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DUAL RESULTS OF FACTORIZATION FOR OPERATORS

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Abstract. We study the duality properties of the well-known DFJP factorization of operators [3] by means of a refinement of it. Given an operator $T\colon X\to Y$ we consider a decomposition $T=jUk$, where $U\colon E\to F$ is an isomorphism, and j, Uk are the factors in the DFJP factorization.

If T^* is the conjugate operator of T, and $\bar T\colon X^{**}/X\to Y^{**}/Y$ is the operator given by $\bar T(x+X):=T^{**}x+Y$ $(x\in X^{**})$, then we show that the decompositions of T^* and $\bar T$ are precisely $k^*U^*j^*=(jUk)^*$ and $\bar j\bar U\bar k$. From this result we derive several consequences. For example, we detect new operator ideals with the factorization property, we characterize operators whose conjugate is Rosenthal, and using a result of Valdivia [11] we show that an operator T such that $\bar T$ has separable range can be decomposed as $T=S+K$, where S^{**} has separable range and K is weakly compact.

0. Introduction

For a (continuous linear) operator $T\in L(X,Y)$ we shall introduce a decomposition $T=jUk$ in which U is an isomorphism, j is an injective tauberian operator, and k is a cotauberian operator with dense range.

This decomposition is inspired by the equivalent versions of the real interpolation method of Banach spaces [2], and it is a refinement of the well-known DFJP factorization introduced in [3] which factorizes T in two factors: j and Uk. Moreover, the factorization of T in two factors jU and k was considered in [5].

We show that $k^*U^*j^*$ coincides with the decomposition of the conjugate operator $T^*\in L(Y^*,X^*)$, and $\bar j\bar U\bar k$ coincides with the decomposition of the operator $\bar T\in L(X^{**}/X,Y^{**}/Y)$. Moreover, if T belongs to a closed operator ideal $\mathscr{A}$, then k and j belong to the injective hull and the surjective hull of $\mathscr{A}$, respectively. In this way the decomposition of an operator makes clear the duality properties and the symmetry of the DFJP factorization.

As an application we obtain necessary conditions for the factorization property for an operator ideal, and we show that some operator ideals defined in terms of T^* or $\bar T$ verify this property. Also we characterize the class of operators whose

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conjugate is Rosenthal as those operators which factor into a Banach space containing no copies of l_1 and no quotients isomorphic to c_0, and using a result of [11] we show that an operator T such that $\bar{T}$ has separable range can be decomposed as $T = S + K$, where S^{**} has separable range and K is weakly compact.

Notations. X, Y, Z will be Banach spaces, and B_X the closed unit ball of X. For an operator $T \in L(X,Y)$ we shall denote $T^* \in L(Y^*, X^*)$ the conjugate operator, and $\bar{T} \in L(X^{**}/X, Y^{**}/Y)$ the operator defined by

$$\bar{T}(z + X) := T^{**}z + Y \qquad (z \in X^{**}).$$

The properties of $\bar{T}$ have been studied in [12].

An operator ideal $\mathscr{A}$ is said to be *closed* if $\mathscr{A}(X,Y)$ is closed in $L(X,Y)$ for every pair of spaces X, Y.

$\mathscr{A}$ is *injective* if given $T \in L(X,Y)$ and an injection (isomorphism into) $J \in L(Y,Z)$ we have $JT \in \mathscr{A}$ implies $T \in \mathscr{A}$.

$\mathscr{A}$ is *surjective* if given $T \in L(X,Y)$ and a surjection (surjective operator) $Q \in L(Z,X)$ we have $TQ \in \mathscr{A}$ implies $T \in \mathscr{A}$.

$\mathscr{A}$ has the *factorization property* if every $T \in \mathscr{A}$ factors through a Banach space Z such that the identity I_Z belongs to $\mathscr{A}$.

For an account of the theory of operator ideals we refer to [9].

1. Construction of the decomposition

In this section, given Banach spaces X and Y, and an operator $T \in L(X,Y)$ we shall construct the decomposition $T = jUk$.

For each positive integer n we denote

$$p_n(x) := 2^n \|Tx\| + 2^{-n} \|x\| \qquad (x \in X),$$

q_n the gauge of the set $2^n TB_X + 2^{-n} B_Y$.

Clearly p_n and q_n are norms in X and Y respectively, equivalent to the initial ones.

We shall consider also the Banach spaces

$$l_2(X, p_n) := \Big\{ (x_n)/x_n \in X,\ \Big(\sum_{n=1}^{\infty} p_n(x_n)^2 \Big)^{1/2} < \infty \Big\} \quad \text{and} \quad l_2(Y, q_n)$$

endowed with its natural norms.

1.1. Lemma. *For every $(x_k) \in l_2(X, p_n)$, the series $\sum_{n=1}^{\infty} Tx_n$ is absolutely convergent in Y. Moreover*

$$N := \Big\{ (x_n) \in l_2(X, p_n)/ \sum_{n=1}^{\infty} Tx_n = 0 \Big\} \text{ is a closed subspace of } l_2(X, p_n).$$

Proof. Note that

$$\|Tx_k\| < 2^{-k}\,\|(x_n)\| \qquad \text{for every } k \text{ and every } (x_n) \in l_2(X, p_n).$$

Hence the series $\sum_{n=1}^{\infty} Tx_n$ is absolutely convergent, and N is closed.

The above lemma allows us to introduce the intermediate spaces E, F of the decomposition, and two of the factors.

$$\begin{aligned}
&E := l_2(X, p_n)/N, \quad \text{and} \\
&k\colon x \in X \to (x, 0, 0, \dots) + N \in E. \\
&F := \big\{(y_k) \in l_2(Y, q_n)/y_k = y_1 \quad \text{for every integer } k\big\}, \quad \text{and} \\
&j\colon (y, y, y, \dots) \in F \to y \in Y.
\end{aligned}$$

Our next result will allow us to connect j and k.

1.2. Theorem. *The map $U\colon E \to F$ defined by*

$$U\big((x_n) + N\big) := \Big(\sum_{n=1}^{\infty} Tx_n, \sum_{n=1}^{\infty} Tx_n, \dots\Big)$$

is an isomorphism of E onto F.

Proof. First we prove that U is well-defined and continuous.

Fix $(x_n) \in l_2(X, p_n)$. Denoting

$$c_m := \Big\|2^{-m}\sum_{k=1}^{m} x_k\Big\| = \Big\|\sum_{k=0}^{m-1} 2^{-k}2^{-(m-k)}x_{m-k}\Big\|$$

we have

$$\Big(\sum_{n=1}^{\infty} c_n^2\Big)^{1/2} \le \sum_{k=0}^{\infty} 2^{-k}\Big(\sum_{n=1}^{\infty} \|2^{-n}x_n\|\Big)^{1/2} \le 2\big\|(x_n)\big\|.$$

Also, if

$$d_m := \Big\|2^{m}\sum_{k=m+1}^{\infty} Tx_k\Big\| = \Big\|\sum_{k=1}^{\infty} 2^{-k}2^{m+k}Tx_{m+k}\Big\|$$

we have

$$\Big(\sum_{n=1}^{\infty} d_n^2\Big)^{1/2} \le \sum_{k=1}^{\infty} 2^{-k}\Big(\sum_{n=1}^{\infty} \|2^{n}Tx_n\|^2\Big)^{1/2} \le \big\|(x_n)\big\|.$$

Note that $\sum_{k=1}^{m} x_k \in 2^m c_m B_X$; hence $\sum_{k=1}^{m} Tx_k \in 2^m c_m T B_X$. Analogously, $\sum_{k=m+1}^{\infty} Tx_k \in 2^{-m} d_m B_Y$.

Then we have $q_m\big(\sum_{n=1}^{\infty} Tx_n\big) \leq \max\{c_m, d_m\}$; hence

$$\Big(\sum_{m=1}^{\infty} q_m\Big(\sum_{n=1}^{\infty} Tx_n\Big)^2\Big)^{1/2} \leq 2\,\|(x_n)\|\,.$$

We conclude that

$$U\big((x_n)+N\big) = \Big(\sum_{n=1}^{\infty} Tx_n, \sum_{n=1}^{\infty} Tx_n, \ldots\Big) \in l_2(Y, q_n) \qquad \text{and} \qquad \|U\| \leq 2.$$

It follows immediately from the definition that U is injective. It remains only to prove that U is surjective.

Given $(y, y, y, \ldots) \in F$ and denoting $b_n := q_n(y)$, for any $\varepsilon > 0$ we have $y \in (1+\varepsilon)b_n(2^n TB_X + 2^{-n}B_Y)$.

Then we can write $y = Tu_n + v_n$ with $\|u_n\| \leq 2^n(1+\varepsilon)b_n$ and $\|v_n\| \leq 2^{-n}(1+\varepsilon)b_n$.

Note that Tu_n converges to y, since b_n converges to 0. We take $x_1 = u_1$ and $x_n := u_n - u_{n-1}$ for $n > 1$.

Obviously $\sum_{n=1}^{\infty} Tx_n$ converges to y in Y. Moreover, $2^{-n}\,\|x_n\| \leq 2(1+\varepsilon)b_n$ and, since $Tx_n = v_{n-1} - v_n$ for $n > 1$, $2^n\|Tx_n\| \leq 3(1+\varepsilon)b_n$.

In this way we obtain $\big(\sum_{n=1}^{\infty} p_n(x_n)^2\big)^{1/2} \leq 4(1+\varepsilon)\big(\sum_{n=1}^{\infty} q_n(y)^2\big) < \infty$; hence $(x_n) \in l_2(X_n, p_n)$ and $U\big((x_n)+N\big) = (y, y, y, \ldots)$.

The proof is finished.

1.3. Definition. We shall call jUk *the decomposition of* T.

It is immediate to check that $jUkx = Tx$ for every $x \in X$.

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "k"] & Y \\ E \arrow[r, "U"] & F \arrow[u, "j"'] \end{tikzcd}$$

Note that j, Uk are the factors, and F the intermediate space in the DFJP decomposition [3], and jU, k are the factors, and E the intermediate space in the factorization considered in [5].

Next we shall show the duality properties of the decomposition.

Given the conjugate $T^* \in L(Y^*, X^*)$ of $T \in L(X, Y)$, for every positive integer n we shall denote the equivalent norms associated with T^* in the following way.

$p_n^*(g) := 2^n\|T^*g\| + 2^{-n}\,\|g\| \qquad (g \in Y^*)$,

q_n^* the gauge of the set $2^nT^*B_{Y^*} + 2^{-n}B_{X^*}$.

Moreover, for subsets $A \subset X$ and $V \subset X^*$, we shall denote by A° and ${}^\circ V$ the polar subsets given by

$$A^\circ := \{f \in X^* / |f(x)| \leq 1 \text{ for } x \in A\} \qquad \text{and}$$
$${}^\circ V := \{x \in X / |f(x)| \leq 1 \text{ for } f \in V\}.$$

1.4. Lemma. *We have $(X, p_n)^* = (X^*, q_n^*)$ and $(Y, q_n)^* = (Y^*, p_n^*)$ isometrically.*

Proof. Denoting $Z := X \times Y$ endowed with the norm $\|(x,y)\| := \|x\| + \|y\|$, we consider the auxiliary operator $S \in L(X, Z)$ defined by $Sx := (2^{-n}x, 2^n Tx)$, and its conjugate $S^* \in L(Z^*, X^*)$, which is given by $S^*(f,g) = 2^{-n}f + 2^n T^* g$.

We have that the unit ball of (X, p_n) coincides with $S^{-1}B_Z$. Then, as Z^* is $X^* \times Y^*$ with the supremum norm, the unit ball of $(X, p_n)^*$ is

$$(S^{-1}B_Z)^\circ = S^* B_{Z^*} = 2^n T^* B_{Y^*} + 2^{-n} B_{X^*},$$

which coincides with the unit ball of (X^*, q_n^*).

The other part of the lemma can be proved in a similar manner.

Next we establish the main duality properties of the decomposition.

1.5. Theorem. *Suppose jUk is the decomposition of T. Then $k^*U^*j^*$ is the decomposition of T^*.*

Proof. We begin showing that j^* coincides with the first term of the decomposition of T^*.

Considering F as a closed subspace of $l_2(Y, q_n)$, we have that j^* acts from Y^* into $l_2(Y, q_n)^* / F^\circ$.

By Lemma 1.4, we can identify $l_2(Y, q_n)^*$ and $l_2(Y^*, p_n^*)$. Moreover it is not difficult to check that under this identification

$$F^\circ = \Big\{(g_k) \in l_2(X^*, p_n^*) / \sum_{n=1}^{\infty} T^* g_n = 0\Big\},$$

and for every $g \in Y^*$ and $(y, y, y, \dots) \in F$ we have

$$(j^*g)(y, y, y, \dots) = g(y) = (g, 0, 0, \dots)(y, y, y, \dots);$$

hence $j^*g = (g, 0, 0, \dots) + F^\circ$, as we wanted to prove.

Next we show that k^* is the third term in the decomposition of T^*.

The operator k^* acts from $E^* = \big(l_2(X, p_n)/N\big)^* = N^\circ$, which can be identified with the subspace

$$\{(f_k) \in l_2(X^*, q_n^*) / f_k = f_1 \text{ for every integer } k\},$$

into X^*. Moreover, for every $(f, f, f, \dots) \in l_2(X^*, q_n^*)$ and $x \in X$ we have $\big(k^*(f, f, f, \dots)\big)(x) = f(x)$; hence $k^*(f, f, f, \dots) = f$.

Finally, U^* is an isomorphism and verifies $k^*U^*j^* = T^*$. Hence it is the second term, since j^* has dense range, and k^* is injective.

1.6. Theorem. *Suppose jUk is the decomposition of T. Then $\bar{j}\bar{U}\bar{k}$ is the decomposition of $\bar{T}$.*

Proof. If Z_n is a sequence of Banach spaces, then the map

$$(w_n) + l_2(Z_n) \in l_2(Z_n^{**})/l_2(Z_n) \to (w_n + Z_n) \in l_2(Z_n^{**}/Z_n)$$

defines a bijective isometry.

Also, for a closed subspace M of X, we can identify

$$M^{**}/M = (M^{\circ\circ} + X)/X, \qquad \text{and} \qquad (X/M)^{**}/(X/M) = X^{**}/(M^{\circ\circ} + X).$$

The operator $\bar{j}$ acts from F^{**}/F into Y^{**}/Y, and we have

$$F^{\circ\circ} = \{(z_k) \in l_2(Y^{**}, q_n)/z_k = z_1 \text{ for every } k\}$$

and

$$F^{**}/F = \{(z_k + Y_k) \in l_2\big((Y, q_n)^{**}\big)/l_2(Y, q_n) \ / \ z_k = z_1 \text{ for every } k\};$$

hence $\bar{j}(z_1 + Y_k) = j^{**}(z_1) + Y = z_1 + Y$, and we conclude that $\bar{j}$ coincides with the third term in the decomposition of $\bar{T}$.

Analogously, we can identify E^{**}/E with $l_2\big((X, p_n)^{**}\big)/\big(N^{\circ\circ} + l_2(X, p_n)\big)$, and for every $z \in X^{**}$ we have

$$\bar{k}(z + X) = k^{**}z + \big(N^{\circ\circ} + l_2(X, p_n)\big) = (z, 0, 0, \ldots) + \big(N^{\circ\circ} + l_2(X, p_n)\big).$$

Hence $\bar{k}$ is the first term in the decomposition of $\bar{T}$, and proceeding as in the last theorem, we can show that $\bar{U}$ is the second term.

We finish the section showing some additional properties of j and k. Recall that $T\colon X \to Y$ is *tauberian* if $T^{**^{-1}}Y = X$ [8], and it is *cotauberian* if T^* is tauberian [10]. We note that T is tauberian if and only if $\bar{T}$ is injective [12], and T is cotauberian if and only if $\bar{T}$ has dense range [10].

1.7. Proposition. *Suppose jUk is the decomposition of T. Then*
(a) *j is tauberian injective,*
(b) *k is cotauberian with dense range.*

Proof. It is enough to note that $j \in L(F, Y)$ is injective, $k \in L(X, E)$ has dense range, and $\bar{j}\bar{U}\bar{k}$ is the decomposition of $\bar{T}$.

2. Applications

In this section we apply the decomposition to obtain some results for operator ideals [9]. In particular we prove the interpolation property (stronger than the factorization property) for some operator ideals.

2.1. Proposition. *Let n be a positive integer.*
(a) $\|kx\| \leq 2^n\|Tx\| + 2^{-n}\|x\|$ *for every* $x \in X$.
(b) $jB_F \subset 2^nTB_X + 2^{-n}B_Y$.

Proof. (a) It is enough to note that

$$(x, 0, \ldots, 0, 0, \ldots) = (0, 0, \ldots, x, 0, \ldots) + (x, 0, \ldots, -x, 0, \ldots)$$

with $(x, 0, \ldots, -x, 0, \ldots) \in N$.

(b) Clearly $(y, y, y, \ldots) \in B_F$ implies $q_n(y) < 1$ for every n; i.e., $y \in 2^nTB_X + 2^{-n}B_Y$ for every n.

Given operator ideals $\mathscr{A}$ and $\mathscr{B}$, the product $\mathscr{A} \circ \mathscr{B}$ is an operator ideal defined by

$$\mathscr{A} \circ \mathscr{B}(X, Y) := \big\{T \in L(X, Y)/T = AB \text{ for some } A \in \mathscr{A}, B \in \mathscr{B}\big\}.$$

2.2. Proposition. *Let $\mathscr{A}$, $\mathscr{B}$ be closed operator ideals.*
(a) *If $\mathscr{A}$ is injective and $T \in \mathscr{A}$, then $k \in \mathscr{A}$.*
(b) *If $\mathscr{B}$ is surjective and $T \in \mathscr{B}$, then $j \in \mathscr{B}$.*
(c) *If $\mathscr{A}$ is injective and $\mathscr{B}$ is surjective, then $\mathscr{B} \cap \mathscr{A} = \mathscr{B} \circ \mathscr{A}$.*

Proof. (a) and (b) can be derived from the last proposition, using the characterization of the closed injective hull of an operator ideal in [7; 20.7.3], and the corresponding characterization of the closed surjective hull, respectively.

(c) is a consequence of (a) and (b):

$$\text{If } T \in \mathscr{A} \cap \mathscr{B} \text{ then } j \in B \text{ and } k \in \mathscr{A}; \quad \text{hence } jUk = T \in \mathscr{B} \circ \mathscr{A},$$

and the converse implication is evident. This part was proved in [6].

We shall consider now for operator ideals a more restrictive property than the factorization property.

2.3. Definition. An operator ideal $\mathscr{A}$ has the *interpolation property* if the identity I_F of the intermediate space in the DFJP factorization of T belongs to $\mathscr{A}$ when $T \in \mathscr{A}$.

The decomposition can be applied to show that some operator ideals have the interpolation property, as in the following result, proved in [6] using real interpolation techniques. The proof we shall give is more elementary.

2.4. Theorem. *Let $\mathscr{A}$ be an injective and surjective operator ideal verifying the Σ_2-condition: For all sequences of Banach spaces (X_n) and (Y_n), an operator $S\colon l_2(X_n) \to l_2(Y_n)$ belongs to $\mathscr{A}$ when the component operators $S_{mn}\colon X_n \to Y_m$ in the matricial representation of S belong to $\mathscr{A}$.*

Then $\mathscr{A}$ has the interpolation property.

Proof. Let Q and J denote the quotient map onto $l_2(X, p_n)/N$ and the injection of F into $l_2(Y, q_n)$ respectively, and suppose jUk is the decomposition of T.

An easy computation shows that the components of the matricial representation of the operator $JUQ\colon l_2(X, p_n) \to l_2(Y, q_n)$ coincide with $T\colon (X, p_n) \to (Y, q_m)$; then $JUQ \in \mathscr{A}$, because $\mathscr{A}$ satisfies the Σ_2-condition; hence $U \in \mathscr{A}$, since $\mathscr{A}$ is injective and surjective.

Before continuing, we present some examples.

2.5. Examples. The following operator ideals have the interpolation property.

(a) Operators with finite dimensional range.

(b) Operators with separable range.

(c) Weakly compact operators.

(d) Rosenthal operators.

(e) Banach–Saks operators.

(f) Decomposing operators.

Proof. (a) and (b) are immediate.

(c), (d), (e) and (f) follow Theorem 2.4 (see [6]).

As an application of the first part, we present two procedures of construction of operator ideals which preserve the interpolation property.

2.6. Theorem. *Let $\mathscr{A}$ be an operator ideal with the interpolation property. The operator ideals defined by*

$$\mathscr{A}^d(X, Y) := \{T \in L(X, Y)/T^* \in \mathscr{A}(Y^*, X^*)\}$$

and

$$\mathscr{A}^{\mathrm{co}}(X, Y) := \{T \in L(X, Y)/\bar{T} \in \mathscr{A}(X^{**}/X, Y^{**}/Y)\}$$

have the interpolation property.

Proof. It is easy to verify that $\mathscr{A}^d$ and $\mathscr{A}^{\mathrm{co}}$ are operator ideals, and the interpolation property follows from Theorems 1.5 and 1.6, respectively.

We derive some consequences.

2.7. Corollary. *A quasi-weakly compact operator T; i.e., an operator such that $\bar{T}$ has finite dimensional range, factors through a quasi-reflexive space.*

This is the main result in [1].

2.8. Corollary. *Let $T \in L(X,Y)$. T^* is Rosenthal if and only if T factors through a Banach space containing no subspaces isomorphic to l_1 and no quotients isomorphic to c_0.*

Proof. It follows from the interpolation property of Rosenthal operators, Theorem 1.5 and the fact [4] that E^* contains no copies of l_1 if and only if E contains no copies of l_1 and no quotients isomorphic to c_0.

2.9. Corollary. *Let $T \in L(X,Y)$. $\bar{T}$ has separable range if and only it is $T = S + A$, with A, $S \in L(X,Y)$, $R(S^{**})$ separable and A weakly compact.*

Proof. Since $\bar{T}$ has separable range, the intermediate space E^{**}/E is separable. Then E is isomorphic to $E_1 \times E_2$, with E_1 reflexive and E_2^{**} separable [11]. Now denoting by P the projection onto E_1 along E_2, it is enough to take $A := jUPk$ and $S := jU(I-P)k$, where jUk denotes the decomposition of T.

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