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ON THE RELATIVE SCHOENFLIES THEOREM

Jouni Luukkainen

University of Helsinki, Department of Mathematics
P.O. Box 4 (Hallituskatu 15), FIN-00014 University of Helsinki, Finland; luukkainen@cc.helsinki.fi

Abstract. We prove generalizations of the relative Schoenflies extension theorem for topological, quasiconformal, or bi-Lipschitz embeddings due to Gauld and Väisälä, and show that maximal dilatations and bi-Lipschitz constants of the extensions can be controlled.

1. *Introduction.* The relative Schoenflies theorem of Gauld and Väisälä [GVä, 2.4] for C-embeddings with C one of the categories TOP (just topological), LQC (locally quasiconformal), and LIP (locally bi-Lipschitz) has turned out to be very important in the theory of LQC manifolds and LIP manifolds. In this version of the Schoenflies theorem the given embedding is assumed and the extending embedding claimed to respect a given set Y $(= \mathbf{R}^k$ or $\mathbf{R}^k_+)$ and to be the identity map on another given set Z $(= \emptyset$ or $Y \cap \mathbf{R}^l)$. But before we continue, we remark that just since respecting a given set is preserved in this result and wanting to reserve the word 'relative' for other purposes, we prefer to talk about the respectful Schoenflies theorem rather than the relative Schoenflies theorem, following here the practice of Siebenmann [S, p. 123] and of our earlier papers [L1] and [L3]. In addition to the numerous applications of the Gauld–Väisälä theorem already in [GVä], we used it (with $C = \mathrm{LQC}$ and $Z = \emptyset$) in our recent paper [L3] for deducing from a respectful extension result for quasiconformal homeomorphisms a similar result for LQC homeomorphisms. Unfortunately, thus, the proof in [GVä] contains some small errors and a more serious gap as observed by the author. The gap is in the fourth paragraph, where the proof is not valid for $Y = \mathbf{R}^1_+$. Fortunately, Väisälä could later fill in the gap. The first goal of this paper is to give a revision of the proof of Gauld and Väisälä. We decided not to give a proof consisting of comments on the original proof, which already refers to the proof of the non-respectful Schoenflies theorem of Gauld and Vamanamurthy [GVa, Theorem 3]. Instead, we give a complete detailed proof without indicating the flaws of [GVä]. A detailed proof has also the advantage that we can then easily prove the theorem, Theorem 3, in a mixed-category form and generalize the result in Theorems 12 and 15 to a form which is more general and more symmetric with respect to the family $\{Y, Z\}$.

In Theorems 17 and 19 we prove quantitative versions of Theorems 3, 12, and 15 where maximal dilatations and bi-Lipschitz constants are under control. The

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proofs are based on the latter theorems themselves and on the known respectful quantitative Schoenflies theorem for embeddings near the identity. These results generalize non-respectful results of Tukia and Väisälä [TV2].

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2. *Notation and terminology.* The boundary of a manifold X is denoted by ∂X. If $0 \le k < n$, we identify $\mathbf{R}^k$ with the subspace $\mathbf{R}^k \times \{0\}$ of $\mathbf{R}^n$. We let $\overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$ denote the one-point compactification of $\mathbf{R}^n$ and equip it with the chordal metric, which is LIP equivalent to the Euclidean metric on $\mathbf{R}^n$. Writing $x = (x_1, \dots, x_n)$ for a point $x \in \mathbf{R}^n$, the following subsets of $\mathbf{R}^n$ are defined: $\mathbf{R}^n_+ = \{\, x \mid x_n \ge 0 \,\}$, $\mathbf{R}^{n,k}_+ = \{\, x \in \mathbf{R}^n_+ \mid x_i = 0 \text{ if } i \le n-k \,\}$ for $1 \le k \le n$, $\mathbf{R}^{n,k}_{++} = \{\, x \in \mathbf{R}^{n,k}_+ \mid x_{n-1} \ge 0 \,\}$ for $2 \le k \le n$, $B^n(r) = \{\, x \bigm| |x| < r \,\}$ and $S^{n-1}(r) = \partial \overline{B}^n(r)$ for $r > 0$, $B^n = B^n(1)$, and $S^{n-1} = S^{n-1}(1)$. We let $\mathscr{Y}_1$ be the set of the subsets $\mathbf{R}^1$, $\mathbf{R}^1_+$, $\mathbf{R}^1_- = -\mathbf{R}^1_+$, and $\mathbf{R}^0$ of $\mathbf{R}^1$, and let $\mathscr{Y}_n$ for $n \ge 2$ be the set of all products $Y_1 \times \cdots \times Y_n \subset \mathbf{R}^n$ where $Y_i \in \mathscr{Y}_1$, $i = 1, \dots, n$.

An embedding $f\colon A \to \overline{\mathbf{R}}^n$ of a set $A \subset \overline{\mathbf{R}}^n$ is said to *respect* a set $Y \subset \overline{\mathbf{R}}^n$ if $f^{-1}Y = A \cap Y$. We let id denote various inclusion maps. By $\mathscr{H}$ we denote the group of all self-homeomorphisms of $\mathbf{R}^1_+$. A homeomorphism $\sigma\colon \mathbf{R}^n \to \mathbf{R}^n$ is radial if there is $\sigma_0 \in \mathscr{H}$ such that $\sigma(0) = 0$ and $\sigma(x) = \sigma_0(|x|)x/|x|$ if $x \ne 0$. For $M = \mathbf{R}^n$ or $\mathbf{R}^n_+$ and an open set $U \subset M$, we let $E(U;M)$ denote the set of all embeddings $f\colon U \to M$ which are open, i.e., respect ∂M, and equip it with the compact-open topology. A set $\mathscr{F} \subset E(U;M)$ is called solid if its closure in $E(U;M)$ is compact.

The following concepts for embeddings between metric spaces are the same as in [L1, 2.1] and [TV1, 1.1 and 2.22]: bi-Lipschitz (abbreviated BL), locally BL (LIP), L-BL, L-Lipschitz, locally L-BL, and locally L-Lipschitz for $L \ge 1$, quasisymmetric (QS), locally QS (LQS), η-QS, and locally η-QS for $\eta \in \mathscr{H}$. The basic theory of LIP embeddings is given in [LuV] (using terminology slightly different from ours) and of LQS embeddings in [TV1] and [V2]. In particular, LIP or LQS embeddings of compact spaces are BL or QS, respectively. Moreover, if an embedding $f\colon A \to \mathbf{R}^1$ of an interval $A \subset \mathbf{R}^1$ is η-QS, then it is K-QS in the usual sense (though possibly decreasing) with $K = \eta(1)$, and conversely, if f is K-QS in the usual sense with $K \ge 1$, then it is η-QS with $\eta = \eta_K \in \mathscr{H}$.

Consider an embedding $f\colon A \to \overline{\mathbf{R}}^n$ of a set $A \subset \overline{\mathbf{R}}^n$ with $n \ge 2$ and $A \subset \operatorname{cl} \operatorname{int} A$. If $K \ge 1$ and for each component G of $\operatorname{int} A$ the homeomorphism $f|G\colon G \to fG$ is K-quasiconformal in the sense of [V1], we say that f is quasiconformal (QC) or K-QC. If every point $x \in A$ has an open neighbourhood U in A such that $f|U$ is QC, then f is called locally QC (LQC). If $Y \in \mathscr{Y}_n$, $\dim Y \ge 2$, and f respects Y, we often say that $f|A \cap Y$ is (K-)QC or LQC if the embedding $A \cap Y \to Y$ defined by f is such. Suppose now for $M = \mathbf{R}^n$ or $\mathbf{R}^n_+$ that A and fA are open in M. If f is locally η-QS, then f is $\eta(1)^{n-1}$-

QC by [V1, 34.2]. Conversely, if f is K-QC, then f is locally η-QS with $\eta \in \mathscr{H}$ depending only on (n, K); the case $M = \mathbf{R}^n$ is proved in [V2, 2.4] and implies the case $M = \mathbf{R}^n_+$ by reflection and [V1, 35.2]. Thus, f is LQC if and only if f is LQS. An LQS embedding $f\colon A \to \overline{\mathbf{R}}^1$ of a set $A \subset \overline{\mathbf{R}}^1$ is also said to be LQC.

3. Theorem. *Let $C, C' \in \{\mathrm{TOP}, \mathrm{LQC}, \mathrm{LIP}\}$, let $1 \le l \le k \le n$ be integers, let $Y \in \{\emptyset, \mathbf{R}^k, \mathbf{R}^k_+\}$, let $Z \in \{\emptyset, Y \cap \mathbf{R}^l\}$, and let $A = \overline{B}^n \setminus B^n(1/2)$. Suppose that $e\colon A \to \mathbf{R}^n$ is a C-embedding such that e respects Y and Z, that $e|A \cap Y$ is a C'-embedding, that $e|A \cap Z = \mathrm{id}$, and that $eA \subset \overline{G}$, where G is the bounded component of $\mathbf{R}^n \setminus eS^{n-1}$. Then there is a C-homeomorphism $\hat{e}\colon \overline{B}^n \to \overline{G}$ respecting Y and Z such that $\hat{e}|\overline{B}^n \cap Y$ is a C'-embedding, that $\hat{e}|\overline{B}^n \cap Z = \mathrm{id}$, and that $\hat{e} = e$ on $\overline{B}^n \setminus B^n(3/4)$.*

4. *Remark.* Of course, except in the cases $(C, C') \in \{(\mathrm{TOP}, \mathrm{LQC}), (\mathrm{TOP}, \mathrm{LIP}), (\mathrm{LQC}, \mathrm{LIP})\}$, Theorem 3 would be the same if C' were not present at all; and without C' the theorem is essentially the same as [GVä, 2.4]. In the proof some extra trouble is only caused by the case $(C, C') = (\mathrm{TOP}, \mathrm{LQC})$, with $n \ge 2$, $\dim Y = 1$, and $Z = \emptyset$.

5. *Proof of Theorem* 3. We denote $D = B^n(3/4)$ and $A_0 = \overline{B}^n \setminus D$. We may assume $e|A \setminus S^{n-1}$ to respect ∂Y. Indeed, note that $e|\operatorname{int} A$ respects ∂Y. Thus, by choosing a radial LIP homeomorphism $\sigma\colon \mathbf{R}^n \to \mathbf{R}^n$ with $\sigma A = \overline{B}^n \setminus B^n(2/3)$ and $\sigma|A_0 = \mathrm{id}$ and by replacing initially e by $\sigma^{-1} e \sigma|A$ and finally $\hat{e}$ by $\sigma \hat{e} \sigma^{-1}|\overline{B}^n$ we arrive at the desired situation.

Consider first the case $n = 1$. We may assume that e is LIP near $S^0(1/2)$, resorting to [LeV, Lemma 1] if $\mathrm{LQC} \in \{C, C'\}$. Now we can define $\hat{e}\colon \overline{B}^1 \to \mathbf{R}^1$ to be the extension of e which is affine on $\overline{B}^1(1/2)$ if $Y \in \{\emptyset, \mathbf{R}^1\}$ and linear on $[-1/2, 0]$ and on $[0, 1/2]$ if $Y = \mathbf{R}^1_+$.

From now on we assume $n \ge 2$. Then the proof is a modification of the proof of Theorem 3 of Gauld and Vamanamurthy [GVa] in [GVa, Section 5]. We closely follow the proof of [GVä, 2.4] in the case $n \ge 2$, $C = \mathrm{LQC}$, $Z \ne \emptyset$, and $Y \ne \mathbf{R}^1_+$. We use Greek letters to denote embeddings which are LIP independently of C and C'.

We start with the following analogue of [GVa, Lemma 7].

Claim A. *There is a C-embedding $f\colon \overline{B}^n \to G$ respecting Y and Z such that $f|\overline{B}^n \cap Y$ is a C'-embedding, that $f|\overline{D} \cap Z = \mathrm{id}$, and that $fB^n \cup e[\operatorname{int} A] = G$.*

Proof. All closures are taken in $\overline{\mathbf{R}}^n$. We divide the proof into two cases.

Case 1: $Y \ne \mathbf{R}^1_+$. Let $x_0 = e(7/8)$. Set $X = \overline{\mathbf{R}}^n$ if $Y = \emptyset$; otherwise, set

$$X = \bigcap(\{\overline{Y}, \partial\overline{Y}, \overline{Z}, \partial\overline{Z}\} \setminus \{\emptyset\}) \subset \overline{\mathbf{R}}^n;$$

note that $\overline{Z} \subset \partial\overline{Y}$ whenever $Y = \mathbf{R}^k_+$ and $l < k$. Then X is a positive-dimensional sphere with $\{x_0, \infty\} \subset X$. It is thus possible to choose a LIP homeomorphism $\alpha\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ respecting $\overline{Y}$ and $\overline{Z}$ such that $\alpha(\infty) = x_0$, that $B_1 = \overline{\mathbf{R}}^n \setminus \alpha B^n$ is a small round ball centred at x_0 and contained in $e[\operatorname{int} A_0]$, and that $\alpha|\overline{D} = \mathrm{id}$ if $Z \neq \emptyset$. Let G_0 denote the bounded component of $\mathbf{R}^n \setminus eS^{n-1}(3/4)$, i.e., $G_0 = G \setminus eA_0$. Then $\overline{G}_0 \cap Z = \overline{D} \cap Z$, and we can choose a point $y_0 \in G_0 \cap X$, e.g., $y_0 = e(2/3)$.

Define a continuous surjection $\beta\colon A_0 \to \overline{B}^n$ by $\beta(x) = (4|x|-3)x/|x|$. Then β defines a LIP homeomorphism $\overline{B}^n \setminus \overline{D} \to \overline{B}^n \setminus \{0\}$, and $\beta S^{n-1}(3/4) = \{0\}$. Clearly, βe^{-1}, defined on eA_0, has a continuous extension to $\overline{G}$ which sends $\overline{G}_0$ onto $\{0\}$. Choose a LIP homeomorphism $\gamma\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ such that $\gamma = \mathrm{id}$ near $\overline{\mathbf{R}}^n \setminus B^n$, γ respects $\overline{Y}$ and $\overline{Z}$, and $\gamma\beta e^{-1}(x_0) = \gamma(1/2) = 0$. Then the point $p = e\beta^{-1}\gamma(0)$ is in X. Define a continuous map $\bar{f}\colon \overline{\mathbf{R}}^n \setminus \{x_0\} \to \overline{\mathbf{R}}^n$ by

$$\bar{f}(x) = \begin{cases} e\beta^{-1}\gamma\beta e^{-1}(x) & \text{if } x \in \overline{G} \setminus \{x_0\} \setminus \overline{G}_0, \\ p & \text{if } x \in \overline{G}_0, \\ x & \text{if } x \in \overline{\mathbf{R}}^n \setminus \overline{G}. \end{cases}$$

Then $\bar{f}$ defines a C-homeomorphism $\hat{f}\colon \overline{\mathbf{R}}^n \setminus \{x_0\} \setminus \overline{G}_0 \to \overline{\mathbf{R}}^n \setminus \overline{G}_0 \setminus \{p\}$ which respects $\overline{Y}$ and $\overline{Z}$. Note that $\hat{f} = \mathrm{id}$ near $\overline{\mathbf{R}}^n \setminus G$ and that $\hat{f}|\hat{f}^{-1}\overline{Y}$ is a C'-embedding.

Choose open round n-balls U, V, W in $\mathbf{R}^n$ such that U and V are centred at p and W at y_0, $\overline{U} \subset V$, $\overline{V} \subset e[\operatorname{int} A_0]$, and $\overline{W} \subset G_0$. Choose a LIP homeomorphism $\delta\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ respecting $\overline{Y}$ and $\overline{Z}$ such that $\delta\overline{W} = \overline{\mathbf{R}}^n \setminus V$ and $\delta|\overline{U} = \mathrm{id}$. Now define a C-embedding $f^*\colon \overline{\mathbf{R}}^n \setminus \{x_0\} \to \overline{\mathbf{R}}^n$ by

$$f^*(x) = \begin{cases} \hat{f}^{-1}\delta\hat{f}(x) & \text{if } x \in \overline{\mathbf{R}}^n \setminus \{x_0\} \setminus \overline{G}_0, \\ x & \text{if } x \in \bar{f}^{-1}U. \end{cases}$$

Here $\hat{f}^{-1}\delta\hat{f}$ is defined because $\delta[\operatorname{im}\hat{f}\,] \subset V \setminus \{p\} \subset \operatorname{im}\hat{f}$, and f^* is well-defined because $\hat{f}^{-1}\delta\hat{f}|\bar{f}^{-1}U \setminus \overline{G}_0 = \mathrm{id}$. Clearly, f^* respects $\overline{Y}$ and $\overline{Z}$, $f^*|\overline{G}_0 = \mathrm{id}$, and $f^*|f^{*-1}\overline{Y}$ is a C'-embedding.

Finally, define a C-embedding $f = f^*\alpha|\overline{B}^n\colon \overline{B}^n \to \mathbf{R}^n$. Then f respects $\overline{Y}$ and $\overline{Z}$, and $f|f^{-1}\overline{Y}$ is a C'-embedding. Since $\alpha = f^* = \mathrm{id}$ on $\overline{D} \cap Z$, we have $f|\overline{D} \cap Z = \mathrm{id}$. Moreover, $f\overline{B}^n \subset \bar{f}^{-1}V \subset G$. Since $fB^n = f^*[\overline{\mathbf{R}}^n \setminus B_1] \supset f^*\overline{G}_0 = \overline{G}_0$, we have $G = fB^n \cup e[\operatorname{int} A_0]$. Thus, f is the desired embedding.

Case 2: $Y = \mathbf{R}^1_+$. This proof is due to Väisälä and published here with his permission. It suffices to make the following modifications to the proof of Case 1.

We omit all conditions related to Y and Z except those involving $\overline{D}$. We do not define X. We require that $\alpha\overline{Y} = [\,0, x_0\,] \subset Y$. We set $y_0 = 0 \in G_0$. We require that $\gamma[\,0, 1/2\,] = [\,0, 1/2\,]$ (with $\gamma(1/2) = 0$). Then $p = x_0$. We require that $V \cap e\beta^{-1}\gamma(1/2, 1\,] = \emptyset$ and that $\delta[\,0, x_0\,] = [\,0, x_0\,]$. Let $I = [\,0, 1\,]$.

It remains to check that f respects Y and Z, i.e., $f^{-1}Y = I$, and that $f|I$ is a C'-embedding. Let $a = e(3/4)$ and $b = e(1)$. Then $eA_0 \cap Y = [\,a, b\,]$ and $\overline{G}_0 \cap Y =$

$[0,a]$. Since βe^{-1} maps (a,x_0) C'-homeomorphically onto $(0,1/2)=\gamma(0,1/2)$, we see that $\hat{f}$ defines a C'-homeomorphism $f_1\colon (a,x_0)\to(a,x_0)$. Moreover, $V\cap \hat{f}(x_0,b]=\emptyset$. Define an open cover $\{I_1,I_2\}$ of I by $I_1=I\cap\alpha^{-1}\bar{f}^{-1}U$ and $I_2=I\setminus\alpha^{-1}\overline{G}_0$.

Consider first $x\in I$. We must show that $y=f(x)\in Y$. If $x\in I_1$, then $y=\alpha(x)\in[0,x_0)$. Suppose $x\in I_2$. Then $\alpha(x)\in(a,x_0)$ and $y=\hat{f}^{-1}\delta\hat{f}\alpha(x)$. Since $\hat{f}(a,x_0)=(a,x_0)$ and $\delta(a,x_0)\subset(a,x_0)$, we get $y\in(a,x_0)$. From this we also see that $f|I\colon I\to Y$ is the C'-embedding $(\alpha|I_1)\cup(f_1^{-1}\delta f_1\alpha|I_2)$.

Consider then $x\in\overline{B}^n$ with $y=f(x)\in Y$. We must show that $x\in I$. If $y\in\overline{G}_0$, then $\alpha(x)=y\in[0,a]$, and therefore $x\in I$. Suppose $y\notin\overline{G}_0$. Then $y=\hat{f}^{-1}\delta\hat{f}\alpha(x)$. Thus $y\in(a,b)\setminus\{x_0\}$. Since $\hat{f}(y)\in V$, we conclude that $y\in(a,x_0)$. Hence, $\delta\hat{f}\alpha(x)\in(a,x_0)$, which implies that $\hat{f}\alpha(x)\in(0,x_0)$, and therefore $\hat{f}\alpha(x)\in(a,x_0)$. Thus, $\alpha(x)\in(a,x_0)$. It follows that $x\in I$, which completes the proof of Claim A. □

Let f be given by Claim A. For $Z=\emptyset$, observe that if X denotes $\mathbf{R}^n$, Y, or ∂Y whenever Y is, respectively, $\emptyset$, $\mathbf{R}^k$, or $\mathbf{R}^k_+$, then $B^n\cap X\not\subset f^{-1}eA$; the initial normalization is needed here if $Y=\mathbf{R}^k_+$. For $Z\neq\emptyset$, note that $\overline{B}^n(1/3)\cap Z\cap f^{-1}eA=\emptyset$ and $\big(\overline{B}^n\setminus\overline{B}^n(1/2)\big)\cap Z\subset f^{-1}e[\operatorname{int}A]$. In all cases, note that $S^{n-1}\subset f^{-1}e[\operatorname{int}A]$.

By these facts, there is a LIP embedding $\varepsilon\colon\overline{D}\to\overline{B}^n$ respecting Y and Z such that $\varepsilon|\overline{D}\cap Z=\mathrm{id}$, $\varepsilon D\cup f^{-1}e[\operatorname{int}A]=\overline{B}^n$, and $\varepsilon B^n(1/3)\cap f^{-1}eA=\emptyset$. Then the C-embedding $g=f\varepsilon\colon\overline{D}\to G$ has the following properties:

(a) $gD\cup e[\operatorname{int}A]=G$,
(b) $g[\overline{D}\setminus B^n(b)]\subset e[B^n\setminus\overline{B}^n(a)]$ for some a, b with $1/2<a<b<3/4$,
(c) $gB^n(1/3)\cap eA=\emptyset$,
(d) g respects Y and Z,
(e) $g|\overline{D}\cap Y$ is a C'-embedding,
(f) $g|\overline{D}\cap Z=\mathrm{id}$.

The inclusion (b) certainly holds with $a=1/2$, $b=3/4$, so it holds for some $a>1/2$, $b<3/4$.

We now apply the procedure of Lemma 9 of [GVa] to the C-embedding $h=g^{-1}e\colon e^{-1}g\overline{D}\to\mathbf{R}^n$. Choose radial LIP homeomorphisms $\zeta,\eta\colon\mathbf{R}^n\to\mathbf{R}^n$ with

$\zeta=\mathrm{id}$ near $\mathbf{R}^n\setminus D$,
$\zeta[\overline{D}\setminus B^n(b)]=\overline{D}\setminus B^n(1/3)$,
$\zeta(x)=x/3b$ for each $x\in\overline{B}^n(b)$,
$\eta(x)=ax$ for each $x\in\mathbf{R}^n\setminus B^n$,
$\eta A=\overline{B}^n(a)\setminus B^n(1/2)$,
$\eta=\mathrm{id}$ near $\overline{B}^n(1/2)$.

Let $A_1 = \overline{B}^n \setminus B^n(a)$, and define a function $\tilde{e}\colon A_1 \to \mathbf{R}^n$ by

$$\tilde{e}(x) = \begin{cases} h\eta(x) & \text{if } x \notin e^{-1}g\overline{D}, \\ h\eta h^{-1}\zeta h(x) & \text{if } x \in e^{-1}g\overline{D} \text{ and } g\zeta h(x) \in eA, \\ \zeta h(x) & \text{if } x \in e^{-1}g\overline{D} \text{ and } g\zeta h(x) \notin eA. \end{cases}$$

Claim B. *The function $\tilde{e}$ is well-defined, $\tilde{e}$ is a C-embedding, $\tilde{e}$ respects Y and Z, $\tilde{e}|A_1 \cap Y$ is a C'-embedding, and $\tilde{e} = \eta\zeta$ on $A_1 \cap Z$.*

Proof. We first verify that $\tilde{e}$ is defined as a function. Let $H = e^{-1}g\overline{D}$. Since $e[\overline{B}^n(a) \setminus B^n(1/2)] \subset gB^n(b)$, we have $\eta A \subset H$. Hence, $\tilde{e}|A_1 \setminus H = h\eta|A_1 \setminus H$ is defined.

Let $H_1 = A_1 \cap H$. From $\zeta h H_1 \subset \zeta\overline{D} = \overline{D}$ it follows that $k = g\zeta h|H_1$ is defined. Let $R_1 = k^{-1}eA \subset H_1$.

Since $\zeta h R_1 \subset hH$, the embedding $\eta h^{-1}\zeta h|R_1$ is defined and maps R_1 to $\eta H \subset \eta A \subset H$. Thus, $\tilde{e}|R_1 = h\eta h^{-1}\zeta h|R_1$ is defined.

Finally, $\tilde{e}|H_1 \setminus R_1 = \zeta h|H_1 \setminus R_1$ is defined.

From $kH_1 \supset g\zeta[\overline{D} \setminus B^n(b)] = g[\overline{D} \setminus B^n(1/3)] \supset gS^{n-1}(3/4) \cup eS^{n-1}(1/2) \subset eA$ it follows that R_1 is the closure of the domain whose boundary components are the $(n-1)$-spheres $S_1 = k^{-1}gS^{n-1}(3/4) = h^{-1}S^{n-1}(3/4)$ and $S_2 = k^{-1}eS^{n-1}(1/2) = h^{-1}\zeta^{-1}hS^{n-1}(1/2)$. Further, $R_2 = \operatorname{cl}(A_1 \setminus H)$ is the closure of the domain whose boundary components are S^{n-1} and S_1. Since $S^{n-1}(a) \subset H_1$ and $kS^{n-1}(a) \subset g\zeta B^n(b) = gB^n(1/3) \subset \mathbf{R}^n \setminus eA$, we conclude that $R_3 = \operatorname{cl}(H_1 \setminus R_1)$ is the closure of the domain whose boundary components are S_2 and $S^{n-1}(a)$.

Since $\zeta = \mathrm{id}$ near $hS_1 = S^{n-1}(3/4)$ and $\eta = \mathrm{id}$ near $h^{-1}\zeta hS_2 = S^{n-1}(1/2)$, we have that $\tilde{e} = h\eta$ near R_2 and that $\tilde{e} = \zeta h$ near R_3. Thus, $\tilde{e}$ is locally a C-embedding. To compare the images of the three closed sets R_1, R_2, and R_3, note that for the images of their boundary components we have

$$\tilde{e}S^{n-1} = h\eta S^{n-1}, \qquad \tilde{e}S_1 \subset h\eta[\operatorname{int} A], \qquad \tilde{e}S_2 = h\eta S^{n-1}(1/2);$$
$$\tilde{e}S_2 = hS^{n-1}(1/2) \subset \mathbf{R}^n \setminus B^n(1/3), \qquad \tilde{e}S^{n-1}(a) \subset \zeta B^n(b) = B^n(1/3).$$

This implies that $\tilde{e}$ is injective and, thus, a C-embedding.

By (d), h and thus $\tilde{e}$, too, respect Y and Z. By (e), $h|h^{-1}Y$ is a C'-embedding. Therefore $\tilde{e}|\tilde{e}^{-1}Y$ is locally and thus also globally a C'-embedding.

We finally verify that $\tilde{e} = \eta\zeta$ on $A_1 \cap Z$. By (f), $h = \mathrm{id}$ on $H \cap Z = \overline{D} \cap A \cap Z$. Consider $x \in A_1 \cap Z$. Suppose first $x \notin H$. Then $\tilde{e}(x) = h\eta(x)$ and $1/2 < |\eta(x)| \le a < 3/4$. Hence, $\tilde{e}(x) = \eta(x)$. We have $\zeta(x) = x$, for otherwise $a \le |x| < 3/4$ implying $x = e^{-1}g(x) \in H$. Thus, $\tilde{e}(x) = \eta\zeta(x)$. Suppose now $x \in R_1$. Then $\tilde{e}(x) = h\eta h^{-1}\zeta h(x)$ and $x \in H$. We get consecutively $h(x) = x$, $|\zeta(x)| \le 3/4$, $h^{-1}\zeta(x) = \zeta(x)$, $|\eta\zeta(x)| < a$, and $h\eta\zeta(x) = \eta\zeta(x)$. Hence, $\tilde{e}(x) = \eta\zeta(x)$. Suppose finally $x \in H_1 \setminus R_1$. Then $\tilde{e}(x) = \zeta h(x) = \zeta(x)$. Since

$|\zeta(x)| \le 3/4$, we have $\zeta(x) = g\zeta h(x) \notin eA \cap Z = A \cap Z$, and therefore $|\zeta(x)| < 1/2$. Hence, $\tilde{e}(x) = \eta\zeta(x)$. □

The embedding $\tilde{e}$ acts in the same way on $S^{n-1}(a)$ as on S^{n-1}; more precisely,

$$\tilde{e}(x) = h\eta(x) = h(ax) \qquad \text{if } x \in S^{n-1}, \tag{6}$$

$$\tilde{e}(x) = \zeta h(x) = h(x)/3b \qquad \text{if } x \in S^{n-1}(a). \tag{7}$$

Note that here $3b > 1$.

Let $A^* = \eta\zeta A_1$. Then $A^* = \eta[\overline{B}^n \setminus B^n(a/3b)] = \overline{B}^n(a) \setminus B^n(a/3b)$ as $a < b$ and $a/3b < 1/3$. The C-embedding $e^* = \tilde{e}\zeta^{-1}\eta^{-1}\colon A^* \to \mathbf{R}^n$ respects Y and Z, is a C'-embedding on $A^* \cap Y$, and the identity on $A^* \cap Z$. Further, e^* acts in the same way on $S^{n-1}(a/3b)$ as on $S^{n-1}(a)$; more precisely,

$$e^*(x) = h(x) = 3b\, e^*(x/3b) \qquad \text{if } x \in S^{n-1}(a). \tag{8}$$

To see this, let $x \in S^{n-1}(a)$. Then $\zeta^{-1}\eta^{-1}(x) = x/a \in S^{n-1}$, which implies $e^*(x) = \tilde{e}(x/a) = h(x)$ by (6), and $\zeta^{-1}\eta^{-1}(x/3b) = x$, which implies $e^*(x/3b) = \tilde{e}(x) = h(x)/3b$ by (7). Thus, (8) obtains. Then it is also clear that 0 belongs to the bounded component of $\mathbf{R}^n \setminus e^*A^*$.

Now by (8) we can uniquely extend e^* to an embedding $e'\colon \overline{B}^n(a) \to \mathbf{R}^n$ such that $e'(0) = 0$ and such that $e'(x) = (3b)^{-j}e^*\big((3b)^j x\big)$ if $x \in \overline{B}^n(a) \setminus \{0\}$ and if $j \ge 0$ is an integer for which $(3b)^j x \in A^*$. Obviously, e' respects Y and Z, and $e'|\overline{B}^n(a) \cap Z = \mathrm{id}$. By (8), $\operatorname{im} e' \subset \overline{D}$ and $ge' = e$ on $S^{n-1}(a)$. We thus obtain a homeomorphism

$$\hat{e} = ge' \cup (e|A_1)\colon \overline{B}^n \to \overline{G}.$$

Then $\hat{e}$ respects Y and Z, is the identity on $\overline{B}^n \cap Z$, and extends $e|A_0$. As some of the above equalities actually hold on larger sets, we have

$$e'(x) = h(3bx)/3b \qquad \text{for each } x \text{ near } S^{n-1}(a/3b), \tag{9}$$

$$\hat{e} = e \qquad \text{near } S^{n-1}(a). \tag{10}$$

We now verify that e' is a C-embedding whenever $C \ne \mathrm{TOP}$. If $C = \mathrm{LIP}$, then e^* is L-BL for some $L \ge 1$ as A^* is compact. Hence, $e'|(3b)^{-j}A^*$ is L-BL for each $j \ge 0$. It easily follows that e' is L-Lipschitz. Choose $r > 0$ with $B^n(r) \subset \operatorname{im} e'$. Then $e'^{-1}|B^n(r)$ is L-Lipschitz, which implies that e'^{-1} is locally L-Lipschitz (in fact, e'^{-1} is easily seen to be even L-Lipschitz). Thus, e' is LIP. If $C = \mathrm{LQC}$, then e' is LQC on $A_2 = \overline{B}^n(a) \setminus B^n\big(a/(3b)^2\big)$ by (9), thus K-QC for some $K \ge 1$ by compactness and [V1, 34.7]. It follows that e' is K-QC outside 0. Hence, e' is K-QC by [V1, 17.3].

By (10) it follows that $\hat{e}$ is a C-embedding.

A similar proof shows that $\hat{e}|\overline{B}^n \cap Y$ is a C'-embedding. Only the verification that $e'|\overline{B}^n(a) \cap Y$ is LQC whenever $C' = \mathrm{LQC}$ and $\dim Y = 1$ needs a separate argument. First, e' is QS on each component of $A_2 \cap Y$ by (9) and [L2, 2.2]. Hence, $e'|\overline{B}^n(a) \cap Y$ is QS by [L2, 2.4 and 2.5]. Thus, $\hat{e}$ satisfies Theorem 3. □

11. *Definitions.* Let $\mathscr{M}_n$ be the family of all non-empty closed topological submanifolds of $\mathbf{R}^n$. For a non-empty subfamily $\mathscr{V} \subset \mathscr{M}_n$ define $\mathscr{V}^\cap = \{\bigcap \mathscr{W} \mid \emptyset \neq \mathscr{W} \subset \mathscr{V}\} \setminus \{\emptyset\}$, $\mathscr{V}^\partial = \{\partial V \mid V \in \mathscr{V}\} \setminus \{\emptyset\}$, and $\mathscr{V}' = \mathscr{V}^\cap \cup \mathscr{V}^\partial \supset \mathscr{V}$. We call $\mathscr{V}$ *admissible* if we can inductively define families $\mathscr{V}^{(j)} \subset \mathscr{M}_n$ for all $j \geq 0$ by setting $\mathscr{V}^{(0)} = \mathscr{V}$ and $\mathscr{V}^{(j+1)} = (\mathscr{V}^{(j)})'$ for $j \geq 0$.

Suppose that $\mathscr{V} \subset \mathscr{Y}_n$ is an admissible family. (We do not know whether every non-empty family $\mathscr{V} \subset \mathscr{Y}_n$ is admissible.) Then $\mathscr{V}^{(j+1)} = \mathscr{V}^{(j)}$ for some $j \geq 0$. Let $\mathscr{V}^* = \mathscr{V}^{(j)}$ and $X_\mathscr{V} = \bigcap \mathscr{V}^* \in \mathscr{V}^*$. On the other hand, let $X_\mathscr{V}^* = X_1 \times \cdots \times X_n \subset \mathbf{R}^n$ where $X_i = \mathbf{R}^1$ if $V_i = \mathbf{R}^1$ for each $V = V_1 \times \cdots \times V_n \in \mathscr{V}$ and $X_i = \mathbf{R}^0$ otherwise. Then $X_\mathscr{V} \supset X_\mathscr{V}^*$, and obviously $X_\mathscr{V}$ is a linear subspace of $\mathbf{R}^n$ if and only if $X_\mathscr{V} = X_\mathscr{V}^*$.

12. Theorem. *Let $n \geq 1$, let M be $\mathbf{R}^n$ or $\mathbf{R}^n_+$, let $\mathscr{V} \subset \mathscr{Y}_n \cap \mathscr{P}(M)$ be an admissible family such that $M \in \mathscr{V}$, that $\partial M \in \mathscr{V}$ if $\partial M \neq \emptyset$, and that $X_\mathscr{V}$ is a non-zero linear subspace of $\mathbf{R}^n$, let $A = \bigl(\overline{B}^n \setminus B^n(1/2)\bigr) \cap M$, and let $e\colon A \to M$ be an embedding respecting each $V \in \mathscr{V}$ such that $eA \subset \overline{G}$, where G is the bounded component of $M \setminus e[S^{n-1} \cap M]$. Then there is a homeomorphism $\hat{e}\colon \overline{B}^n \cap M \to \overline{G}$ respecting each $V \in \mathscr{V}$ such that $\hat{e} = e$ on $\bigl(\overline{B}^n \setminus B^n(3/4)\bigr) \cap M$ and such that the following conditions hold for each $V \in \mathscr{V}$:*

(i) *If $e|A \cap V = \mathrm{id}$, then $\hat{e}|\overline{B}^n \cap V = \mathrm{id}$.*

(ii) *If $C \in \{\mathrm{LQC}, \mathrm{LIP}\}$ and if $e|A \cap V$ is a C-embedding, then $\hat{e}|\overline{B}^n \cap V$ is a C-embedding.*

13. *Remark.* Theorem 12 generalizes Theorem 3 whenever the case $Y = \mathbf{R}^1_+$ is excluded. Theorem 12 applies, in particular, if $M = \mathbf{R}^n_+$ and $\mathscr{V} = \{M, \partial M, Y\}$ where either $Y = \mathbf{R}^{n,k}_+$ with $2 \leq k \leq n$ or $Y = \mathbf{R}^{n,k}_{++}$ with $3 \leq k \leq n$. But the cases $Y = \mathbf{R}^{n,1}_+$ with $n \geq 2$ and $Y = \mathbf{R}^{n,2}_{++}$ with $n \geq 2$ remain open problems; here $X_\mathscr{V} = \mathbf{R}^0$ as also in Theorem 15, which treats the case $Y = \mathbf{R}^1_+$.

14. *Proof of Theorem* 12. If $n = 1$, the theorem reduces to the special case $n = 1$ and $Y = \mathbf{R}^1$ of Theorem 3. Suppose $n \geq 2$. Only obvious slight modifications to the proof of Theorem 3 are needed. Since e respects ∂M, it defines an open embedding $\operatorname{int}_M A \to M$, which then respects each $V \in \mathscr{V}^*$ and thus, in particular, $X_\mathscr{V}$. The initial normalization now guarantees that $e|A \setminus S^{n-1}$ similarly respects each $V \in \mathscr{V}^*$. We may assume that $\mathbf{R}^1 \subset X_\mathscr{V}$.

In Claim A we require that there is an embedding $f\colon \overline{B}^n \cap M \to G$ which respects each $V \in \mathscr{V}$ such that $f[B^n \cap M] \cup e[\operatorname{int}_M A] = G$ and such that (i) and (ii) are true for each $V \in \mathscr{V}$ when $\hat{e}|\overline{B}^n \cap V$ is replaced by $f|\overline{D} \cap V$ in (i) and by $f|\overline{B}^n \cap V$ in (ii).

In the proof of Claim A (Case 1), we define $X = \overline{X}_\mathscr{V}$. We choose α with $\alpha\overline{V} = \overline{V}$ if $V \in \mathscr{V}$ and with $\alpha|\overline{D} = \mathrm{id}$ if $e|A \cap X = \mathrm{id}$. Since $\{0, 1/2\} \subset X$, we can choose γ such that $\gamma\overline{V} = \overline{V}$ if $V \in \mathscr{V}$. Since $\{y_0, p\} \subset X$, we can choose δ such that $\delta\overline{V} = \overline{V}$ if $V \in \mathscr{V}$. Then $\hat{f}$, f^*, and f respect $\overline{V}$ if $V \in \mathscr{V}$. In (i),

$\overline{G}_0 \cap V = \overline{D} \cap V$, and so f satisfies (i). Clearly f satisfies (ii).

After Claim A, we choose a LIP embedding $\varepsilon\colon \overline{D} \cap M \to \overline{B}^n \cap M$ which respects each $V \in \mathscr{V}$ such that $\varepsilon|\varepsilon^{-1}V = \mathrm{id}$ in (i), that $\varepsilon[D \cap M] \cup f^{-1}e[\mathrm{int}_M A] = \overline{B}^n \cap M$, and that $\varepsilon[B^n(1/3) \cap M] \cap f^{-1}eA = \emptyset$. This is possible because $B^n \cap X_{\mathscr{V}} \not\subset f^{-1}eA$ and because $\overline{B}^n(1/3) \cap V \cap f^{-1}eA = \emptyset$ and $\left(\overline{B}^n \setminus \overline{B}^n(1/2)\right) \cap V \subset f^{-1}e[\mathrm{int}_M A]$ whenever $V \in \mathscr{V}$ is as in (i).

The rest of the proof now goes as for Theorem 3. □

15. Theorem. *The modification of Theorem* 12 *holds where* $M = \mathbf{R}^n_+$ *with* $n \geq 1$ *and* $\mathscr{V} = \{M, \partial M, \mathbf{R}^1_+\}$ *and where* $V \neq \mathbf{R}^1_+$ *in* (ii) *if* $n = 2$.

Proof. If $n = 1$, the proof goes as for Theorem 3. If $n \geq 2$, the proof is as for Theorem 12. In the proof of Claim A—Case 1 if $n = 2$, Case 2 if $n \geq 3$—we require that the embeddings α, γ, $\hat{f}$, δ, f^*, and f respect M and ∂M but not necessarily $\mathbf{R}^1_+$. If $n = 2$, we can choose $\alpha(0) = 0 \in G_0$; then f respects $\mathbf{R}^1_+$. If $n \geq 3$, the existence of γ follows from $\dim \partial M \geq 2$. □

16. *Quantitative versions.* The following theorem is a quantitative version of Theorem 3. There we construct extensions $\hat{e}$ whose maximal dilatation or bi-Lipschitz constant has an upper bound which depends only on n and the respective constant of e. The theorem is a respectful version of the quantitative quasiconformal Schoenflies theorem [TV2, 5.4] and of the quantitative bi-Lipschitz Schoenflies theorem [TV2, 5.10]. To prove it we modify the proofs of these two theorems; in particular, we follow the proof of [TV2, 5.3] simplifying it in our special case. The idea of the proof is after a normalization to glue e to a modification of the extension u given by Theorem 3 for an embedding e_0 which belongs to a fixed finite set of embeddings satisfying the same assumptions as e and which is close enough to e. The modification is done by composing u with the extension of $e_0^{-1}e$ given by the respectful quantitative (canonical) Schoenflies theorem for embeddings near id proved in [L1, 2.13]. In [TV2] the auxiliary result used is the non-elementary deformation theorem of Sullivan. The finite set is found by the solidity of the set of all embeddings in question. In Theorem 19 we give quantitative versions of Theorems 12 and 15. In Theorem 21 the case $eS^{n-1} = S^{n-1}$ is considered by the aid of a respectful quasiconformal extension result of the author in [L3].

17. Theorem. *To Theorem* 3 *with* $C = C' = \mathrm{TOP}$ *we can add any one of the following conditions:*

(a) *Let* $L \geq 1$ *and* e *be* L-BL. *Then* $\hat{e}$ *is* $\hat{L}$-BL *with* $\hat{L} = \hat{L}(n, L) \geq 1$.

(b) *Let* $n \geq 2$, $L \geq 1$, *and* e *be locally* L-BL. *Then* $\hat{e}$ *is locally* $\hat{L}$-BL *with* $\hat{L} = \hat{L}(n, L) \geq 1$.

(c) *Let* $n \geq 2$, $K \geq 1$, *and* e *be* K-QC. *Then* $\hat{e}$ *is* $\hat{K}$-QC *with* $\hat{K} = \hat{K}(n, K) \geq 1$.

(c$'$) *As* (c) *but let, in addition,* (c1) e *be* LIP *or* (c2) $e|A \cap Y$ *be* LIP *or* (c3) $\lambda \geq 1$ *and* $e|A \cap Y$ *be locally* λ-BL, *or* (c4) $\lambda \geq 1$ *and* $e|A \cap Y$ *be* λ-BL. *Then, in* *addition and respectively,* (c1) $\hat{e}$ *is* LIP *or* (c2) $\hat{e}|\overline{B}^n \cap Y$ *is* LIP *or* (c3) $\hat{e}|\overline{B}^n \cap Y$ *is locally* $\hat{\lambda}$-BL, *or* (c4) $\hat{e}|\overline{B}^n \cap Y$ *is* $\hat{\lambda}$-BL *with* $\hat{\lambda} = \hat{\lambda}(n, K, \lambda) \geq 1$, *provided that in* (c3) *and* (c4) *we allow* $\hat{K}$ *to depend also on* λ.

(d) *If* $n = 1$, $\eta \in \mathscr{H}$, *and* e *is* η-QS, *then* $\hat{e}$ *is* $\hat{\eta}$-QS *with* $\hat{\eta} \in \mathscr{H}$ *depending only on* η.

(d$'$) *As* (d) *but let, in addition,* (d1) e *be* LIP *or* (d2) $e|A \cap Y$ *be* LIP, *or* (d3) $\lambda \geq 1$ *and* $e|A \cap Y$ *be locally* λ-BL. *Then, in addition and respectively,* (d1) $\hat{e}$ *is* LIP *or* (d2) $\hat{e}|\overline{B}^1 \cap Y$ *is* LIP, *or* (d3) $\hat{e}|\overline{B}^1 \cap Y$ *is* $\hat{\lambda}$-BL *with* $\hat{\lambda} = \hat{\lambda}(\eta, \lambda) \geq 1$, *provided that in* (d3) *we allow* $\hat{\eta}$ *to depend also on* λ.

Proof. We begin by applying Theorem 3 with $3/4$ replaced by $2/3$. This is possible by choosing a radial BL homeomorphism $\sigma\colon \mathbf{R}^n \to \mathbf{R}^n$ with $\sigma A = A$ and $\sigma S^{n-1}(3/4) = S^{n-1}(2/3)$ and by replacing initially e by $\sigma^{-1} e \sigma | A$ and finally $\hat{e}$ by $\sigma \hat{e} \sigma^{-1} | \overline{B}^n$. Thus, let $U = B^n \setminus \overline{B}^n(2/3)$, and let e^* in place of $\hat{e}$ be the extension of $e|\overline{U}$ that Theorem 3 gives if (C, C') is $(\mathrm{LIP}, \mathrm{TOP})$ in (a), (b), (c1), and (d1), $(\mathrm{LQC}, \mathrm{TOP})$ in (c) and (d), and $(\mathrm{LQC}, \mathrm{LIP})$ in (c2)–(c4) and (d2)–(d3). We may assume that $e^*(0) = 0$. In fact, this is the case if $Z \neq \emptyset$ or if $Y = \mathbf{R}^1_+$. Otherwise, letting X denote $\mathbf{R}^n$, Y, or ∂Y whenever Y is, respectively, $\emptyset$, $\mathbf{R}^k$, or $\mathbf{R}^k_+$, we have $y = e^*(0) \in X$; then by replacing initially e by $e - y$ and e^* by $e^* - y$ and finally $\hat{e}$ by $\hat{e} + y$ we may assume that $y = 0$. In (c)–(c2) and (d)–(d2) we may assume also in the case $Z = \emptyset$ that $e(4/5) = 4/5$ by replacing initially e by αe and e^* by αe^* and finally $\hat{e}$ by $\alpha^{-1} \hat{e}$ for a suitable linear similarity $\alpha\colon \mathbf{R}^n \to \mathbf{R}^n$ respecting Y. In (c3), (c4), and (d3) we may assume that $Y \neq \emptyset$. We let $\mathscr{C}$ denote the set of all the 11 conditions (a)–(d3). We may allow $\hat{L}$, $\hat{K}$, $\hat{\eta}$, and $\hat{\lambda}$ to depend also on Y and Z.

Let $\mathscr{F}$ be the set of all embeddings $f\colon \overline{B}^n \to \mathbf{R}^n$ respecting Y and Z such that $f|\overline{B}^n \cap Z = \mathrm{id}$ and $f(0) = 0$. We define a set $\mathscr{F}_\nu \subset \mathscr{F}$ for each $(\nu) \in \mathscr{C}$ as follows. For $L \geq 1$ we let $\mathscr{F}_a = \{ f \in \mathscr{F} \mid f|\overline{U} \text{ is } L\text{-BL},\ f \text{ is BL} \}$ if $n \geq 1$ and $\mathscr{F}_b = \{ f \in \mathscr{F} \mid f|\overline{U} \text{ is locally } L\text{-BL},\ f \text{ is BL} \}$ if $n \geq 2$. For $n \geq 2$, $K \geq 1$, and $\lambda \geq 1$ we let $\mathscr{F}_{c0} = \{ f \in \mathscr{F} \mid f|\overline{U} \text{ is } K\text{-QC},\ f \text{ is QC} \}$, $\mathscr{F}_c = \{ f \in \mathscr{F}_{c0} \mid f(4/5) = 4/5 \}$, $\mathscr{F}_{c1} = \{ f \in \mathscr{F}_c \mid f \text{ is LIP} \}$, $\mathscr{F}_{c2} = \{ f \in \mathscr{F}_c \mid f|\overline{B}^n \cap Y \text{ is LIP} \}$, $\mathscr{F}_{c3} = \{ f \in \mathscr{F}_{c0} \mid f|\overline{U} \cap Y \text{ is locally } \lambda\text{-BL},\ f|\overline{B}^n \cap Y \text{ is BL} \}$, and $\mathscr{F}_{c4} = \{ f \in \mathscr{F}_{c3} \mid f|\overline{U} \cap Y \text{ is } \lambda\text{-BL} \}$. For $n = 1$, $\eta \in \mathscr{H}$, and $\lambda \geq 1$ we let $\mathscr{F}_{d0} = \{ f \in \mathscr{F} \mid f|\overline{U} \text{ is } \eta\text{-QS},\ f \text{ is QS} \}$, $\mathscr{F}_d = \{ f \in \mathscr{F}_{d0} \mid f(4/5) = 4/5 \}$, $\mathscr{F}_{d1} = \{ f \in \mathscr{F}_d \mid f \text{ is LIP} \}$, $\mathscr{F}_{d2} = \{ f \in \mathscr{F}_d \mid f|\overline{B}^1 \cap Y \text{ is LIP} \}$, and $\mathscr{F}_{d3} = \{ f \in \mathscr{F}_{d0} \mid f|\overline{U} \cap Y \text{ is locally } \lambda\text{-BL},\ f|\overline{B}^1 \cap Y \text{ is BL} \}$. For each $(\nu) \in \mathscr{C}$ we define $\mathscr{G}_\nu = \{ f|U \mid f \in \mathscr{F}_\nu \}$. Then $e^* \in \mathscr{F}_\nu$ and hence $e|U \in \mathscr{G}_\nu$ in each $(\nu) \in \mathscr{C}$.

We now show that $\mathscr{G}_\nu$ is solid for each $(\nu) \in \mathscr{C}$. If $g \in \mathscr{G}_a$, then g is L-BL and $gU \subset \overline{B}^n(2L)$; thus, $\mathscr{G}_a$ is solid by Ascoli's theorem. If $g \in \mathscr{G}_b$, then g is $(L\pi/2)$-Lipschitz, $gU \subset \overline{B}^n(L\pi)$, and $g|T$ is L_T-BL for each compact set $T \subset U$ with $L_T = L_T(n, T, L) \geq 1$ by [TV2, 2.17]; thus, $\mathscr{G}_b$ is solid by Ascoli's theorem. If $g \in \mathscr{G}_c$, then g is K-QC and omits 0 and ∞, $g(4/5) = 4/5$, and $d\big(gS^{n-1}(4/5)\big) \geq 4/5$; thus, $\mathscr{G}_c$ and hence $\mathscr{G}_{c1}$ and $\mathscr{G}_{c2}$, too, are solid by [V1, 19.3, 20.5, and 21.9]. If $g \in \mathscr{G}_{c3}$ and $T = \overline{B}^n(4/5) \setminus B^n(3/4)$, then g is K-QC and omits 0 and ∞, $g|T$ is η-QS with $\eta \in \mathscr{H}$ depending only on (n, K) by [V2, 2.7], $g|[3/4, 4/5]$ is L-BL with $L = L(k, \lambda) \geq 1$ by [TV2, 2.17], and $|g(3/4)| \leq d(gT) \leq 2\eta\big(|(-4/5) - 3/4|/|4/5 - 3/4|\big)|g(4/5) - g(3/4)| \leq \eta(31)L/10$; thus, $\mathscr{G}_{c3}$ and hence $\mathscr{G}_{c4}$, too, are solid by [V1, 19.3, 20.5, and 21.9] again. If $g \in \mathscr{G}_d$, then g is η-QS and $g(-3/4) < 0 < g(3/4) < g(4/5) = 4/5$; thus, $\mathscr{G}_d$ and hence $\mathscr{G}_{d1}$ and $\mathscr{G}_{d2}$, too, are solid by [TV1, 3.4, 3.6, and 3.7]. Finally, if $g \in \mathscr{G}_{d3}$, then g is η-QS, $|g(3/4)| \leq |g(-3/4) - g(3/4)| \leq \eta(30)|g(4/5) - g(3/4)|$ and $|g(4/5) - g(3/4)| \leq \lambda/20$; thus, $\mathscr{G}_{d3}$ is solid by [TV1, 3.4, 3.6, and 3.7] again.

Choose numbers $2/3 < r_2' < r_1' < 3/4 < r_1 < r_2 < 1$ independently of n, let $R_i = B^n(r_i) \setminus \overline{B}^n(r_i')$ for $i = 1, 2$, and let $B_1 = B^n(r_1)$. We apply, as we may, [L1, 2.13] with the radii $1/3$, $2/3$, 1 replaced by r_1', $3/4$, r_1, respectively; let $\mathscr{P}$ be the neighbourhood of id in $E(R_1; \mathbf{R}^n)$ and $\varphi\colon \mathscr{P} \to E(B_1; \mathbf{R}^n)$ the continuous map thus obtained. Choose $\varepsilon > 0$ with $\varepsilon \leq d(R_1, \mathbf{R}^n \setminus R_2)$ such that the set $N = \{\, h \in E(R_1; \mathbf{R}^n) \bigm| |h(x) - x| < \varepsilon \text{ if } x \in R_1 \,\}$ is contained in $\mathscr{P}$.

Consider a condition $(\nu) \in \mathscr{C}$. Since $\mathscr{G}_\nu$ is solid, there is $\delta > 0$ such that $|g(x) - g(y)| \geq \delta$ whenever $g \in \mathscr{G}_\nu$, $x, y \in \overline{R}_2$, and $|x - y| \geq \varepsilon$. Furthermore, there is a finite set $\mathscr{U}_\nu \subset \mathscr{F}_\nu$ such that the sets $N_u = \{\, g \in \mathscr{G}_\nu \bigm| |g(x) - u(x)| < \delta$ if $x \in R_1 \,\}$ for $u \in \mathscr{U}_\nu$ cover $\mathscr{G}_\nu$. In (a) and (b) we choose $L_0 = L_0(\mathscr{U}_\nu) \geq L$ such that each $u \in \mathscr{U}_\nu$ is L_0-BL. In (c)–(c4) we choose $K_0 = K_0(\mathscr{U}_\nu) \geq K$ such that each $u \in \mathscr{U}_\nu$ is K_0-QC. In (d)–(d3) we choose $\eta_0 \in \mathscr{H}$ depending only on $\mathscr{U}_\nu$ such that each $u \in \mathscr{U}_\nu$ is η_0-QS. In (c3), (c4), and (d3) we choose $\lambda_0 = \lambda_0(\mathscr{U}_\nu) \geq \lambda$ such that $u|\overline{B}^n \cap Y$ is λ_0-BL for each $u \in \mathscr{U}_\nu$.

Let $(\nu) \in \mathscr{C}$, and let e satisfy the assumptions of (ν). We now construct $\hat{e}$ satisfying the conclusions of (ν). Choose $u \in \mathscr{U}_\nu$ with $e|U \in N_u$. Since $d(uR_1, u[\partial \overline{R}_2]) \geq \delta$, we get that $eR_1 \subset uR_2$. Setting $h = u^{-1}e|R_1$ we thus obtain an embedding $h\colon R_1 \to R_2$, which belongs to N and hence to $\mathscr{P}$. This yields an embedding $h_1 = \varphi(h)\colon B_1 \to \mathbf{R}^n$ such that $h_1 = h$ on $B_1 \setminus B^n(3/4)$. We have that h and hence h_1, too, respect Y and Z, that $h|R_1 \cap Z = \mathrm{id}$, and that, hence, $h_1|B_1 \cap Z = \mathrm{id}$. Moreover, in (a), h is L^2-BL and h_1 thus L_1-BL with $L_1 = L_1(L) \geq 1$; in (b), h is locally L^2-BL and h_1 thus locally L_1-BL with L_1 as above; in (c)–(c4), h is K^2-QC and h_1 thus K_1-QC with $K_1 = K_1(n, K) \geq 1$; in (d)–(d3), h is $\eta'\eta$-QS, where $\eta' \in \mathscr{H}$ satisfies $\eta'(t) = \eta^{-1}(t^{-1})^{-1}$ for $t > 0$, and h_1 thus η_1-QS with $\eta_1 \in \mathscr{H}$ depending only on η; in (c1) and (d1), h and h_1 are LIP; in (c2) and (d2), $h|R_1 \cap Y$ and $h_1|B_1 \cap Y$ are LIP; in (c3) and (d3), $h|R_1 \cap Y$ is locally λ^2-BL and $h_1|B_1 \cap Y$ thus locally λ_1-BL with $\lambda_1 = \lambda_1(\lambda) \geq 1$; and finally in (c4), $h|R_1 \cap Y$ is λ^2-BL and $h_1|B_1 \cap Y$ thus λ_1-BL with λ_1 as above. We define a homeomorphism $\hat{e}\colon \overline{B}^n \to \overline{G}$ by

$$\hat{e}(x) = \begin{cases} e(x) & \text{if } x \in \overline{B}^n \setminus B^n(3/4), \\ u\big(h_1(x)\big) & \text{if } x \in B_1. \end{cases}$$

Then $\hat{e}$ respects Y and Z, and $\hat{e}|\overline{B}^n \cap Z = \mathrm{id}$. Moreover, $\hat{e}$ satisfies (ν): In (a) and (b), $\hat{L} = L_0 L_1$ applies, by the lemma at the end of the proof of [TV2, 3.4] when (a) is in question; in (c)–(c4), $\hat{K} = K_0 K_1$ applies; in (d)–(d3), by [L2, 2.2] there is $\hat{\eta} \in \mathscr{H}$ depending only on $(\eta, \eta_0\eta_1)$ such that $\hat{e}$ is $\hat{\eta}$-QS; in (c1) and (d1), $\hat{e}$ is LIP; in (c2) and (d2), $\hat{e}|\overline{B}^n \cap Y$ is LIP; and finally in (c3), (c4), and (d3), $\hat{\lambda} = \lambda_0\lambda_1$ applies. □

18. *Remarks.* 1. In the conditions (b) and (c) of Theorem 17, $n = 1$ must really be excluded if being K-QC is interpreted as being K-QS on each component of A. For example, define for each $t > 0$ an embedding $e\colon \overline{B}^1 \setminus B^1(1/2) \to \mathbf{R}^1$ by $e(x) = x - t$ if $x < 0$ and by $e(x) = x$ if $x > 0$. Then $e|[-1, -1/2]$ and $e|[1/2, 1]$ are isometric. If now $\hat{e}\colon \overline{B}^1 \to \mathbf{R}^1$ is an extension of $e|S^0$ such that $\hat{e}$ is locally L-BL (and thus L-BL) or such that $\hat{e}$ is K-QC and, in addition, $\hat{e}(0) = 0$ or $\hat{e}(3/4) = e(3/4)$, then neither L nor K can be independent of t.

2. The condition (b) of Theorem 17 easily implies that if in it, in addition, $\lambda \geq 1$ and $e|S^{n-1} \cap Y$ is λ-BL, then $\hat{e}|\overline{B}^n \cap Y$ is $\hat{\lambda}$-BL with $\hat{\lambda} = \max(\hat{L}, \lambda)$.

3. In the conditions (c) and (c′) of Theorem 17, quasisymmetry can be substituted for quasiconformality; this follows from (c) and (c′) by [V3, 3.12].

19. Theorem. *To Theorems* 12 *and* 15 *with* $n \geq 2$ *we can add any one of the conditions* (a), (b), *and* (c) *of Theorem* 17.

Proof. With the following changes the proof is otherwise essentially the same as that of Theorem 17. Let $\mathscr{V}_0 = \{\, V \in \mathscr{V} \mid e|A \cap V = \mathrm{id} \,\}$. Let $\mathscr{V}_1 = \{\, V \in \mathscr{V} \mid e|A \cap V$ is LIP $\}$; however, if $n = 2$ and $\mathscr{V}_1 = \{\mathbf{R}^1_+\}$, redefine $\mathscr{V}_1 = \emptyset$. We may allow $\hat{L}$ and $\hat{K}$ to depend also on $\mathscr{V}_0$ and $\mathscr{V}_1$. Replace every occurring subset of $\mathbf{R}^n$ by its intersection with M. Let e^* be the extension of $e|\overline{U}$ given by the pertinent one of Theorems 12 and 15. If $X_{\mathscr{V}} = \mathbf{R}^0$ or $\mathscr{V}_0 \neq \emptyset$, then $e^*(0) = 0$; otherwise, $y = e^*(0) \in X_{\mathscr{V}}$, and by replacing $(e, e^*, \hat{e})$ by $(e - y, e^* - y, \hat{e} + y)$ we may assume that $y = 0$. In Theorem 12 we may assume that $4/5 \in X_{\mathscr{V}}$. Thus, in (c) we may assume that $e(4/5) = 4/5$ by replacing $(e, e^*, \hat{e})$ by $(\alpha e, \alpha e^*, \alpha^{-1}\hat{e})$ for a suitable linear similarity $\alpha\colon M \to M$ respecting each $V \in \mathscr{V}$. Let $\mathscr{C} = \{(\mathrm{a}), (\mathrm{b}), (\mathrm{c})\}$.

Let $\mathscr{F}$ be the set of all embeddings $f\colon \overline{B}^n \cap M \to M$ respecting each $V \in \mathscr{V}$ such that $f|\overline{B}^n \cap V = \mathrm{id}$ for each $V \in \mathscr{V}_0$, that $f|\overline{B}^n \cap V$ is LIP for each $V \in \mathscr{V}_1$, and that $f(0) = 0$. Define $\mathscr{F}_\nu$ and $\mathscr{G}_\nu$ for $(\nu) \in \mathscr{C}$ as earlier; then $e^* \in \mathscr{F}_\nu$ and $e|U \in \mathscr{G}_\nu$ in each $(\nu) \in \mathscr{C}$. In the case $M = \mathbf{R}^n_+$, if $g \in \mathscr{G}_b$ or $g \in \mathscr{G}_c$, then the extension $B^n \setminus \overline{B}^n(2/3) \to \mathbf{R}^n$ of g by reflection is again, respectively, locally L-BL or K-QC; hence, it can be proved as earlier that $\mathscr{G}_\nu$ is solid in $E(U; M)$ for each $(\nu) \in \mathscr{C}$.

We let $\mathscr{P}$ be the neighbourhood of id in $E(R_1; M)$ and $\varphi\colon \mathscr{P} \to E(B_1; M)$ the continuous map given by [L1, 2.13]. We have that $\hat{e}$ satisfies (ν). Moreover, $\hat{e}$ respects each $V \in \mathscr{V}$, $\hat{e}|\overline{B}^n \cap V = \mathrm{id}$ for each $V \in \mathscr{V}_0$, and $\hat{e}|\overline{B}^n \cap V$ is LIP for each $V \in \mathscr{V}_1$; hence, $\hat{e}$ satisfies the pertinent one of Theorems 12 and 15. □

20. *Remark.* Concerning the assumption $n \geq 2$ in Theorem 19, note that the quantitative version of the case $M = \mathbf{R}^1$ of Theorem 12 is already included in Theorem 17 and that, as it is easy to see, in the case $M = \mathbf{R}^1_+$ neither (a) nor (d) can be added to Theorem 15.

21. Theorem. *Let $n \geq 3$, let $Y = \mathbf{R}^k$ with $1 \leq k \leq n$ or $Y = \mathbf{R}^k_+$ with $2 \leq k \leq n$, let $K \geq 1$, let $0 < r < 1$, and let $f\colon \overline{B}^n \setminus B^n(r) \to \overline{B}^n$ be a K-QC embedding respecting Y such that $fS^{n-1} = S^{n-1}$. Then there is a K_1-QC homeomorphism $g\colon \overline{B}^n \to \overline{B}^n$ respecting Y and extending $f|S^{n-1}$ with $K_1 = K_1(n,K) \geq 1$.*

Proof. All closures are taken in $\overline{\mathbf{R}}^n$. Let $Y_1 = \overline{\mathbf{R}}^{n,k}_+$ if $Y = \mathbf{R}^k$ and $Y_1 = \overline{\mathbf{R}}^{n,k}_{++}$ if $Y = \mathbf{R}^k_+$. Choose Möbius transformations φ and ψ of $\overline{\mathbf{R}}^n$ such that $\varphi\overline{B}^n = \psi\overline{B}^n = \overline{\mathbf{R}}^n_+$, that $\varphi[\overline{B}^n \cap Y] = \psi[\overline{B}^n \cap Y] = Y_1$, and that $\varphi f \psi^{-1}(\infty) = \infty$. Since $\varphi f \psi^{-1}|\psi[\overline{B}^n \setminus B^n(r)]$ is K-QC and respects Y_1, its restriction $f_1\colon \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ is K-QC by [G] and respects Y_1. Hence, by [L3, 3.1] there is a K_1-QC homeomorphism $g_1\colon \overline{\mathbf{R}}^n_+ \to \overline{\mathbf{R}}^n_+$ extending f_1 and respecting Y_1 with $K_1 = K_1(n,K) \geq 1$. Then $g = \varphi^{-1} g_1 \psi\colon \overline{B}^n \to \overline{B}^n$ is the desired homeomorphism. □

22. *Remark.* The absolute case $Y = \mathbf{R}^n$ of Theorem 21 is due to Tukia and Väisälä [TV3, 3.14]. For $n = 2$, Theorem 21 with $K_1 = K_1(K,r)$ follows from Theorem 17, but, as an example of Näätänen [N, 5.11] shows, K_1 cannot be independent of r.

References

[GVä] Gauld, D. B., and J. Väisälä: Lipschitz and quasiconformal flattening of spheres and cells. - Ann. Acad. Sci. Fenn. Ser. A I Math. 4, 1978/1979, 371–382.

[GVa] Gauld, D. B., and M. K. Vamanamurthy: Quasiconformal extensions of mappings in n-space. - Ann. Acad. Sci. Fenn. Ser. A I Math. 3, 1977, 229–246.

[G] Gehring, F. W.: Dilatations of quasiconformal boundary correspondences. - Duke Math. J. 39, 1972, 89–95.

[LeV] Lehto, O., and K. I. Virtanen: On the existence of quasiconformal mappings with prescribed complex dilatation. - Ann. Acad. Sci. Fenn. Ser. A I Math. 274, 1960, 1–24.

[L1] Luukkainen, J.: Respectful deformation of bi-Lipschitz and quasisymmetric embeddings. - Ann. Acad. Sci. Fenn. Ser. A I Math. 13, 1988, 137–177.

[L2] Luukkainen, J.: Deformation of one-dimensional quasisymmetric embeddings. - Complex analysis, Joensuu 1987, eds. I. Laine, S. Rickman and T. Sorvali, Lecture Notes in Math. 1351, Springer-Verlag, Berlin, 1988, 236–242.

[L3] Luukkainen, J.: Respectful quasiconformal extension from dimension $n-1$ to n. - Ark. Mat. (to appear).

[LuV] Luukkainen, J., and J. Väisälä: Elements of Lipschitz topology. - Ann. Acad. Sci. Fenn. Ser. A I Math. 3, 1977, 85–122.

[N] Näätänen, M.: Dilatation estimates for quasiconformal extensions. - Israel J. Math. 29, 1978, 346–356.

[S] Siebenmann, L. C.: Deformation of homeomorphisms on stratified sets. - Comment. Math. Helv. 47, 1972, 123–163.

[TV1] Tukia, P., and J. Väisälä: Quasisymmetric embeddings of metric spaces. - Ann. Acad. Sci. Fenn. Ser. A I Math. 5, 1980, 97–114.

[TV2] Tukia, P., and J. Väisälä: Lipschitz and quasiconformal approximation and extension. - Ann. Acad. Sci. Fenn. Ser. A I Math. 6, 1981, 303–342.

[TV3] Tukia, P., and J. Väisälä: Quasiconformal extension from dimension n to $n+1$. - Ann. of Math. (2) 115, 1982, 331–348.

[V1] Väisälä, J.: Lectures on n-dimensional quasiconformal mappings. - Lecture Notes in Math. 229, Springer-Verlag, Berlin, 1971.

[V2] Väisälä, J.: Quasi-symmetric embeddings in Euclidean spaces. - Trans. Amer. Math. Soc. 264, 1981, 191–204.

[V3] Väisälä, J.: Quasisymmetry and unions. - Manuscripta Math. 68, 1990, 101–111.

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