $\big[(b+1)B, \eta\big]$ on the real axis (the slit may have length zero).

Subcase β: Γ is an analytic Jordan curve surrounding $w = 0$, $w = B$ and $w = (b+1)B$.

We handle the former subcase first. In Subcase α, F as given by (18) and (19) has a double zero on the unit circle so that (19) becomes

$$F(z) = \frac{A(z+1)^2}{(z-\zeta)(1-\zeta z)}. \tag{37}$$

Upon considering (18) along with (37) we have three parameters; A, B, b. To express them in terms of R and ζ we require three relations among the variables. Evaluating $\lim_{z\to 0} F(z)$ in each of the expressions (18) and (37) gives us the first relation:

$$b + 1 = \frac{-A}{\zeta}. \tag{38}$$

For the second relation we calculate $G(w)$:

$$\begin{aligned} G(w) &= \int_{(b+1)B}^{w} \sqrt{\frac{w-(b+1)B}{w-B}}\,\frac{dw}{w} \\ &= \log w + \log\frac{4}{bB} + 2\sqrt{\frac{A}{\zeta}}\arg\left(\sqrt{\frac{A}{\zeta}} + i\right) + O\left(\frac{1}{w}\right), \end{aligned}$$

where the fundamental function element is chosen and (38) has been used. Since $\mathrm{Re}\,\{G(w)\}$ is Green's function for the complement of $\overline{f(\mathbf{D})}$, we see that

$$\log R = \log \frac{-bB}{4} + 2\sqrt{\frac{A}{\zeta}}\arg\left(\sqrt{\frac{A}{\zeta}} + i\right). \tag{39}$$

To establish a third relation we use the equality of the integrals,

$$\int_{f(\varepsilon)}^{B} \sqrt{\frac{w-(b+1)B}{B-w}}\,\frac{dw}{w} = \sqrt{A}\int_{\varepsilon}^{\zeta} \frac{z+1}{\sqrt{(\zeta-z)(1-\zeta z)}}\,\frac{dz}{z}.$$

Upon calculation it is clear that the first integral is

$$-i\log\frac{b}{2i\sqrt{-(b+1)}+b+2} - \sqrt{-(b+1)}\,\log\frac{bf(\varepsilon)}{4(b+1)B} + O(\varepsilon),$$

while the second integral is

$$-\sqrt{\frac{A}{\zeta}}\,\log\frac{(1-\zeta)^2}{4\zeta}\,\varepsilon + O(\varepsilon).$$

Since the two integrals are equal for all small $\varepsilon > 0$, we let ε approach zero and use the fact that $f'(0) = 1$ to conclude

$$\sqrt{\frac{\zeta}{A}}\,\arg\frac{b}{2i\sqrt{-(b+1)}+b+2} = \log\left[\frac{b}{(b+1)B}\cdot\frac{\zeta}{(1-\zeta)^2}\right]. \tag{40}$$

Employing (39) to eliminate B and (38) to eliminate b we find

$$\log R = \log\frac{\zeta}{(1-\zeta)^2} + \log\left(\frac{A}{\zeta}+2+\frac{\zeta}{A}\right) + 2\left(\sqrt{\frac{A}{\zeta}} - \sqrt{\frac{\zeta}{A}}\right)\arg\left(\sqrt{\frac{A}{\zeta}}+i\right) - \log 4. \tag{41}$$

A calculation reveals that this expression for $\log R$ is monotone with respect to A. Thus (41) determines A for given values of R. Therefore, equations (39) and (41) determine B parametrically in terms of R and ζ for Subcase α.

To determine for which values of R and ζ Subcase α may occur, we will determine for what values of R the "no-slit" domain described above has inner radius $r \geq 1$. If $r = 1$, then the "no-slit" domain possibly corrresponds to an extremal function. If $r > 1$ the "no-slit" domain can be modified by the addition of a horizontal radial slit of suitable length to reduce the inner radius of the domain to exactly 1. For $r < 1$ this subcase may not occur.

Our work will be simplified by obtaining an explicit expression for A in terms of ζ alone for the "no-slit" region. To do this we make use of a special contour integral. In the w-plane, let Γ_ε be the union of the two line segments $\big[(b+1)B, f(-\varepsilon)\big]$, $\big[f(\varepsilon), B\big]$ and the lower semicircle joining $w = f(-\varepsilon)$ to $w = f(\varepsilon)$. Note that $f(-\varepsilon) = -f(\varepsilon) + O(\varepsilon^2)$ so the semicircle construction will remain valid when we allow ε to approach zero later. In the z-plane let γ_ε be the union of the segments $[-1, -\varepsilon]$, $[\varepsilon, \zeta]$ and the lower semicircle joining $z = -\varepsilon$ to $z = \varepsilon$. Then the semicircle in the w-plane is path-homotopic to the image under f of the prescribed semicircle in the z-plane. Since $f(-1) = (b+1)B$ for the "no-slit" region, it is clear that

$$\int_{\Gamma_\varepsilon} \sqrt{\frac{(b+1)B - w}{w - B}}\, \frac{dw}{w} = \sqrt{A} \int_{\gamma_\varepsilon} \frac{z+1}{\sqrt{(\zeta - z)(1 - \zeta z)}}\, \frac{dz}{z}.$$

A calculation reveals that the first and second integrals are

$$\pi + \sqrt{-(b+1)}\pi i + O(\varepsilon)$$

and

$$2\sqrt{\frac{A}{\zeta}}\, \log \frac{1 - \sqrt{\zeta}}{1 + \sqrt{\zeta}} + \sqrt{\frac{A}{\zeta}}\, \pi i + O(\varepsilon),$$

respectively. Now by letting ε approach zero and comparing imaginary parts, we again obtain (38), while a comparison of real parts tells us that

$$A = \frac{\pi^2 \zeta}{4\left[\log\left(\dfrac{1 - \sqrt{\zeta}}{1 + \sqrt{\zeta}}\right)\right]^2}. \tag{42}$$

We define

$$\Psi(\zeta) = \frac{\pi}{2 \log\left(\dfrac{1 + \sqrt{\zeta}}{1 - \sqrt{\zeta}}\right)} \tag{43}$$

and combine (39) and (42) to establish the relation

$$B = \frac{4R}{\Psi(\zeta)^2 + 1} e^{-2\Psi(\zeta) \arg(\Psi(\zeta) + i)}. \tag{44}$$

The same calculation as that used to obtain (40) tells us for the "no-slit"domain that

$$\sqrt{\frac{\zeta}{A}}\, \arg \frac{b}{2i\sqrt{-(b+1)} + b + 2} + \log \frac{(b+1)B}{b} \cdot \frac{(1 - \zeta)^2}{\zeta} = \log r.$$

Using (44) we see that $r \geq 1$ exactly when

$$\log R \geq \log \frac{\zeta}{(1-\zeta)^2} + \log \frac{\left(\Psi(\zeta)^2+1\right)^2}{4\Psi(\zeta)^2} + 2\frac{\Psi(\zeta)^2-1}{\Psi(\zeta)} \arg\left(\Psi(\zeta)+i\right). \tag{45}$$

Thus the extremal function can have the form of Subcase α only if R satisfies (45). If R does not satisfy (45), then the extremal function takes the form of Subcase β or one of the cases $b=-1$, $b>-1$. Note that when, in Subcase α, the solution region has no slit, then

$$B = \frac{\zeta}{(1-\zeta)^2}\left[\frac{\Psi^2+1}{\Psi^2} e^{-(2/\Psi)\arg(\Psi+i)}\right],$$

which compares favorably with the Koebe distortion theorem since the term in square brackets takes values between 0 and 1 for all $\zeta \in (0,1)$. This is easily proven by noting that the expression in brackets has limits 1 and 0 at $\zeta = 1$, respectively; while it is monotone with respect to ζ between those two values. Note that letting ζ approach zero in (45) leads to the inequality $R \geq \varepsilon^2\pi^2/64$, which agrees with previous work in [6]. This completes Subcase α.

In Subcase β, as in the case $b>-1$, both R and B will be expressed in terms of a third parameter z_0; further, the solutions will involve the use of elliptic integrals. Note that work in the previous cases and Subcase α above implies that Subcase β certainly occurs for values of R and ζ lying between the curve (26) and the curve obtained by taking equality in (45).

Again $F(z)$ is given by (18) and (19) so

$$b+1 = \frac{Az_0}{\zeta}. \tag{46}$$

Now let $w_{-1} = f(-1)$, the point where Γ meets the negative real axis, and consider

$$H_0(w) = \int_{w_{-1}}^{w} \sqrt{\frac{w-(b+1)B}{w-B}}\,\frac{dw}{w} = \tilde{G}(w) - \tilde{G}(w_{-1}),$$

where

$$\tilde{G}(w) = \int_{(b+1)B}^{w} \sqrt{\frac{w-(b+1)B}{w-B}}\,\frac{dw}{w}.$$

Since Γ satisfies (10) we see that $H_0(w)$ is the analytic completion of Green's function for the complement of $\overline{f(\mathbf{D})}$. Considering (45) and the analysis leading up to it, we have

$$\log R = \log\frac{-bB}{4} + 2\sqrt{\frac{-Az_0}{\zeta}}\arg\left(\sqrt{\frac{-Az_0}{\zeta}}+i\right) + \operatorname{Re}\left\{\tilde{G}(w_{-1})\right\},$$

where (46) has been used. Also, by (19),

$$\tilde{G}(w_{-1}) = \int_{(b+1)B}^{w_{-1}} \sqrt{\frac{w-(b+1)B}{w-B}}\,\frac{dw}{w} = \sqrt{A}\int_{z_0}^{-1} \sqrt{\frac{(z-z_0)(1-z_0 z)}{(z-\zeta)(1-\zeta z)}}\,\frac{dz}{z}.$$

Now a series of substitutions similar to those used to obtain (33) shows that

$$\tilde{G}(w_{-1}) = \sqrt{A}\,\frac{2(1-z_0)}{1+\zeta}K\left(\frac{t_0}{\tau}\right) - \sqrt{A}\,\frac{8z_0}{(1+\zeta)(1-z_0)}\Pi\left(t_0^2, \frac{t_0}{\tau}\right),$$

where as before, K and Π are normal complete elliptic integrals of the first and third kind, respectively. Hence

$$\log R = \log\frac{-bB}{4} + 2\sqrt{\frac{-Az_0}{\zeta}}\arg\left(\sqrt{\frac{-Az_0}{\zeta}} + i\right) + \tilde{G}(w_{-1}). \tag{47}$$

We now establish a third relation that will determine A (and thus b by (46)) in terms of z_0 and ζ. Let $\mathbf{T}$ be the unit circle and as usual, let Γ be the boundary of $f(\mathbf{D})$. Then the equivalence of (18) and (19) implies that

$$\int_{\mathbf{T}} \sqrt{\frac{A(z-z_0)(1-z_0 z)}{(\zeta-z)(1-\zeta z)}}\,\frac{dz}{z} = \int_{\mathbf{T}} \sqrt{\frac{(b+1)B-w}{w-B}}\,\frac{dw}{w}. \tag{48}$$

By calculating the residue at infinity of the integrand in the second integral, we see that its value is 2π. The first integral, which is elliptic, will be left in open form after a simplifying substitution. With $z = e^{i\theta}$, the left side of (48) becomes

$$2\sqrt{\frac{-Az_0}{\zeta}}J(z_0,\zeta),$$

where

$$J(z_0,\zeta) = \int_0^{\pi} \sqrt{\frac{\cos\theta - \frac{1}{2}(z_0+z_0^{-1})}{\frac{1}{2}(\zeta+\zeta^{-1}) - \cos\theta}}\,d\theta. \tag{49}$$

Note that $J(z_0,\zeta)$ is positive for $-1 < z_0 < 0$ and $0 < \zeta < 1$ and

$$J(-1,\zeta) = 2\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}. \tag{50}$$

Equation (48) implies that

$$b+1 = \frac{Az_0}{\zeta} = \frac{-\pi^2}{J(z_0,\zeta)^2}. \tag{51}$$

We may now view (47) as a relation expressing B in terms of R, ζ and z_0.

To express B parametrically in terms of R and ζ alone we need one further relation. It is obtained by observing that since (18) and (37) are equivalent,

$$\int_{f(\varepsilon)}^{B}\sqrt{\frac{(b+1)B-w}{w-B}}\,\frac{dw}{w}=\int_{\varepsilon}^{\zeta}\sqrt{\frac{A(z-z_0)(1-z_0z)}{(\zeta-z)(1-\zeta z)}}\,\frac{dz}{z}. \tag{52}$$

After dividing both sides by

$$\sqrt{-(b+1)}=\sqrt{\frac{-Az_0}{\zeta}},$$

the left side of this expression may be rewritten as

$$\int_{f(\varepsilon)/B}^{1}\left(\sqrt{\frac{1-w(b+1)^{-1}}{1-w}}-1\right)\frac{dw}{w}-\log f(\varepsilon)+\log B,$$

while the right side is equal to

$$\log\zeta-\log\varepsilon+\int_{\varepsilon}^{\zeta}\left(\sqrt{\frac{(z-z_0)(z-z_0^{-1})}{(\zeta-z)(\zeta^{-1}-z)}}-1\right)\frac{dz}{z}.$$

We now let ε approach 0 and use the fact that $f'(0)=1$ to see that

$$\begin{aligned}\log B=&\int_{0}^{\zeta}\left(\sqrt{\frac{(z-z_0)(z-z_0^{-1})}{(\zeta-z)(\zeta^{-1}-z)}}-1\right)\frac{dz}{z}+\log\zeta\\&-\int_{0}^{1}\left(\sqrt{\frac{1-w(b+1)^{-1}}{1-w}}-1\right)\frac{dw}{w}.\end{aligned}\tag{53}$$

Together, equations (53) and (47) indicate that R and B are related parametrically in terms of z_0 for a fixed value of ζ.

The solution to the problem will be complete once it is shown that R varies monotonically with z_0. This monotonicity will imply that the extremal function is unique when it falls into Subcase β. Further it will imply that all the cases we have described actually occur and the solution is unique in each case.

To prove monotonicity, it is sufficient in view of the lemma in Section 5 to show that $\partial B/\partial z_0<0$. Toward this end we differentiate with respect to z_0 in (53) to obtain

$$\begin{aligned}\frac{\partial\log B}{\partial z_0}=&\tfrac12(z_0^{-2}-1)\int_0^{\zeta}\frac{dz}{\sqrt{\tilde q(z)}}\\&-\tfrac12\frac{\partial b}{\partial z_0}\int_0^1\frac{dw}{\big[-(b+1)\big]^{3/2}\sqrt{(1-w)(w-b+1)}},\end{aligned}\tag{54}$$

where

$$\tilde{q}(z) = (\zeta - z)\left(\frac{1}{\zeta} - z\right)(z - z_0)\left(z - \frac{1}{z_0}\right).$$

Although we could now obtain an expression for $\partial b/\partial z_0$ directly from (51), our calculations will be facilitated if we instead use the following relation,

$$\int_{(b+1)B}^{f(-\varepsilon)} \sqrt{\frac{w-(b+1)B}{B-w}}\,\frac{dw}{w} = \int_{z_0}^{-\varepsilon} \sqrt{\frac{A(z-z_0)(1-z_0 z)}{(\zeta - z)(1-\zeta z)}}\,\frac{dz}{z}.$$

We proceed just as with (52) above to obtain

$$\begin{aligned}\log B = &-\int_{z_0}^{0}\left(\sqrt{\frac{(z-z_0)(z-z_0^{-1})}{(\zeta-z)(\zeta^{-1}-z)}} - 1\right)\frac{dz}{z} + \log\frac{z_0}{b+1}\\ &+\int_{b+1}^{0}\left(\sqrt{\frac{1-w(b+1)^{-1}}{1-w}} - 1\right)\frac{dw}{w}.\end{aligned} \tag{55}$$

After differentiating (55) and comparing the result with (54) we conclude that

$$\frac{\partial b}{\partial z_0} = \frac{1}{\pi}(z_0^{-2} - 1)\big[-(b+1)\big]^{3/2}\int_{z_0}^{\zeta}\frac{dz}{\sqrt{\tilde{q}(z)}}.$$

This in turn allows us to conclude that

$$\frac{\partial \log B}{\partial z_0} = \tfrac{1}{2}(z_0^{-2} - 1)\left[\frac{2}{\pi}\arctan\sqrt{-(b+1)}\int_{z_0}^{\zeta}\frac{dz}{\sqrt{\tilde{q}(z)}} - \int_{z_0}^{0}\frac{dz}{\sqrt{\tilde{q}(z)}}\right]. \tag{56}$$

Though an analytic proof is yet to be found, computer calculations have shown numerically that the expression in (56) is negative over a wide range of values of z_0 and ζ. The method of numerical integration used relies upon a modified Newton–Cotes formula known as Weddle's rule ([2, Chapter 3]) after a change of variable. A few sample values are shown in Table 1 below. These data suggest that B and thus R varies monotonically with respect to z_0 in Subcase β. Thus the data also indicate that the extremal function for all values of R and ζ is unique.

	$\zeta = .9$	$\zeta = .5$	$\zeta = .1$
$z_0 = -.99$	-1.7422	-1.1367	-0.3092
$z_0 = -.9$	-0.8747	-0.5450	-0.1408
$z_0 = -.5$	-0.2206	-0.1204	-0.0271
$z_0 = -.1$	-0.0108	-0.0051	-0.0010
$z_0 = -.01$	-0.00012	-0.00005	-0.00001

Table 1. Table entries are numerical values for the bracketed terms in (56) above.

7. Summary of results

Before stating our results in the form of a theorem, we summarize the five types of extremal functions which may occur.

I. The extremal function f maps $\mathbf{D}$ onto a region bounded by a Jordan curve which is analytic except for a corner at the point $w = (b+1) \cdot |f(\zeta)|$ along with a radial slit from $(b+1) \cdot |f(\zeta)|$ toward the origin. The length of this slit is determined by the fact that $f'(0) = 1$. Extremal functions of this type are unique when they occur. $|f(\zeta)|$ is given implicitly by (39) and (41). This solution type may occur only for values of R and ζ lying above the curve (45).

II. The extremal function f maps $\mathbf{D}$ onto a region similar to that described for Type I solutions except that the slit has length zero. Again $|f(\zeta)|$ is given by (39) and (41). This type of solution may occur only for values of R and ζ lying on (45).

III. The extremal function f maps $\mathbf{D}$ onto a region bounded by an analytic Jordan curve surrounding the point $w = (b+1) \cdot |f(\zeta)|$, where $b > -1$. Extremal functions of this type are unique when they occur and they may occur only for values of R and ζ lying below the curve (26). $|f(\zeta)|$ is given parametrically in terms of complete elliptic integrals by equations (31) and (35).

IV. The extremal function f maps $\mathbf{D}$ onto the interior of an ellipse with foci at $w = 0$ and $w = |f(\zeta)|$. This type of solution occurs only for R and ζ lying on the curve (26). $|f(\zeta)|$ is given directly in terms of complete elliptic integrals by (30).

V. The extremal function f maps $\mathbf{D}$ onto a region bounded by an analytic Jordan curve surrounding the point $w = (b+1) \cdot |f(\zeta)|$, where $b < -1$. $|f(\zeta)|$ is given parametrically in terms of complete elliptic integrals by (47) and (53).

We have established the following results.

Theorem. *Given $\zeta \in \mathbf{D}$ and $R \in (1, \infty)$, each function f maximizing $|g(\zeta)|$ over the class S_R maps $\mathbf{D}$ onto a region bounded by arcs lying on trajectories of the quadratic differential* (10). *In* (10) *the negative parameter b is determined by R and ζ. For all values of R and ζ, $\arg f(\zeta) = \arg \zeta$. For values of R and ζ lying below the curve* (26) *each extremal function is either of Type III or Type V. On the curve* (26) *itself each extremal function is of Type IV or Type V. In the region between the curves* (26) *and* (45) *the solution is of Type V. On the curve* (45) *the solution is of Type II or Type V. Finally, above the curve* (45) *the solution is of Type I or Type V.*

As mentioned in Section 6, numerical evidence indicates that extremal functions of Type V may occur *only* in the region between the curves (26) and (45). This numerical evidence also suggests that the extremal function is unique for each value of R and ζ. For each type of solution, the value of the derivative of the extremal function at ζ may be obtained using the calculations in Section 3.

It should be mentioned that one can show for values of R and ζ lying on the curve (26) that the extremal function is of Type IV and is unique. This is done by defining R_f to be the transfinite diameter of $f(\mathbf{D})$ and then finding the maximum of

$$\frac{|f(\zeta)|}{R_f}$$

over the class S. The extremal functions for this problem all map $\mathbf{D}$ onto the interior of an ellipse.

Figure 4. A graph of the curves (26) and (45) as drawn using Matlab. Roman numerals are used to indicate which solution types may occur in each region of the (R, ζ) half-strip.

Note added in proof. During the publication process it was brought to the author's attention that certain of the results (particularly those established in Section 2) were previously established by A. Solynin and M. Gavrilyuk in their article "Moduli estimates for certain classes of univalent functions" in the Russian language publication *Dynamic Problems* in *Mechanics of a Continuous Medium* 63, 1983. Their approach employs the method of extremal lengths. Our approach is new as are the results involving the distribution of solutions for various values of R and ζ and the quantitative estimates of solutions.

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