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SOME RESULTS ON THE POMPEIU PROBLEM

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Abstract. A domain $\Omega \subset \mathbf{R}^2$ is said to have the Pompeiu property if $f \equiv 0$ is the only continuous function in $\mathbf{R}^2$ such that the integral of f over $\sigma(\Omega)$, for every rigid motion σ of $\mathbf{R}^2$, vanishes. It has been conjectured that the disc is the only bounded simply connected domain, modulo sets of Lebesgue measure zero, in which the Pompeiu property fails. In this paper we obtain some results which support that conjecture. In the first part of the paper we show that the disc is the only quadrature domain in which the Pompeiu property fails. In the second part of the paper we prove a result claiming nonexistence, under certain conditions, of solutions to a family of overdetermined Cauchy problems. This result is used to obtain the Pompeiu property for a wide variety of domains, including "kth roots" of ellipses and domains which are mapped conformally onto the unit disc by a rational function other than a Möbius transformation.

1. Introduction

In this paper we consider the Pompeiu problem: *characterize the bounded domains in* $\mathbf{R}^2$ *which have the Pompeiu property* (see Definition 1.1 below). It was (almost?) conjectured in [GS1] that the disc is the only bounded simply connected domain, modulo sets of measure zero, without the Pompeiu property. Our main results, Theorem 2.1 (although this is not just a statement about simply connected domains; see the remark at the end of Section 3), Theorems 3.1, 3.2 (or perhaps Corollaries 3.1 and 3.2) and 3.3, support this conjecture. We refer the reader to [B], [BST] and [GS1] for a more extensive introduction to the problem and a discussion of previous results. Also, see [Gu] and [S] for a discussion of quadrature domains.

Definition 1.1. *A domain* $\Omega \in \mathbf{R}^2$ *has the* Pompeiu property *if* $f \equiv 0$ *is the only continuous function in* $\mathbf{R}^2$ *such that*

$$\int_{\sigma(\Omega)} f(x,y)\,dx\,dy = 0 \qquad \textit{for every rigid motion } \sigma \textit{ of } \mathbf{R}^2. \tag{1.1}$$

Note that the Pompeiu property is invariant with respect to adding or taking away sets of Lebesgue measure zero.

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By a theorem of L. Brown, B.M. Schreiber and B.A. Taylor (see [BST]) a bounded domain Ω has the Pompeiu property if and only if $\hat{\mu}$, the Fourier–Laplace transform of the area measure μ of Ω, does not vanish identically on

$$M_\alpha = \{(\zeta_1, \zeta_2) \in \mathbf{C}^2 : \zeta_1^2 + \zeta_2^2 = \alpha\} \tag{1.2}$$

for any α in $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. Also, S.A. Williams showed (see [Wi1]) that if Ω is a bounded simply connected Lipschitz domain then it has the Pompeiu property if and only if there is no solution to the following overdetermined Cauchy problem

$$\begin{cases} \Delta u + \alpha u = 1 & \text{in } \Omega \\ u(x,y) = \left|\nabla u(x,y)\right| = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where the Laplace operator Δ is $(\partial^2/\partial x^2) + (\partial^2/\partial y^2)$, for any $\alpha \in \mathbf{C}^*$ (in fact, it suffices to consider $\alpha > 0$, cf. [B]). As the referee so kindly remarked, in this formulation the problem essentially goes back to the original book on "The theory of sound" by Lord Rayleigh. It later became known as the Schiffer problem and in this context the conjecture mentioned above is known as Schiffer's conjecture (see [GS1]). Another result of Williams is that any Lipschitz domain in which the Pompeiu property fails must have a nonsingular analytic boundary (see [Wi2]). Consequently, as regards the Pompeiu problem, assuming that the boundary of a domain is nonsingular and analytic is not excessive.

In Section 2 we use the formulation of Brown, Schreiber and Taylor to show that the disc is the only bounded quadrature domain without the Pompeiu property. In Section 3 we obtain, using an idea due to H.S. Shapiro, a result claiming nonexistence of solutions to a family of overdetermined Cauchy problems, which includes (1.3), under certain conditions (see Proposition 3.1). This idea was used successfully by G. Johnsson in [J] to prove that ellipsoids in $\mathbf{R}^n$ have the Pompeiu property (Definition 1.1 immediately extends to arbitrary dimensions). We also find that kth roots of ellipses, loosely speaking, and a family of domains, the conformal maps onto the unit disc of which have certain properties (see Theorem 3.2), satisfy these conditions and therefore, by the result of Williams, have the Pompeiu property. We conclude the paper by showing some examples of domains that satisfy the hypotheses of Theorem 3.2.

2. Quadrature domains

Let Ω be a bounded quadrature domain, i.e. Ω is a bounded domain such that there is a distribution ν with finite support in Ω and with the property that $\mu - \nu$, where μ is the area measure on Ω, annihilates the space of integrable holomorphic functions in Ω (see e.g. [Gu] or [S]). It is known that Γ, the boundary of Ω, is algebraic. Furthermore, there is a meromorphic function S in Ω, called the Schwarz function of Ω, which extends analytically across every nonsingular point of Γ and which satisfies

$$S(z) = \bar{z} \tag{2.1}$$

on Γ. We assume, as we may since the Pompeiu property is invariant with respect to adding or subtracting sets of Lebesgue measure zero, that Γ has no isolated points. By Stokes' theorem, we have

$$\begin{aligned} \hat{\mu}(\zeta_1, \zeta_2) &= \int_\Omega e^{-i(\zeta_1 x + \zeta_2 y)}\, d\mu(x, y) \\ &= \int_\Omega \exp\Big(-\frac{i}{2}\big((\zeta_1 - i\zeta_2)z + (\zeta_1 + i\zeta_2)\bar{z}\big)\Big)\, d\mu(z) \\ &= \frac{1}{2i(\zeta_1 + i\zeta_2)} \int_\Gamma \exp\Big(-\frac{i}{2}\big((\zeta_1 - i\zeta_2)z + (\zeta_1 + i\zeta_2)\bar{z}\big)\Big)\, dz, \end{aligned} \tag{2.2}$$

where $z = x + iy$. Pick α in $\mathbf{C}^*$, parameterize M_α by

$$\begin{cases} \zeta_1 - i\zeta_2 = -\zeta \\ \zeta_1 + i\zeta_2 = -\dfrac{\alpha}{\zeta} \end{cases} \tag{2.3}$$

and let $f(\zeta; \alpha)$ be the restriction of $2i(\zeta_1 + i\zeta_2)\hat{\mu}(\zeta_1, \zeta_2)$ to M_α, i.e.

$$f(\zeta; \alpha) = \int_\Gamma \exp\Big(\frac{i}{2}\Big(\zeta z + \frac{\alpha}{\zeta}\bar{z}\Big)\Big)\, dz. \tag{2.4}$$

Clearly, $f(\cdot; \alpha)$ is analytic in $\mathbf{C}^*$.

Thus, by the theorem of Brown, Schreiber and Taylor mentioned above, Ω fails to have the Pompeiu property if and only if there is an $\alpha \in \mathbf{C}^*$ such that $f(\cdot, \alpha)$ is identically zero. Following [B] and [GS1] we wish to study the asymptotic properties of $f(\zeta; \alpha)$ as $\zeta \to \infty$ to deduce one of our main results (Theorem 2.1 below).

Proposition 2.1. *Suppose that S has a simple pole and nonconstant regular part at a point z_1 and suppose that z_1 is an extreme point of Δ, the convex hull of the set of poles of S. If $\mathrm{Re}(ie^{i\theta}z) = c$ is a supporting line for Δ through z_1—we may assume that $\mathrm{Re}(ie^{i\theta}z) < c$ for every other point of Δ—then, as $r \to \infty$,*

$$\begin{aligned} f(re^{i\theta}; \alpha) &= \frac{2\pi i\sqrt{a_{-1}\alpha}}{re^{i\theta}} \exp\Big(\frac{i}{2}\Big(re^{i\theta}z_1 + \frac{a_0\alpha}{re^{i\theta}}\Big)\Big) \cdot \\ &\cdot \Big(J_1\big(\sqrt{a_{-1}\alpha}\big) + \Big(\frac{i}{re^{i\theta}}\Big)^{k+1} \frac{a_k\alpha}{2}\big(\sqrt{a_{-1}\alpha}\big)^{k+1} J_{k+1}\big(\sqrt{a_{-1}\alpha}\big) + O\Big(\frac{1}{r^{k+2}}\Big)\Big), \end{aligned} \tag{2.5}$$

where the a_j's appearing are the coefficients in the Laurent series expansion of S at z_1, k is the least positive integer such that $a_k \neq 0$ and J_l is the lth order Bessel function of the first kind.

Remark. By a complex number $C = |C|e^{i \arg C}$, $\arg C \in [0, 2\pi)$, raised to a noninteger β we mean $|C|^{\beta} e^{i\beta \arg C}$.

The proof of Proposition 2.1 is based on the following

Lemma 2.1. *If γ is a closed curve homologous (relative $\mathbf{C}^*$) to a circle around the origin then*

$$\begin{aligned}(2.6)\qquad &\int_\gamma \exp\Big(\frac{i}{2}\Big(\zeta z + \frac{\alpha}{\zeta}\Big(\frac{a_{-1}}{z} + a_k z^k\Big)\Big)\Big)\, dz \\ &= \sum_{j=0}^{\infty} \frac{2\pi i^{jk+1}}{j!}\Big(\frac{i a_k \alpha}{2}\Big)^j \big(\sqrt{a_{-1}\alpha}\big)^{jk+1} J_{jk+1}\big(\sqrt{a_{-1}\alpha}\big)\frac{1}{\zeta^{j(k+1)+1}}.\end{aligned}$$

Proof. Clearly, we have

$$\begin{aligned}(2.7)\qquad &\int_\gamma \exp\Big(\frac{i}{2}\Big(\zeta z + \frac{\alpha}{\zeta}\Big(\frac{a_{-1}}{z} + a_k z^k\Big)\Big)\Big)\, dz \\ &= \sum_{j=0}^{\infty} \frac{1}{j!}\Big(\frac{i a_k \alpha}{2\zeta}\Big)^j \int_\gamma z^{jk} \exp\Big(\frac{i}{2}\Big(\zeta z + \frac{\alpha a_{-1}}{\zeta z}\Big)\Big)\, dz.\end{aligned}$$

We may choose γ to be the circle $\{|z| = |\sqrt{a_{-1}\alpha}\,|/|\zeta|\}$. If we parameterize γ we obtain

$$\begin{aligned}(2.8)\qquad &\int_\gamma \exp\Big(\frac{i}{2}\Big(\zeta z + \frac{\alpha}{\zeta}\Big(\frac{a_{-1}}{z} + a_k z^k\Big)\Big)\Big)\, dz = \sum_{j=0}^{\infty} \frac{i}{j!}\Big(\frac{i a_k \alpha}{2\zeta}\Big)^j \Big(\frac{\sqrt{a_{-1}\alpha}}{\zeta}\Big)^{jk+1} \times \\ &\times \int_{-\pi}^{\pi} \exp\big(i\sqrt{a_{-1}\alpha}\cos\theta\big)\big(\cos(jk+1)\theta + i\sin(jk+1)\theta\big)\, d\theta.\end{aligned}$$

Since $\sin(jk+1)\theta$ is odd and $\cos(jk+1)\theta$ is even the lemma follows from an integral representation of J_{jk+1} (see [O], p. 360). □

This lemma will be useful in the proof of the proposition.

Proof of Proposition 2.1. Using the Schwarz function of Ω we may write

$$(2.9)\qquad f(\zeta;\alpha) = \int_\Gamma \exp\Big(\frac{i}{2}\Big(\zeta z + \frac{\alpha}{\zeta} S(z)\Big)\Big)\, dz.$$

Since the integrand now is holomorphic in $\Omega \setminus \{z_1, \ldots, z_n\}$, where $\{z_1, \ldots, z_n\}$ are the poles of S, we can write

$$(2.10)\qquad f(\zeta;\alpha) = \sum_{m=1}^{n} f_m(\zeta;\alpha),$$

where

$$(2.11)\qquad f_m(\zeta;\alpha) = \int_{\Gamma_m} \exp\Big(\frac{i}{2}\Big(\zeta z + \frac{\alpha}{\zeta} S(z)\Big)\Big)\, dz$$

and Γ_m is some sufficiently small circle around z_m.

We first deal with $f_1(\zeta;\alpha)$. Write the Schwarz function as

$$S(z) = \frac{a_{-1}}{z - z_1} + a_0 + a_k(z - z_1)^k + h(z), \tag{2.12}$$

where h is a holomorphic function near z_1 which is $O\big((z - z_1)^{k+1}\big)$. We split (2.11) into two integrals and make the change of variables $z' = z - z_1$

$$\begin{aligned} f_1(\zeta;\alpha) &= \int_{\Gamma_1} \exp\Big(\frac{i}{2}\Big(\zeta z + \frac{\alpha}{\zeta}\Big(\frac{a_{-1}}{z - z_1} + a_0 + a_k(z - z_1)^k\Big)\Big)\Big)\, dz \\ &\quad + \int_{\Gamma_1} \exp\Big(\frac{i}{2}\Big(\zeta z + \frac{\alpha}{\zeta}\Big(\frac{a_{-1}}{z - z_1} + a_0 + a_k(z - z_1)^k\Big)\Big)\Big)\Big(e^{\frac{i\alpha}{2\zeta}h(z)} - 1\Big)\, dz \\ &= \exp\Big(\frac{i}{2}\Big(\zeta z_1 + \frac{a_0\alpha}{\zeta}\Big)\Big)\Big(\int_{\tilde\Gamma_1} \exp\Big(\frac{i}{2}\Big(\zeta z + \frac{\alpha}{\zeta}\Big(\frac{a_{-1}}{z} + a_k z^k\Big)\Big)\, dz \\ &\quad + \int_{\tilde\Gamma_1} \exp\Big(\frac{i}{2}\Big(\zeta z + \frac{\alpha}{\zeta}\Big(\frac{a_{-1}}{z} + a_k z^k\Big)\Big)\Big)\Big(e^{\frac{i\alpha}{2\zeta}h(z+z_1)} - 1\Big)\, dz\Big), \end{aligned} \tag{2.13}$$

where $\tilde\Gamma_1$ is some circle around the origin. As ζ tends to infinity we let the radius of the circle $\tilde\Gamma_1$ tend to zero as $|\zeta|^{-1}$. The imaginary part of $\zeta z + \alpha(a_{-1}/z + a_k z^k)/\zeta$ then remains uniformly bounded and, hence, the last integral in (2.13) will be $O(1/\zeta^{k+3})$. It follows from Lemma 2.1 that, as ζ tends to infinity,

$$\begin{aligned} f_1(\zeta;\alpha) &= \frac{2\pi i\sqrt{a_{-1}\alpha}}{\zeta} \exp\Big(\frac{i}{2}\Big(\zeta z_1 + \frac{a_0\alpha}{\zeta}\Big)\Big)\Big(J_1(\sqrt{a_{-1}\alpha}) \\ &\quad + \Big(\frac{i}{\zeta}\Big)^{k+1}\frac{a_k\alpha}{2}\big(\sqrt{a_{-1}\alpha}\big)^{k+1} J_{k+1}(\sqrt{a_{-1}\alpha}) + O\Big(\frac{1}{\zeta^{k+2}}\Big)\Big). \end{aligned} \tag{2.14}$$

Now, we look at $f_m(\zeta;\alpha)$ for $m > 1$. Write the Schwarz function as

$$S(z) = \sum_{j=1}^{p} \frac{c_{-j}}{(z - z_m)^j} + h(z), \tag{2.15}$$

where h is holomorphic near z_m. The exponent of the integrand in (2.11) can be written as

$$\frac{i}{2}\zeta^\beta\Big(\zeta^{1-\beta}z + \frac{\alpha c_{-p}}{\big(\zeta^{1-\beta}(z - z_m)\big)^p}\Big) + \frac{i\alpha}{2\zeta}\Big(\sum_{j=1}^{p-1} \frac{c_{-j}}{(z - z_m)^j} + h(z)\Big), \tag{2.16}$$

if we let $\beta = (p-1)/(p+1)$. If we, as above, make the change of variables $z' = z - z_m$ and integrate over a circle $\tilde\Gamma_m$, the radius of which tends to zero as $|\zeta|^{-(1-\beta)}$ when ζ tends to infinity, then we get

$$f_m(\zeta;\alpha) = e^{\frac{i}{2}\zeta z_m} \int_{\tilde\Gamma_m} \exp\Big(\frac{i}{2}\zeta^\beta\varphi(z,\zeta) + \psi(z,\zeta)\Big)\, dz, \tag{2.17}$$

where $|\varphi(z,\zeta)|$ and the real part of $\psi(z,\zeta)$ remain uniformly bounded on $\tilde\Gamma_m$ (in fact, $|\psi(z,\zeta)|$ tends to zero) as ζ tends to infinity. We deduce that there is a constant b_m such that $f_m(re^{i\theta};\alpha)$, as r tends to infinity, is $O\big(r^{-1+\beta}\exp(\frac{1}{2}\operatorname{Re}(ire^{i\theta}z_m) +b_m r^{\beta})\big)$. Since $\beta<1$ the proposition follows from (2.10), (2.14) and the fact that $\operatorname{Re}(ie^{i\theta}z_m)<\operatorname{Re}(ie^{i\theta}z_1)$. □

We also need an asymptotic expansion of $f(\zeta;\alpha)$ in case there are no points z_1 that satisfy the hypothesis of Proposition 2.1. To this end we use a result due to N. Garofalo and F. Segala. In proving their Theorem 3.1 of [GS1] the authors prove (the argument in Section 3 of [GS1] modulo some unfortunate misprints) that the saddle point method (see [De]) is stable under small perturbations in the following sense:

Lemma 2.2. *Suppose that $\varphi(z)$ is analytic in a punctured disc $D_R^*=\{0<|z|<R\}$ and that $\psi(z;\sigma)$, for $(z;\sigma)$ in $D_R^*\times(-\delta,\delta)$, is analytic in z and real-analytic in σ. If there is a curve γ in D_R^*, which is homologous to a circle around the origin in D_R^*, through a point z_0 such that z_0 is a nondegenerate critical point (Morse point) of φ, i.e. $\varphi'(z_0)=0$ and $\varphi''(z_0)\neq 0$, and such that*

$$\operatorname{Re}\big(\varphi(z)\big)<\operatorname{Re}\big(\varphi(z_0)\big) \tag{2.18}$$

for all $z\in\gamma\setminus\{z_0\}$ then, as $\sigma\to 0$,

$$\begin{aligned}\int_\gamma &\exp\big(\sigma^{-\varrho}\big(\varphi(z)+\sigma\psi(z;\sigma)\big)\big)\,dz\\ &+\sigma^{\varrho/2}\exp\big(\sigma^{-\varrho}\big(\varphi(z_0)+\nu(\sigma)\big)\big)\Big(\frac{\sqrt{2\pi}}{i\sqrt{\varphi''(z_0)}}+O(\sigma^{\tau})\Big),\end{aligned} \tag{2.19}$$

where $\nu(\sigma)\to 0$ as $\sigma\to 0$ and $\tau=\min(\frac{1}{2}\varrho,1)$, for all $\varrho>0$.

From now on we will use the notation

$$\varphi(z;\theta)=e^{i\theta}\Big(z+\frac{1}{z^p}\Big), \tag{2.20}$$

$$\beta=\frac{p-1}{p+1} \tag{2.21}$$

and

$$\varepsilon=\frac{1}{p+1}. \tag{2.22}$$

To keep the notation reasonably simple we do not indicate the dependence on the integer p. This should cause no confusion. Using Lemma 2.2 we can prove

Proposition 2.2. *Suppose that S has a pole of order $p>1$ at a point z_1 and suppose that z_1 is an extreme point of Δ, the convex hull of the set of poles of S. If $\mathrm{Re}(ie^{i\theta}z)=c$ is a supporting line for Δ through z_1—as in Proposition 2.1 we assume that $\mathrm{Re}(ie^{i\theta}z)<c$ for every other point of Δ—and if θ', defined by*

$$\theta' = \beta\theta + \frac{\pi}{2} + \varepsilon \arg(\alpha a_{-p}), \tag{2.23}$$

where a_j is the j th coefficient in the Laurent series expansion of S at z_1, is such that

$$\theta' \neq \frac{(2k+1)\pi}{p+1} \tag{2.24}$$

for all $k\in\mathbf{Z}$—such θ can always be found if z_1 is an extreme point of Δ—then, as $r\to\infty$,

$$\begin{aligned} f(re^{i\theta};\alpha) = {} & \frac{|\alpha a_{-p}|^{\varepsilon/2}e^{i\omega}}{r^{1-\beta/2}} \exp\Big(\frac{i}{2}re^{i\theta}z_1 + r^{\beta}\frac{|\alpha a_{-p}|^{\varepsilon}}{2}\big(\varphi(z_0,\theta')+\nu(r)\big)\Big)\times \\ & \times\Big(\frac{\sqrt{2\pi}}{i\sqrt{\varphi''(z_0,\theta')}} + O\Big(\frac{1}{r^{\tau}}\Big)\Big), \end{aligned} \tag{2.25}$$

where $\omega=\theta'-\theta-\frac{1}{2}\pi$, $\nu(r)\to 0$ as $r\to\infty$, $\tau=\min(1-\beta,\frac{1}{2}\beta)$ and $z_0= p^{1/(p+1)}e^{-2\pi ik/(p+1)}$ if

$$\frac{(2k-1)\pi}{p+1} < \theta' < \frac{(2k+1)\pi}{p+1}. \tag{2.26}$$

Proof. We proceed as in the proof of Proposition 2.1 and write $f(\zeta;\alpha)$ as (2.10). First, we look at $f_1(\zeta;\alpha)$. We write

$$S(z) = \frac{a_{-p}}{(z-z_1)^p} + \sum_{j=-p+1}^{\infty} a_j(z-z_1)^j. \tag{2.27}$$

We make the change of variables $z'=z-z_1$ in (2.11) and obtain

$$f_1(\zeta;\alpha) = e^{i\zeta z_1/2}\int_{\tilde\Gamma_1} \exp\Big(\frac{i}{2}\Big(\zeta z + \frac{\alpha}{\zeta}\Big(\frac{a_{-p}}{z^p} + \sum_{j=-p+1}^{\infty} a_j z^j\Big)\Big)\Big)\,dz, \tag{2.28}$$

where $\tilde\Gamma_1$ is some circle around the origin. Now, make the change of variables

$$z' = \frac{\zeta^{1-\beta}z}{(\alpha a_{-p})^{\varepsilon}}. \tag{2.29}$$

We get

$$f_1(\zeta;\alpha)=\frac{(\alpha a_{-p})^{\varepsilon}}{\zeta^{1-\beta}}e^{i\zeta z_1/2}\int_{\gamma}\exp\Big(\frac{i}{2}\zeta^{\beta}\Big(\alpha a_{-p})^{\varepsilon}\times \\ \times\Big(\varphi(z;0)+\frac{\alpha^{1-\varepsilon}}{(a_{-p})^{\varepsilon}\zeta^{1-\beta}}\tilde{\psi}(z,\zeta^{1-\beta})\Big)\Big)\,dz, \tag{2.30}$$

where γ is some circle around the origin and

$$\tilde{\psi}(z,\zeta)=\frac{1}{\zeta^{p-1}}\sum_{j=-p+1}^{\infty}a_j\Big(\frac{(\alpha a_{-p})^{\varepsilon}z}{\zeta}\Big)^{j}. \tag{2.31}$$

Denote the integral in (2.30) by I. Setting $\zeta=re^{i\theta}$ and

$$\sigma=\frac{2^{(1-\beta)/\beta}}{|\alpha a_{-p}|^{\varepsilon(1-\beta)/\beta}r^{1-\beta}} \tag{2.32}$$

we get

$$I=\int_{\gamma}\exp(\sigma^{-\beta/(1-\beta)}\big(\varphi(z;\theta')+\sigma\psi(z,\sigma)\big)\,dz, \tag{2.33}$$

where θ' is given by (2.23) and

$$\psi(z;\sigma)=C\tilde{\psi}\Big(z,\frac{|\alpha a_{-p}|^{\varepsilon(1-\beta)/\beta}e^{-i(1-\beta)\theta}}{2^{(1-\beta)/\beta}}\sigma\Big). \tag{2.34}$$

C is some suitable constant which is of no interest to us. The important thing is that $\psi(z;\sigma)$ is analytic in z and real-analytic in σ in $D_R^*\times(-\delta,\delta)$ for any $R>0$, provided δ is small enough. Suppose that we could find a curve γ, homologous to a circle around the origin relative $\mathbf{C}^*$, which passes through a nondegenerate critical point z_0 of φ and which is such that

$$\mathrm{Re}\,\big(\varphi(z)\big)<\mathrm{Re}\,\big(\varphi(z_0)\big) \tag{2.35}$$

for all $z\in\gamma\setminus\{z_0\}$. Lemma 2.2 would then give the asymptotic behavior of I as σ tends to zero. Using (2.30) and (2.32) we would obtain

$$f_1(re^{i\theta};\alpha)=\frac{|\alpha a_{-p}|^{\varepsilon/2}e^{i\omega}}{r^{1-\beta/2}}\exp\Big(\frac{i}{2}re^{i\theta}z_1+r^{\beta}\frac{|\alpha a_{-p}|^{\varepsilon}}{2}\big(\varphi(z_0,\theta')+\nu(r)\big)\Big) \\ \times\Big(\frac{\sqrt{2\pi}}{i\sqrt{\varphi''(z_0,\theta')}}+O\Big(\frac{1}{r^{\tau}}\Big)\Big), \tag{2.36}$$

where ω, ν and τ are as stated in the proposition above. Thus, we need

Lemma 2.3. *The points*

$$z'_j = p^{1/(p+1)} e^{2\pi i j/(p+1)} \tag{2.37}$$

are nondegenerate critical points of the function $\varphi(\cdot;\theta)$. *Let* γ *be the circle* $\{|z| = p^{1/(p+1)}\}$ *and suppose that*

$$\frac{(2k-1)\pi}{p+1} < \theta' < \frac{(2k+1)\pi}{p+1}. \tag{2.38}$$

for some integer k. *Then*

$$\mathrm{Re}\,\big(\varphi(z)\big) < \mathrm{Re}\,\big(\varphi(z'_{-k})\big) \tag{2.39}$$

for all $z \in \gamma \setminus \{z'_{-k}\}$.

Proof. The statement about the points z'_j being nondegenerate critical points is easy to verify. To prove the second part of the lemma we set

$$g(t) = \mathrm{Re}\,\Big(\varphi\Big(p^{1/(p+1)} e^{it};\theta\Big)\Big) \tag{2.40}$$

and $t_0 = -2\pi k/(p+1)$. The proof is finished once we show that g, restricted to any interval of length 2π that contains t_0 as an interior point, attains its maximum at t_0 and nowhere else. Note that

$$g(t) = p^{1/(p+1)}\Big(\cos(t+\theta) + \frac{1}{p}\cos(pt-\theta)\Big). \tag{2.41}$$

We find that there are two sets of critical points of g

$$\begin{cases} t_m^1 = \dfrac{2m\pi}{p+1} \\ t_m^2 = \dfrac{2\theta + (2m+1)\pi}{p-1}, \end{cases} \tag{2.42}$$

where $m \in \mathbf{Z}$. The important thing to notice is that $\cos(t_m^1 + \theta) = \cos(pt_m^1 - \theta)$ and $\cos(t_m^2 + \theta) = -\cos(pt_m^2 - \theta)$. Hence, at these points we have

$$\begin{cases} g(t_m^1) = p^{1/(p+1)}\cos(t_m^1 + \theta)\Big(1 + \dfrac{1}{p}\Big) \\ g(t_m^2) = p^{1/(p+1)}\cos(t_m^2 + \theta)\Big(1 - \dfrac{1}{p}\Big). \end{cases} \tag{2.43}$$

Now we see that g attains a local maximum at t_0 and that the only points, inside any interval of length 2π containing t_0 as an interior point, at which the value of g can be larger than or equal to $g(t_0)$ are the points t_m^2. However, it is easy to check that if $g(t_m^2) > 0$ then $g''(t_m^2) > 0$ so g must have a local minimum at such a point. Consequently, since $g(t_0)$ obviously is larger than zero, if g would assume its maximum at a point t_m^2 then this point would be a local minimum. This is clearly a contradiction and, hence, the proof is finished. □

To conclude the proof of the proposition we note that $\mathrm{Re}(ie^{i\theta}z) = c$, by hypothesis, is a supporting line for Δ which passes through z_1, so the exact same argument that concluded the proof of Proposition 2.1 is applicable here. □

Now, we are ready to prove one of the main results of this paper. If Δ, the convex hull of the set of poles of S, has an extreme point z_1 which is a pole of order $p > 1$ then it follows from Proposition 2.2 that $f(\zeta;\alpha)$ does not vanish identically for any $\alpha \in \mathbf{C}^*$. This means, by the theorem of Brown, Schreiber and Taylor mentioned above, that Ω has the Pompeiu property. Suppose, on the other hand, that all extreme points of Δ are simple poles and let z_1 be such a point. Unless z_1 is the only point of Δ the regular part of S at z_1 is not constant and, hence, we may apply Proposition 2.1. Since J_1 and J_{k+1}, for a positive integer k, have no common zeros (see [Wa], p. 484) it is clear that $f(\zeta;\alpha)$ does not vanish identically for any $\alpha \in \mathbf{C}^*$ and, consequently, Ω has the Pompeiu property also in this case. A well known fact of quadrature domains is that if the Schwarz function of Ω has a simple pole at some point $z_1 \in \Omega$ and no other poles in Ω then Ω is a disc centered at z_1 (see e.g. [S], p. 51). Also, since the Schwarz function of a disc with radius R centered at z_1 is $S(z) = R/(z - z_1)$ it follows immediately from Lemma 2.1 that a disc does not have the Pompeiu property (this is a well known result, see e.g. [BST]). Hence, we have the following

Theorem 2.1. *The only bounded quadrature domains without the Pompeiu property are discs.*

Remark. If R is a rational function, univalent in the unit disc $\mathbf{D}$ and without poles in $\overline{\mathbf{D}}$, then $R(\mathbf{D})$ is a bounded quadrature domain. Consequently, Theorem 2.1 contains the results in [GS1]. Also, there are quadrature domains of any conformal type (see [Gu])—in fact, any domain bounded by a finite number of analytic curves can be approximated, in some sense, by quadrature domains of the same conformal type—so Theorem 2.1 has something to say about multiply connected domains as well.

3. Overdetermined Cauchy problems

We now turn to the study of some overdetermined Cauchy problems. We identify $\mathbf{R}^2$ with the real plane in $\mathbf{C}^2$, i.e. $\mathbf{R}^2 = \{(X,Y) \in \mathbf{C}^2 : \mathrm{Im}(X) = \mathrm{Im}(Y) = 0\}$. Following [KS] we make the following

Definition 3.1. *If Ω is a domain in $\mathbf{R}^2$ the Vekua hull $\hat{\Omega}$ of Ω is defined by*

$$\hat{\Omega} = \{(X,Y) \in \mathbf{C}^2 : X + iY, \bar{X} + i\bar{Y} \in \Omega\}. \tag{3.1}$$

Furthermore, the Vekua class $V(\Omega)$ of Ω is the class of real-analytic functions in Ω which extend holomorphically to $\hat{\Omega}$.

We let $\mathscr{L}$ be a linear partial differential operator of the type

$$\mathscr{L} = \Delta + a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} + c(x,y), \tag{3.2}$$

where Δ is the Laplace operator as in (1.3) and a, b and c are real-analytic functions in some domain ω. At this point it is convenient to make the linear change of coordinates

$$\begin{cases} z = X + iY \\ w = X - iY. \end{cases} \tag{3.3}$$

In these coordinates a point (z_0, w_0) is in the Vekua hull of a domain Ω' if and only if z_0 and $\bar{w}_0$ are in Ω'. Also, $\mathbf{R}^2 \cong \mathbf{C}$ can be identified with the (anti)-complex line $\{w = \bar{z}\}$. Note that $\mathscr{L}$ is well defined in a neighborhood of ω in $\mathbf{C}^2$ (acting on holomorphic functions). In terms of the coordinates (3.3) we have

$$\mathscr{L} = 4\Big(\frac{\partial^2}{\partial z \partial w} + d(z,w)\frac{\partial}{\partial z} + e(z,w)\frac{\partial}{\partial w} + g(z,w)\Big), \tag{3.4}$$

where d, e and g are holomorphic in the corresponding neighborhood. Recall from the theory of holomorphic partial differential equations that a point (z_0, w_0) on a complex analytic curve $\big\{\varphi(z,w) = 0\big\}$ is called characteristic with respect to $\mathscr{L}$ if

$$\frac{\partial \varphi}{\partial z}(z_0, w_0)\frac{\partial \varphi}{\partial w}(z_0, w_0) = 0. \tag{3.5}$$

Suppose that Ω is a bounded simply connected domain and suppose that there is a point z_0 on the boundary of Ω and a neighborhood U_0 of z_0 such that $\Gamma = \partial\Omega \cap U_0$ is a nonsingular analytic curve, i.e.

$$\Gamma = \big\{z \in \mathbf{C} : \varphi(z, \bar{z}) = 0\big\}, \tag{3.6}$$

where φ is some real-analytic function in U_0. It is well known that Γ has a Schwarz function s (see e.g. [S]), i.e. a function, holomorphic in some neighborhood of Γ, that satisfies

$$s(z) = \bar{z} \tag{3.7}$$

on Γ. We have

Proposition 3.1. *Suppose that there is a point $z_1 \in \mathbf{C} \setminus \overline{\Omega}$ and a simple curve $\gamma\colon [0,1] \mapsto (\mathbf{C} \setminus \overline{\Omega}) \cup \{z_0\}$ connecting z_0 to z_1 such that s extends as a holomorphic function to a neighborhood of γ—by a slight abuse of notation we identify γ with its image—in which the following holds:*
(1) $s'(z_1) = 0$ and s' does not vanish at any other point on γ;
(2) $\bar{s}$ maps $\gamma \setminus \{z_0\}$ into Ω.
If a, b, c in (3.2) and f are in the Vekua class of some neighborhood of $\Omega \cup \gamma$ and $f\big(z_1, s(z_1)\big) \neq 0$ then the solution to the Cauchy problem

$$\begin{cases} \mathscr{L}u = f \\ u(x,y) = |\nabla u(x,y)| = 0 \quad \text{on } \Gamma \end{cases} \tag{3.8}$$

does not extend real-analytically to Ω.

Remark. Proposition 3.1 cannot be used on quadrature domains, because the derivative of the Schwarz function of a quadrature domain does not vanish outside the domain. This can be seen from results in e.g. [E] or [Gu]. However, we know from the results in this paper that the overdetermined Cauchy problems of Williams fail to have solutions in every quadrature domain which is not a disc. It would be interesting to know if any of the overdetermined Cauchy problems considered in this section has a solution in a quadrature domain other than a disc. In order for this to be nontrivial we should ask that a, b, c extend as entire functions of two complex variables and that $f \equiv 1$. Because if E is any entire function of two complex variables such that $\Gamma = \big\{E(z,\bar{z}) = 0\big\}$ bounds a simply connected domain Ω and h is an entire function of one complex variable then $u(z) = h(z)E(z,\bar{z})^2$ extends as an entire function of two complex variables into $\mathbf{C}^2$ with vanishing Cauchy data on the complexified variety $\hat{\Gamma} = \big\{E(z,w) = 0\big\}$. Consequently, the solution of (3.8) in this case, with $f = \Delta u$, always extends into Ω. It is easy to check, however, that this choice of f always vanishes at the characteristic points of $\hat{\Gamma}$.

Proof. Let Ω_1 be a simply connected neighborhood of $\gamma \setminus \{z_1\}$ in which s is holomorphic and s' does not vanish. Also, choose it so small that $\Omega' = \Omega \cup \Omega_1$ is simply connected, $\partial\Omega \cap \Omega_1 \subseteq \Gamma$ and a, b, c and f are in the Vekua class of some neighborhood of $\Omega' \cup \{z_1\}$. We make the change of coordinates (3.3) in $\mathbf{C}^2$ and think of $\mathbf{C}$ as the complex line $\{w = \bar{z}\}$. Let $\mathscr{C}$ and $\mathscr{D}$ be the complex analytic curves defined by $\big\{\big(z, s(z)\big) : z \in \Omega_1\big\}$ and $\big\{\big(\overline{s(\bar{z})}, z\big) : \bar{z} \in \Omega_1\big\}$. By (3.7) these fit together to form a complex analytic curve $\hat{\Gamma}$, the intersection of which with $\mathbf{C}$ contains Γ. In view of hypothesis (2) $\hat{\Gamma}$ is contained in $\widehat{\Omega'}$. Clearly, $\hat{\Gamma}$ is everywhere regular and nowhere characteristic w.r.t. $\mathscr{L}$. By the Cauchy–Kowalevskaya theorem there is a unique holomorphic solution u to the Cauchy problem

$$\begin{cases} \mathscr{L}u = f \\ u = |\nabla u| = 0 \quad \text{on } \hat{\Gamma} \end{cases} \tag{3.9}$$

in some neighborhood of $\hat{\Gamma}$. We claim that the restriction of u to $\mathbf{C}$ extends real-analytically to Ω_1. To see this we let $R(z,w;z',w')$ be the Riemann function for the operator $\mathscr{L}$ (see [Ga], [Ha] or [He]). In view of our hypotheses on the coefficients of $\mathscr{L}$ the Riemann function R is holomorphic in $\widehat{\Omega'} \times \widehat{\Omega'}$ (see [He], p. 179). We introduce the function

$$h(z',w') = \frac{1}{4}\int_{z^*}^{z'}\int_{w^*}^{w'} f(z,w)R(z,w;z',w')\,dz\,dw, \tag{3.10}$$

where z^* and $\bar{w}^*$ are arbitrary points in Ω'. It is clear that h is holomorphic in $\widehat{\Omega'}$ and that $v = u - h$ solves the Cauchy problem (see [Ga], p. 141)

$$\begin{cases} \mathscr{L}v = 0 \\ v + h = |\nabla(v+h)| = 0 \quad \text{on } \hat{\Gamma} \end{cases}. \tag{3.11}$$

To establish our claim it suffices to show that v extends real-analytically to Ω_1. Let (z',w') be a point close to $\hat{\Gamma}$ and let p and q be the points of intersection between $\hat{\Gamma}$ and the complex lines $\{z = z'\}$ and $\{w = w'\}$, respectively. By Riemann's formula (see [Ha], [He] or [K]) the solution v of (3.11) can be represented as follows

$$\begin{aligned} v(z',w') &= \frac{1}{2}\big(v(p)R(p;z',w') + v(q)R(q;z',w')\big) \\ &\quad + \int_p^q \big(P(z,w;z',w')\,dw - Q(z,w;z',w')\,dz\big), \end{aligned} \tag{3.12}$$

where (recall the functions d and e from (3.4))

$$\begin{cases} P(z,w;z',w') = \dfrac{1}{2}\Big(R(z,w;z',w')\dfrac{\partial v}{\partial w}(z,w) - v(z,w)\dfrac{\partial R}{\partial w}(z,w;z',w') \\ \qquad\qquad + d(z,w)v(z,w)R(z,w;z',w')\Big) \\ Q(z,w;z',w') = \dfrac{1}{2}\Big(R(z,w;z',w')\dfrac{\partial v}{\partial z}(z,w) - v(z,w)\dfrac{\partial R}{\partial z}(z,w;z',w') \\ \qquad\qquad - e(z,w)v(z,w)R(z,w;z',w')\Big), \end{cases} \tag{3.13}$$

and the integration in (3.12) is carried out along a curve from p to q on $\hat{\Gamma}$ close to the point (z',w'). Note that the points p and q are well defined and equal to $\big(z', s(z')\big)$ and $\big(\overline{s(\bar{w}')}, w'\big)$ whenever z' and $\bar{w}'$ are in Ω_1, i.e. in a neighborhood of Ω_1 in $\mathbf{C}^2$. Also note that the first homology group $H_1(\hat{\Gamma},\mathbf{Z})$ of $\hat{\Gamma}$ is trivial—this follows from the fact that $\hat{\Gamma} = \mathscr{C} \cup \mathscr{D}$, where $\mathscr{C}$ and $\mathscr{D}$ are homeomorphic to Ω_1 and $\mathscr{C} \cap \mathscr{D}$ is connected, and the Mayer–Vietoris sequence (see [M], p. 186])—and hence, by Stokes' theorem, integration of closed 1-forms (in particular holomorphic differentials) from p to q on $\hat{\Gamma}$ makes sense, i.e. is independent of the curve connecting p and q. The facts that $\hat{\Gamma} \subset \widehat{\Omega'}$ and $v+h = \big|\nabla(v+h)\big| = 0$ on $\hat{\Gamma}$ in combination with the representation (3.12) now imply that v extends holomorphically to some neighborhood of Ω_1 in $\mathbf{C}^2$ and, consequently, the claim above is established. This argument is also sketched in [K]. If we assume that s is univalent in Ω_1 the claim follows from Theorem 4.1 of [KS].

If we assume, in order to get a contradiction, that u extends real-analytically into Ω then, by Corollary 3.5 of [KS], u extends holomorphically to $\widehat{\Omega'}$. Now, since s is holomorphic in a neighborhood Ω_2 of z_1, we may enlarge $\hat{\Gamma}$ slightly by adding the complex analytic curve segment $\big(z, s(z)\big)$ for $z \in \Omega_2$. We denote this enlarged complex analytic curve by $\hat{\Gamma}'$. Define the curve

$$\hat{\gamma}(t) = \big(\gamma(t), s\big(\gamma(t)\big)\big), \qquad t \in [0,1]. \tag{3.14}$$

Clearly, $\hat{\gamma}$ is a curve on $\hat{\Gamma}'$ from $(z_0, \bar{z}_0)$ to $\big(z_1, s(z_1)\big)$. By hypothesis, both $\gamma(t)$ and $\overline{s\big(\gamma(t)\big)}$ belong to Ω' for $t \in [0,1)$. This means that $\hat{\gamma}(t)$ belongs to $\widehat{\Omega'}$ for $t \in [0,1)$. Consequently, $\hat{\gamma}(1) = \big(z_1, s(z_1)\big)$ is on the boundary of $\widehat{\Omega'}$. Since $s'(z_1) = 0$ the point $\big(z_1, s(z_1)\big) \in \hat{\Gamma}'$ is characteristic w.r.t. $\mathscr{L}$. The bicharacteristic of $\mathscr{L}$ tangent to $\hat{\Gamma}'$ at $\big(z_1, s(z_1)\big)$, denoted β_1, is the complex line $\big\{w = s(z_1)\big\}$. Unless $\big(z_1, s(z_1)\big)$ is an exceptional point (in the sense of Leray, see [L], [GKL] or [J]) Leray's local theory regarding existence of solutions to characteristic Cauchy problems (again see [L], [GKL], or Theorem 2.3 of [J]) states that there is a unique function v, holomorphic in $U \setminus \beta_1$ for some neighborhood U of the point, that satisfies $\mathscr{L}v = f$ with vanishing Cauchy data on $\hat{\Gamma}' \setminus \big\{\big(z_1, s(z_1)\big)\big\}$. Since f does not vanish at $\big(z_1, s(z_1)\big)$ it follows from a theorem of Shapiro (see [S], Section 9.2) that v must develop a singularity there. As a consequence of Hartogs' theorem (Corollary 2.8 of [J]) the whole complex line $\beta_1 \cap U$ must be singular—in two dimensions, as in this paper, this can also be seen using elementary methods (cf. [S], Section 9.3). Since $\hat{\gamma} \setminus \big\{\big(z_1, s(z_1)\big)\big\}$ is contained in $\widehat{\Omega'}$ it follows that u and v must agree on $\widehat{\Omega'} \cap (U \setminus \beta_1)$. Also, $\widehat{\Omega'} \cap \beta_1 \cap U$ is not empty—the complex line $\beta_1 \cap U$ is parameterized by $\big(z, s(z_1)\big)$ for z near z_1 and since z_1 is a boundary point of Ω' it follows that there are points on β_1, arbitrarily close to $\big(z_1, s(z_1)\big)$, that are contained in $\widehat{\Omega'}$. We have reached a contradiction to the fact that u is holomorphic in $\widehat{\Omega'}$. Consequently, we just need to establish that $\big(z_1, s(z_1)\big)$ is not an exceptional point. By Lemma 2.2 of [J] it suffices to show that β_1 is not contained in a characteristic subvariety of $\hat{\Gamma}'$ near $\big(z_1, s(z_1)\big)$. But this is clear since any characteristic subvariety of $\hat{\Gamma}'$ consists of isolated points—the characteristic points w.r.t. $\mathscr{L}$ on $\hat{\Gamma}'$ correspond to zeros of s'. □

Remark. The author acknowledges helpful correspondence from D. Khavinson in connection with this proof.

Example 3.1. We introduce a family of algebraic curves on which we can use Proposition 3.1. Let $P_k(z,w)$ be defined by

$$P_k(z,w) = Az^{2k} + 2Bz^k w^k + Aw^{2k} - 1, \tag{3.15}$$

where k is a positive integer and

$$\begin{cases} A = \dfrac{1}{4}\Big(\dfrac{1}{a^2} - \dfrac{1}{b^2}\Big) \\ B = \dfrac{1}{4}\Big(\dfrac{1}{a^2} + \dfrac{1}{b^2}\Big), \end{cases} \tag{3.16}$$

for some positive numbers a and b (we may assume that $a > b$). Let Γ_k be the intersection between the zero locus of P_k and $\mathbf{C}$.

The curve Γ_1 is an ellipse with halfaxes a and b. For $k > 1$ the curve Γ_k is the inverse image under $z \mapsto z^k$ of the above mentioned ellipse Γ_1. The mapping properties of the algebraic function $S_k(z)$, one branch of which is the Schwarz function s_k of Γ_k, defined by $P_k(z,w) = 0$ were investigated in [E]. If $k = 1$ the Schwarz function s_1 is holomorphic outside a line segment connecting the foci F_- and F_+ of Γ_1 and s_1' vanishes at two points z_- and z_+ located on the real axis, symmetrically about the imaginary axis, outside $\Omega_1 = \{$*the domain inside* $\Gamma_1\}$. Also, $\bar{s}_1$ maps the segment of the real axis connecting e.g. z_- and Γ_1 to the segment of the real axis connecting F_- and Γ_1. Consequently, we may choose z_1 to be z_-, γ to be the segment of the real axis connecting z_1 to the point z_0 on Γ_1 and apply Proposition 3.1 to the ellipse.

For $k > 1$ the Schwarz function s_k of Γ_k satisfies

$$s_k(z) = \big(s_1(z^k)\big)^{1/k} \tag{3.17}$$

for a suitable choice of branch of the kth root. Moreover, the points z_- and z_+ mentioned above are located on a confocal ellipse Γ_1^T (see [E] for an exact location of Γ_1^T) outside Γ_1 and the only zeros of s_1 are the points of intersection between this ellipse and the imaginary axis. We deduce from (3.17) that any choice of points and connecting curve among the inverse images of z_1, z_0 and γ under $z \mapsto z^k$ will suffice to satisfy the hypotheses of Proposition 3.1. Let Γ_k^T be the inverse image of Γ_1^T under $z \mapsto z^k$. We have proved the following

Theorem 3.1. *Suppose that A and $B > \max(A,0)$ are real numbers. Let $\Omega_k = \{$the domain inside $\Gamma_k\}$ and $\Omega_k^T = \{$the domain inside $\Gamma_k^T\}$, where Γ_k and Γ_k^T are as defined above, for a positive integer k. If a, b, c in (3.2) and f are in the Vekua class of some neighborhood of $\overline{\Omega_k^T}$ and if $f(z_1,w_1) \neq 0$, where $(z_1,w_1) = \big((z_*)^{1/k}, (F_*)^{1/k}\big)$ for some choice of branch of the kth root and $* \in \{-,+\}$, then the overdetermined Cauchy problem*

$$\begin{cases} \mathscr{L}u = f & \text{in } \Omega_k \\ u = |\nabla u| = 0 & \text{on } \Gamma_k \end{cases} \tag{3.18}$$

has no solution.

We could choose $a \equiv b \equiv 0$, $c \equiv \alpha$ for some $\alpha \in \mathbf{C}^*$ and $f \equiv 1$. By the result of Williams mentioned in the introduction we have the following

Corollary 3.1. *The domain Ω_k has the Pompeiu property.*

We can use Proposition 3.1 to prove a result in which the assumptions are made on the conformal map of the domain onto the unit disc $\mathbf{D}$ instead of on the Schwarz function of the boundary. Using this result we construct another class of domains in which the overdetermined Cauchy problems corresponding to (3.18) have no solution.

Again, suppose that Ω is a bounded simply connected domain. Let φ be a conformal map of Ω onto $\mathbf{D}$. Let us also assume that $\Gamma = \partial\Omega$ is a nonsingular analytic curve so that φ extends holomorphically to some neighborhood Ω' of $\overline{\Omega}$.

Theorem 3.2. *Suppose that φ takes in $\Omega' \setminus \overline{\Omega}$ no value in $\overline{\mathbf{D}}$. If there is a point z_1 in Ω' such that $\varphi'(z_1) = 0$ then the overdetermined Cauchy problem*

$$\begin{cases} \mathscr{L}u = f & \text{in } \Omega \\ u = |\nabla u| = 0 & \text{on } \Gamma, \end{cases} \tag{3.19}$$

where a, b, c and f are in the Vekua class of some neighborhood of Ω' and f does not vanish in the Vekua hull of this neighborhood (in fact, it suffices to assume that f does not vanish at a certain point; cf. Proposition 3.1), has no solution.

Proof. We assert that the Schwarz function s of Γ extends holomorphically to $\Omega' \setminus V$, where V is some closed set contained in Ω, maps $\Omega' \setminus \overline{\Omega}$ into $\{z : \bar{z} \in \Omega\}$ and that $s'(z_1) = 0$. The theorem then follows from Proposition 3.1. To show the assertion we observe that the following identity

$$s(z) = \overline{\varphi^{-1}\big(1/\overline{\varphi(z)}\big)}, \tag{3.20}$$

where φ^{-1} is the inverse of φ defined in some neighborhood U of $\overline{\mathbf{D}}$, holds on Γ (both sides are equal to $\bar{z}$). Also, since φ maps $\Omega' \setminus \overline{\Omega}$ to the outside of $\overline{\mathbf{D}}$ the right side of (3.20) is well defined and holomorphic in $\Omega' \setminus V$, where V is the image under φ^{-1} of the closed set inside $\mathbf{D}$ which is reflected ($w \mapsto 1/\bar{w}$) to the outside of U, and, hence, s is holomorpic in $\Omega' \setminus V$ and the identity (3.20) holds there. Clearly, V is contained in Ω and $\bar{s}$ maps $\Omega' \setminus \overline{\Omega}$ into Ω. We just need to check that $s'(z_1) = 0$, but this follows immediately from (3.20) by differentiating both sides and recalling that $\varphi'(z_1) = 0$. □

As above this theorem has the corollary:

Corollary 3.2. *If Ω satisfies the hypotheses of Theorem 3.2 then Ω has the Pompeiu property.*

Next, we give some examples of domains that satisfy the hypotheses of Theorem 3.2.

Example 3.2. An immediate example is obtained by choosing a function f which is holomorphic in some neighborhood of the closed unit disc $\overline{\mathbf{D}}$, univalent in $\mathbf{D}$ and which has a vanishing derivative at $z = 1$, e.g. $f(z) = (z-1)^2$. Then, any domain Ω with $\overline{\Omega} \subset \mathbf{D}$ for which $f(\Omega)$ is a disc satisfies the hypothesis of Theorem 3.2. Using $f(z) = (z-1)^2$ we deduce, from Corollary 3.2, that the inverse image (under f) of any disc D with $\overline{D}$ contained in the conchoid $f(\mathbf{D})$ has the Pompeiu property.

Example 3.3. If we specialize Theorem 2.1 to simply connected domains it can be reformulated as follows: Any domain which is the conformal image of the unit disc under a rational function, other than a Möbius transformation, has the Pompeiu property. In this example we show that if the rational function "goes the other way" (see Theorem 3.3) then Ω satisfies the hypotheses of Theorem 3.2 and, consequently, it has the Pompeiu property.

Let Ω be a bounded simply connected domain with nonsingular analytic boundary and suppose that f, the conformal map of Ω onto $\mathbf{D}$, is a rational function, but not a Möbius transformation. Since the boundary of Ω is regular f is univalent in some neighborhood V of $\overline{\Omega}$. Consider f as a holomorphic map of the Riemann sphere (or complex projective line) $\mathbf{P}$ onto itself and let Ω' be the component of $f^{-1}(\mathbf{D}')$, where $\mathbf{D}' = \mathbf{P} \setminus \overline{\mathbf{D}}$, that meets V. Since Ω' is maximal each point in $\mathbf{D}'$ has a neighborhood the inverse image (under f) of which is relatively compact in Ω', i.e. the closure is a compact subset of Ω'. This means that f makes Ω' into a complete covering surface of $\mathbf{D}'$ (see [AS], p. 42). Since f is no Möbius transformation and f is univalent in V it follows that $f^{-1}(\mathbf{D})$ consists of two disjoint open sets (not necessarily connected, though, since $f^{-1}(\mathbf{D})$ may have several components) the closures of which are also disjoint. Consequently, the boundary of Ω' is disconnected and, thus, Ω' is not simply connected. The fact that $\mathbf{D}'$ is simply connected implies (the monodromy theorem in [AS], p. 31, and the argument on p. 41 in [AS]) that the covering f must be ramified in at least two points. This means that there is at least one finite point z_1 in Ω' such that $f'(z_1) = 0$. We state this as a lemma:

Lemma 3.1. *Let f be a rational function, other than a Möbius transformation, and suppose that f maps a domain V conformally onto a neighborhood of the unit disc $\mathbf{D}$. Then there is a neighborhood U of $\overline{V}$ such that f takes in $U \setminus \overline{V}$ no value in $\mathbf{D}$ and a point z_1 in $U \setminus \overline{V}$ such that $f'(z_1) = 0$.*

This lemma shows that the following is true:

Theorem 3.3. *Let Ω be a bounded simply connected domain with nonsingular analytic boundary and suppose that f, the conformal map of Ω onto $\mathbf{D}$, is a rational function other than a Möbius transformation. Then Ω satisfies the hypotheses of Theorem 3.2 and, in particular, Ω has the Pompeiu property.*

We remark that the arguments in this example carry through even if we just assume that f is meromorphic in some domain $U \subset \mathbf{P}$ with $\overline{\Omega} \subset U$. In this case, however, we need to postulate that Ω' is relatively compact in U.

4. Concluding remarks

At the time when the work in this paper was carried out the author was unaware of the paper [GS2] and, consequently, some results may overlap. The main result, concerning the Pompeiu problem, in that paper states that if a holomorphic parameterization of $\partial\Omega$, e.g. the conformal map φ of $\mathbf{D}$ onto Ω extended across and restricted to $\partial\mathbf{D}$ (by assumption, Ω has a nonsingular analytic boundary), extends holomorphically to a neighborhood of $\partial\mathbf{D}$ with a nondegenerate critical point then, under certain additional hypotheses (we refer the reader to [GS2] for the exact formulation), Ω has the Pompeiu property. This result seems to complement our Theorem 3.2, although the approach taken in [GS2] is the same as in Section 2 of this paper, in an interesting way. Recall that one hypothesis on the domain in that theorem is that φ^{-1} extends holomorphically with a critical point outside Ω. However, note that Theorem 3.2 concerns a larger class of overdetermined Cauchy problems than the eigenvalue problems of Williams ($a \equiv b \equiv 0$, $c \equiv \alpha$ and $f \equiv 1$ in (3.19)). An interesting observation is that if $\varphi(z) = e^z$ then neither result applies. Also, Garofalo and Segala claim to have strong evidence that Ω has the Pompeiu property if $\partial\Omega$ is parameterized by a rational function unless, of course, Ω is a disc. This should be compared to Theorem 2.1 in this paper which contains the slightly weaker statement that any simply connected domain which is the conformal image of the unit disc under a rational function, other than a Möbius transformation, has the Pompeiu property.

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