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A CELLULAR PARAMETRIZATION FOR CLOSED SURFACES WITH A DISTINGUISHED POINT

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Abstract. The convex hull construction in Minkowski space is used here to parametrize Dirichlet fundamental polygons by L-lengths of the edges of the corresponding convex hull. The basic conditions have simple geometric interpretations by projection to the Poincaré model, and the presentation is done also in terms of concepts in the Poincaré model. The parametrization leads to a cell-decomposition of conformal structures of closed surfaces of genus g with a distinguished point. A connection between the entries of the matrix of a Möbius transformation and the corresponding L-length, with distinguished point at the origin, is obtained. A necessary and sufficient condition for discreteness is obtained in terms of the matrices of the generators of the group.

1. Introduction

The convex hull construction in Minkowski space for closed surfaces was presented by Näätänen and Penner [3]. Here we use this construction to parametrize Dirichlet fundamental polygons by L-lengths of the edges of the corresponding convex hull. Since the basic conditions have simple geometric interpretations by projection to the Poincaré model, the presentation is done also in terms of concepts in the Poincaré model. The parametrization leads to a cell-decomposition of conformal structures of closed surfaces of genus g with a distinguished point; each top-dimensional (i.e. $(6g-4)$-dimensional) cell corresponding to a fixed combinatorial type of side-pairings of the Dirichlet polygon. The boundaries of the top-dimensional cells correspond to the "degenerate" Dirichlet polygons, and are characterized by Ptolemy equations in Minkowski space [3]. Examples are done for $g = 2$. A connection between the entries of the matrix of a Möbius transformation and the corresponding L-length, with distinguished point at the origin, is derived in Chapter 6, and in Chapter 7, a necessary and sufficient condition for discreteness, written explicitly in the case of genus 2, is obtained in terms of the matrices of the generators of the group. We wish to thank R. Penner, T. Kuusalo, T. Nakanishi for helpful discussions, T. Sorvali and P. Tukia for comments. The pictures have been drawn by J. Haataja and M. Nikunen.

As mentioned above, the parametrization introduced in this work was initially derived by using the hyperboloid model for hyperbolic plane and constructing there the Euclidean convex hull for the orbit of the distinguished point, following ideas

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in [3] and [4]. Then two conditions, called the face condition and the closing condition, were instrumental. (The first condition is derived in Chapter 3.) By projecting into the Poincaré model it turns out that the above conditions have simple geometric interpretations. (For the face condition, it is given in Note 3.4, and the closing condition amounts to the angles of certain triangles at the distinguished point adding up to 2π.)

Since the Poincaré model is more commonly used, we have chosen to write this paper so that it can also be read without need to get into the calculations of Chapter 3, starting from Note 3.4. The subsequent chapters are written in terms of the Poincaré model, but in the formulas L-lengths—which initially refer to edges of the convex hull—are used since they make the formulas simpler than the use of hyperbolic lengths would. To translate the formulas into hyperbolic metric, it is only needed to replace each L by $\sqrt{2}\sinh(l/2)$, where l is the corresponding hyperbolic length.

In Chapters 4–7 the results are derived in Poincaré model; short comments are made to indicate the geometric setting in the hyperbolic model.

2. Minkowski space and hyperbolic geometry

Let M denote Minkowski three-space, so that M is a real vector space of dimension three with a bilinear pairing $\langle\cdot,\cdot\rangle$ of signature $(1,2)$. M admits a basis (e_0, e_1, e_2) with $\langle e_i, e_j\rangle = 0$ if $i \neq j$ and $-\langle e_0, e_0\rangle = \langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 1$. The upper sheet

$$H = \{v \in M : \langle v, v\rangle = -1 \text{ and } \langle v, e_0\rangle < 0\}$$

of the hyperboloid inherits a Riemannian metric from the pairing and gives a model for the hyperbolic plane. The positive light-cone is

$$L^+ = \{v \in M : \langle v, v\rangle = 0 \text{ and } \langle v, e_0\rangle < 0\}.$$

The group of linear isomorphisms of M preserving the form, the orientation on M, and the sheet H is denoted by $SO^+(1,2)$, this group corresponds to the Möbius group of orientation-preserving isometries of the hyperbolic plane. An explicit isometry of H with the Poincaré disk model of the hyperbolic plane is given by radial projection from $(-1,0,0)$ to the unit disk D about the origin in the plane at height zero; if $u_1, u_2 \in H$, and ϱ denotes the hyperbolic distance between the projections $\bar{u}_1, \bar{u}_2 \in D$, then

$$\cosh \varrho(\bar{u}_1, \bar{u}_2) = -\langle u_1, u_2\rangle.$$

The Klein model K of the hyperbolic plane is identical with the horizontal unit disk in M at height one, and the natural map $H \to K$ is given by radial projection from the origin.

Suppose that S is an affine plane in M. We say that S is elliptic (parabolic, hyperbolic) if the conic section $S \cap L^+$ has the corresponding attribute. The restriction of the form $\langle \cdot, \cdot \rangle$ to S may be definite, degenerate or of type $(1,1)$. If

$$S = \{u \in M : \langle u, s \rangle = r\}$$

for some $0 \neq s \in M$ and $r \in \mathbf{R}$, then these cases correspond to $\langle s, s \rangle < 0$ (elliptic), $\langle s, s \rangle = 0$ (parabolic); and $\langle s, s \rangle > 0$ (hyperbolic), respectively. In the definite case S has an induced Euclidean structure. For $u_i, u_j \in H$ we denote ${L_{ij}}^2 = -\langle u_i, u_j \rangle - 1$. We call L_{ij} the L-length of the segment (or edge) connecting u_i and u_j. An isometry $I \neq g \in SO^+(1,2)$ preserves an elliptic (parabolic, hyperbolic) affine plane in M if and only if g is elliptic (parabolic, hyperbolic; respectively). If u_1, u_2, u_3 are non-collinear points in M, then we denote by $\pi(u_1, u_2, u_3)$ the affine plane through u_1, u_2, u_3.

3. Auxiliary lemmas for the convex hull

In this chapter, the results in terms of L-lengths turn out to be quite similar to the ones obtained in [4] for λ-lengths. The first lemma gives conditions for the ellipticity of a plane through three distinct points of H.

Lemma 3.1. *Let $u_1, u_2, u_3 \in H$, and $\lambda_{12}, \lambda_{23}, \lambda_{13} \in \mathbf{R}_+$ be such that $-\lambda_{ij}^2 = \langle u_i, u_j \rangle$, for $\{i,j,k\} = \{1,2,3\}$, and let $S = \pi(u_1, u_2, u_3)$. We denote $L_{ij} = \sqrt{\lambda_{ij}^2 - 1}$. Then S is elliptic if and only if the three strict triangle inequalities hold for L_{12}, L_{23}, L_{13}, S is parabolic if and only if*

$$L_{ij} = L_{jk} + L_{ik}$$

for some i, j, k, where $\{i,j,k\} = \{1,2,3\}$, and S is hyperbolic if and only if some non-strict triangle inequality fails amongst L_{12}, L_{23}, L_{13}.

Proof. The tangent space to S is spanned by $v_1 = u_1 - u_3$ and $v_2 = u_2 - u_3$. In order to exhibit the determinant of the bilinear form $\langle \cdot, \cdot \rangle$ on S we calculate $\langle v_i, v_i \rangle = \langle u_i, u_i \rangle - 2\langle u_i, u_3 \rangle + \langle u_3, u_3 \rangle = -2 + 2\lambda_{i3}^2$ for $i = 1, 2$, $\langle v_1, v_2 \rangle = \langle u_1, u_2 \rangle - \langle u_1, u_3 \rangle - \langle u_2, u_3 \rangle + \langle u_3, u_3 \rangle = -1 - \lambda_{12}^2 + \lambda_{13}^2 + \lambda_{23}^2$.

The determinant of the form factors as

$$\left(\sqrt{\lambda_{12}^2 - 1} + \sqrt{\lambda_{23}^2 - 1} - \sqrt{\lambda_{13}^2 - 1}\right)\left(\sqrt{\lambda_{12}^2 - 1} + \sqrt{\lambda_{13}^2 - 1} - \sqrt{\lambda_{23}^2 - 1}\right)$$
$$\left(\sqrt{\lambda_{23}^2 - 1} + \sqrt{\lambda_{13}^2 - 1} - \sqrt{\lambda_{12}^2 - 1}\right)\left(\sqrt{\lambda_{12}^2 - 1} + \sqrt{\lambda_{23}^2 - 1} + \sqrt{\lambda_{13}^2 - 1}\right).$$

After substituting $L_{ij} = \sqrt{\lambda_{ij}^2 - 1}$, this reads

$$(L_{12} + L_{23} - L_{13})(L_{12} + L_{13} - L_{23})(L_{23} + L_{13} - L_{12})(L_{12} + L_{23} + L_{13}).$$

At most one of these factors is not strictly positive, hence the claim follows.

The second lemma provides a way to construct a triangle with all vertices in H, starting from a fixed edge and prescribing L-lengths for the two remaining edges.

Lemma 3.2. *Let $u_1, u_2 \in H$, and $\lambda_{12}, \lambda_{23}, \lambda_{13} \in \mathbf{R}_+$ be such that $\langle u_1, u_2 \rangle = -\lambda_{12}^2$. If all triangle inequalities hold for L_{12}, L_{23}, L_{13}, then there exists a unique $u_3 \in H$ on each side of the plane $\pi(0, u_1, u_2)$ so that*

$$\langle u_2, u_3 \rangle = -\lambda_{23}^2, \qquad \langle u_1, u_3 \rangle = -\lambda_{13}^2.$$

Proof. The linear subspace W spanned by u_1, u_2 has type $(1,1)$ (consider the basis $u_1 + u_2, u_1 - u_2$), and so $W^\perp$ has type $(0,1)$. Let e be a vector in $W^\perp$ with $\langle e, e \rangle = 1$ and solve the coefficients $\alpha_1, \alpha_2, \beta$ of $u_3 = \alpha_1 u_1 + \alpha_2 u_2 + \beta e$: By using the equations

$$\begin{aligned} -\lambda_{23}^2 &= \langle u_2, u_3 \rangle = -\alpha_1 \lambda_{12}^2 - \alpha_2, \\ -\lambda_{13}^2 &= \langle u_1, u_3 \rangle = -\alpha_1 - \alpha_2 \lambda_{12}^2, \end{aligned}$$

$$-1 = \langle u_3, u_3 \rangle = -\alpha_1^2 - \alpha_2^2 - 2\alpha_1\alpha_2\lambda_{12}^2 + \beta^2 = -\lambda_{23}^2\alpha_2 - \lambda_{13}^2\alpha_1 + \beta^2$$

we get, after substituting $L_{ij}^2 = \lambda_{ij}^2 - 1$,

$$\begin{aligned} \alpha_1 &= \frac{L_{23}^2 L_{12}^2 + L_{23}^2 + L_{12}^2 - L_{13}^2}{L_{12}^4 + 2L_{12}^2}, \\ \alpha_2 &= \frac{L_{13}^2 L_{12}^2 + L_{13}^2 + L_{12}^2 - L_{23}^2}{L_{12}^4 + 2L_{12}^2}, \end{aligned}$$

$$\begin{aligned} \beta^2 = \Big(&(L_{23} + L_{13} - L_{12})(L_{23} + L_{12} - L_{13})(L_{13} + L_{12} - L_{23})(L_{12} + L_{13} + L_{23}) \\ &+ 2L_{12}^2 L_{13}^2 L_{23}^2\Big) / \big(L_{12}^2(L_{12}^2 + 2)\big). \end{aligned}$$

To see that $u_3 \in H$ (instead of $-H$), note that $(u_1 + u_2)^\perp$ is of type $(0,2)$ separating H from $-H$. The condition for $x \in H$ is $\langle x, u_1 + u_2 \rangle < 0$ and $\langle x, x \rangle = -1$. We have $\langle u_3, u_1 + u_2 \rangle = -\lambda_{23}^2 - \lambda_{13}^2 < 0$. The sign of β determines which side of $\pi(0, u_1, u_2)$ u_3 lies on.

The third lemma gives a necessary and sufficient condition for four points in H to be coplanar, in terms of the L-lengths of the segments connecting the points. We also derive the condition corresponding to the face condition in [4].

Lemma 3.3. *Suppose that $u_1, u_2, u_3, u_4 \in H$ are such that any three are linearly independent, u_1 and u_4 lie on different sides of $\pi(0, u_2, u_3)$ and let*

$$-\lambda_{ij}^2 = \langle u_i, u_j \rangle, \qquad L_{ij}^2 = \lambda_{ij}^2 - 1$$

for $i < j$. Then u_4 lies in $\pi(u_1, u_2, u_3)$ if and only if

$$L_{12}L_{13}(L_{24}^2 + L_{34}^2 - L_{23}^2) + L_{24}L_{34}(L_{12}^2 + L_{13}^2 - L_{23}^2) = 0. \tag{1}$$

Furthermore, u_4 lies above $\pi(u_1, u_2, u_3)$ if and only if

$$L_{12}L_{13}(L_{24}^2 + L_{34}^2 - L_{23}^2) + L_{24}L_{34}(L_{12}^2 + L_{13}^2 - L_{23}^2) > 0. \tag{2}$$

Proof. Since u_1, u_2, u_3 are linearly independent, we may write

$$\pi_1 = \pi(u_1, u_2, u_3) = \{x \in M | \langle x, s_1 \rangle = -1\}$$

for some $0 \neq s_1 \in M$. Similarly,

$$\pi_2 = \pi(u_2, u_3, u_4) = \{x \in M | \langle x, s_2 \rangle = -1\}$$

for some $0 \neq s_2 \in M$. We assume $\pi_1 \neq \pi_2$ and denote $L = \pi_1 \cap \pi_2 \cap \pi(0, u_2, u_3)$. The line L divides the plane π_1 into two half-planes, which lie on different sides of π_2; above if $\langle x, s_2 \rangle \leq -1$ and below if $\langle x, s_2 \rangle \geq -1$. The situation is symmetric with respect to π_1 and π_2. By assumption, u_1 and u_2 lie on different sides of $\pi(0, u_2, u_3)$. Hence there are two possibilities: either u_1 lies above π_2 and u_4 above π_1, or u_1 below π_2 and u_4 below π_1. To distinguish between these we note that the sets

$$K_1 = \{x \in M | \langle x, s_1 \rangle \leq -1,\ \langle x, s_2 \rangle \leq -1\}$$
$$K_2 = \{x \in M | \langle x, s_1 \rangle \geq -1,\ \langle x, s_2 \rangle \geq -1\}$$

are convex. Hence it suffices to study whether the intersection point u_0 of $\pi(0, u_2, u_3)$ and the line connecting u_1 and u_4 lies in K_1 or K_2.

As in the proof of Lemma 3.2 we can choose $e \in \pi(0, u_2, u_3)^{\perp}$, $\langle e, e \rangle = 1$, and denote

$$u_1 = \beta e + \alpha_2 u_2 + \alpha_3 u_3, \qquad u_4 = \beta' e + \alpha_2' u_2 + \alpha_3' u_3.$$

By assumption, $\beta\beta' < 0$. Since $u_0 = tu_1 + (1-t)u_4$ and $\langle u_0, e \rangle = 0$, $t = |\beta'|/(|\beta| + |\beta'|)$. The condition $u_0 \in K_1$ is equivalent to

$$\frac{\alpha_2 + \alpha_3 - 1}{|\beta|} + \frac{\alpha_2' + \alpha_3' - 1}{|\beta'|} \geq 0,$$

which is, by calculations similar to those in the proof of Lemma 3.2, equivalent to

$$\frac{a/c}{\sqrt{1 - (a/c)^2}} + \frac{a'/c'}{\sqrt{1 - (a'/c')^2}} \geq 0,$$

where

$$a = \frac{L_{12}^2 + L_{13}^2 - L_{23}^2}{2 + L_{23}^2}, \qquad c = \frac{2L_{12}^2 L_{13}^2}{2 + L_{23}^2}$$
$$a' = \frac{L_{24}^2 + L_{34}^2 - L_{23}^2}{2 + L_{23}^2}, \qquad c' = \frac{2L_{24}^2 L_{34}^2}{2 + L_{23}^2}.$$

Since the last inequality is equivalent to

$$a/c + a'/c' \geq 0,$$

we get inequality (2).

The equality (1) corresponds to the case $\pi_1 = \pi_2$.

Note 3.4. In order to see the geometric meaning of (2) in D we consider the projection from H to D. Since the projection of $\pi(u_1, u_2, u_3) \cap H$ lies on the circumscribing circle of $\overline{u}_1, \overline{u}_2, \overline{u}_3$, the geometric condition corresponding to (2) is that $\overline{u}_4$ lies outside the circumscribing circle of $\overline{u}_1, \overline{u}_2, \overline{u}_3$ and $\overline{u}_1$ lies outside the circumscribing circle of $\overline{u}_2, \overline{u}_3, \overline{u}_4$.

Hence the face condition is equivalent to the following condition in D: For two triangles intersecting exactly on an edge, the third vertex of each triangle lies outside the circumscribing circle of the other triangle.

As in [4, Lemma 5.2] we can show that inequalities of type (2) for all adjacent triangles in triangulations we shall consider imply all strict triangle inequalities. By Lemma 3.1, all corresponding planes (for example $\pi(u_1, u_2, u_3)$ and $\pi(u_2, u_3, u_4)$) are then elliptic. In D, the geometric meaning of the plane $\pi(u_1, u_2, u_3)$ being elliptic is that $\pi(u_1, u_2, u_3) \cap H$ projects onto a circle, which lies in D.

In Chapter 5 we use the fact that the inequality (2) is homogeneous and that it is similar to the face condition of [4].

4. The circumscribing circle

Let T be a hyperbolic triangle with lengths of sides a, b, e. We assume that its circumscribing circle $C \subset D$, and calculate a formula for the hyperbolic radius R in terms of the L-lengths $L_A = \sqrt{2}\sinh(a/2)$, $L_B = \sqrt{2}\sinh(b/2)$, $L_E = \sqrt{2}\sinh(e/2)$, see Figure 1. The formula for R turns out to be very similar to Heron's formula.

Lemma 4.1. *The radius R of the circumscribing circle of T fulfils*

$$\frac{1}{\sinh R} = \frac{\sqrt{(L_A + L_B - L_E)(L_A + L_E - L_B)(L_B + L_E - L_A)(L_A + L_B + L_E)}}{\sqrt{2} L_A L_B L_E}.$$

Proof. We first assume that the center P of C is inside T. We draw perpendiculars from P to the sides of T and connect P to the vertices of T. Then six right-angled triangles are formed, equivalent in pairs. We denote the angle at P opposite to $a/2$ by α, opposite to $b/2$ by β, and opposite to $e/2$ by γ. By hyperbolic trigonometry,

$$\sinh\frac{a}{2} = \sinh R \sin\alpha$$
$$\sinh\frac{b}{2} = \sinh R \sin\beta.$$

Since

$$\gamma = \pi - (\alpha+\beta), \qquad \sinh\frac{e}{2} = \sinh R\,(\sin\alpha\cos\beta + \cos\alpha\sin\beta).$$

Substituting for the trigonometric functions and squaring yields after a calculation

$$\sinh^2 R = \frac{4\sinh^2\frac{1}{2}a\sinh^2\frac{1}{2}b\sinh^2\frac{1}{2}e}{4\sinh^2\frac{1}{2}a\sinh^2\frac{1}{2}b - (\sinh^2\frac{1}{2}e - \sinh^2\frac{1}{2}a - \sinh^2\frac{1}{2}b)^2}.$$

Substituting for L_A, L_B, L_E we get the claimed result.

In case the center P of C is outside T, if the side with length a divides C into two parts, one containing P and the other containing T, $\gamma = \alpha - \beta$ and we get

$$\sinh\frac{e}{2} = \sinh R(\sin\alpha\cos\beta - \cos\alpha\sin\beta).$$

The formula for R is the same as the one obtained above.

Figure 1.

Note 4.1. If the vertices of T are projected into the hyperboloid model H and joined by Euclidean edges, a triangle in an elliptic plane is obtained, since $C \subset D$. This triangle has L-lengths of sides L_A, L_B, L_E.

We continue to use the notation of Lemma 4.1 and Figure 1 and calculate formulas for trigonometric functions of γ and φ in terms of L_A, L_B, L_E:

Lemma 4.2.

$$\cos\gamma = \frac{|L_A^2 + L_B^2 - L_E^2|}{2L_AL_B}$$

$$\sin\varphi = \frac{|L_A^2 + L_B^2 - L_E^2|}{\sqrt{2}\sqrt{2+L_E^2}L_AL_B}.$$

Proof. By hyperbolic trigonometry,

$$\sinh\frac{e}{2} = \sinh R \sin\gamma.$$

Solving for $\sin\gamma$, squaring and substituting the formula for $\sinh R$ we get

$$\cos\gamma = \frac{|-\sinh^2\frac{1}{2}e + \sinh^2\frac{1}{2}a + \sinh^2\frac{1}{2}b|}{2\sinh\frac{1}{2}a\sinh\frac{1}{2}b}.$$

Substituting the formulas for L_A, L_B, L_E yields the first claim. The other claim follows from the formulas

$$\cos\gamma = \cosh\frac{e}{2}\sin\varphi, \qquad \cosh\frac{e}{2} = \sqrt{1+\sinh^2\frac{e}{2}} = \sqrt{\frac{2+L_E^2}{2}}.$$

5. Parametrization of Dirichlet polygons by L-lengths of the edges in the convex hull

Example 1. We derive a set of L-lengths that are naturally associated with a Dirichlet polygon to illustrate the following theorems. We start by considering an example. Let G be the Fuchsian group with Dirichlet polygon $D(0)$ an 18-gon with pattern 1 (Figure 8 in [1]), depicted in Figure 2 as the inner polygon, and the points in $G(0)$ labeling adjacent Dirichlet polygons forming the vertices of the outer polygon. Then 18 triangles with a vertex at 0 are formed. We choose generators A, B, C, D among the side-pairings of $D(0)$ as indicated in Figure 2 and derive formulas for the rest of the side-pairing transformations in terms of the chosen generators. The relation is $ABA^{-1}B^{-1} = CDC^{-1}D^{-1}$. The numbers $1, \ldots, 6$ indicate equivalence classes of vertices.

Figure 2.

The Dirichlet polygons adjacent to the uppermost vertex labelled 1 have centers $0, AB(0), ABA^{-1}B^{-1}(0)$. The hyperbolic triangle with these vertices has lengths of sides $\varrho\big(0, AB(0)\big), \varrho\big(0, ABA^{-1}B^{-1}(0)\big)$, and $\varrho\big(AB(0), ABA^{-1}B^{-1}(0)\big) = \varrho\big(0, BA(0)\big)$. We denote by $L_{AB}, L_{ABA^{-1}B^{-1}}, L_{BA}$ the corresponding L-lengths, i.e. for example $L_{AB} = \sqrt{2}\sinh\big[\varrho\big(0, AB(0)\big)/2\big]$.

Similarly, for the vertices in the equivalence class of the vertex labelled 2, we get L-lengths L_A, L_B, L_{AB}, for the equivalence class of the vertex labelled 3 the L-lengths are L_A, L_B, L_{BA} and for the vertex labelled 4 we get $L_{CD}, L_{DC}, L_{CDC^{-1}D^{-1}} = L_{ABA^{-1}B^{-1}}$. The vertices labelled 5 and 6 are obtained by symmetry from 2 and 3 by interchanging A, B, to C, D.

In the generic case we are considering, by the definition of the Dirichlet polygon, the vertices are centers of the circumscribing circles of the above 18 triangles, and for each pair of adjacent triangles (intersecting on an edge), the third vertex of a triangle lies outside the circumscribing circle of the other triangle. By Note 3.4 the L-lengths fulfil an inequality of type (2) for each edge. For example, for two adjacent triangles with common side of L-length L_{AB} and remaining sides $L_{BA}, L_{ABA^{-1}B^{-1}}$, and L_A, L_B, respectively, the inequality (2) from Lemma 3.3 reads:

$$L_A L_B(L_{BA}^2 + L_{ABA^{-1}B^{-1}}^2 - L_{AB}^2) + L_{BA}L_{ABA^{-1}B^{-1}}(L_A^2 + L_B^2 - L_{AB}^2) > 0.$$

Hence, for each 18-gonal Dirichlet polygon we get 9 positive numbers fulfilling inequalities of type (2), one for each side-pair, and, in addition, the condition arising from the fact that the angles of the 18 triangles at the center of the polygon add up to 2π.

In terms of the convex hull, its vertices are obtained by projecting the orbit of 0 into the hyperboloid model, and the 18 triangles correspond to triangular faces of the convex hull. The above numbers are the L-lengths of the edges, see [3]. Since the edges of the 18 triangles correspond to extremal edges in the convex hull, inequalities of type (2) hold for each pair of adjacent triangles.

The process desribed above can be reversed. We first give the ideas in Minkowski space, but continue then in the Poincaré model. In Minkowski space, the reverse amounts to constructing first a convex hull with a chosen combinatorial pattern in the sense described later, and having edges with prescribed L-lengths—which have to be properly rescaled to fulfil the closing condition—then projecting to D to get a tesselation of D by triangles (in the generic case). The group generated by Möbius transformations mapping the triangles onto each other according to the pattern can be shown to have a Dirichlet polygon with pattern 1 parametrized by the given (rescaled) L-lengths in the manner indicated in Example 1. In order to construct the convex hull we choose nine positive numbers fulfilling each a condition of type (2), (to ensure that the corresponding edge will be extremal in the convex hull when two triangles are glued together on the edge), the combinatorics given by the wanted identification pattern (for example as in Figure 2) and scale the nine numbers by multiplying with the unique t that will make the construction close without gaps or overlaps at the vertices; (the existence of such a multiplier is proved in Lemma 5.1). Then glue together according to the pattern triangles with vertices in H and with the rescaled L-lengths of sides. By using Lemma 3.3, Note 3.4, and Lemmas 3.2 and 5.1, we see that a convex hull that will project homeomorphically to D and a tesselation by triangles as in Figure 2 is obtained.

To reverse in D the process described in Example 1, we use Note 3.4 and construct directly the tesselation of D by triangles. This is done in Theorem 5.2. Lemma 5.1 is applicable, since the calculations for the closing condition are done in D.

Lemma 5.1. *For each 9-tuple of positive numbers $L_1, \dots, L_9$ fulfilling conditions of type (2) according to the combinatorics of a chosen pattern, there exists a unique $t > 0$ such that in D, the angles at the common vertex of the associated 18 triangles add up to 2π when the numbers above are scaled by multiplying by t.*

Proof. Let $z \in D$, $L_1, \dots, L_9 > 0$, $t > 0$. By Note 3.4 we can construct 6 triangles representing the congruence classes, with L-lengths of sides $tL_1, \dots, tL_9$. We then start gluing them together at the vertex z. In our generic case for genus 2 there are 18 triangles, 3 from each congruence class, to consider. The angle-sum at z is obtained by adding up the angles of the 6 triangles representing the conguence classes. In order for the construction to close at z, the sum should give 2π. Now, as $t \to 0$, the hyperbolic triangles approach Euclidean, each with angle-sum π, i.e. the total angle-sum approaches 6π. In order to show that there exists a unique $t > 0$ such that the angle-sum equals 2π after rescaling, we finish by proving that the total angle-sum decreases monotonously to 0 as t grows: Consider a triangle T with L-lengths of sides tL_1, tL_2, tL_3 and connect the vertices of T to the center of its circumscribing circle. Let the angle thus formed, adjacent to the side labelled by L-length tL_3 be φ. Then $\sin\varphi$ has the formula of Lemma 4.2:

$$\sin\varphi = \frac{|L_1^2 + L_2^2 - L_3^2|}{\sqrt{2}\sqrt{2 + t^2 L_3^2}\, L_1 L_2},$$

which decreases to zero monotonously as $t \to \infty$. Since all angles at the vertices of T (as sums of two angles of the above type) behave similarly, their sum decreases monotonously to zero as $t \to \infty$.

Theorem 5.2. *Let an identification pattern for a Fuchsian group of genus 2 and a basepoint be given. Then for any positive numbers $L_1, \ldots, L_9$ satisfying homogeneous conditions of type (2) for all pairs of adjacent triangles, there exists a tesselation of D by triangles with uniquely rescaled L-lengths $tL_1, \ldots, tL_9$.*

The group of Möbius transformations preserving the tesselation according to the pattern has as its Dirichlet polygon with center z an 18-gon with the chosen identification pattern and parameters the rescaled L-lengths.

Proof. We normalize so that the basepoint is 0 and choose the pattern of Figure 2, use Note 3.4 to make triangles with the chosen numbers as L-lengths of sides, glue them together with combinatorics of Figure 2 and use Lemma 5.1 to rescale L-lengths to ensure that the angles add up to 2π at 0. We label the Möbius transformations preserving the tesselation as in Figure 2. Let $G = \langle A, B, C, D; ABA^{-1}B^{-1} = CDC^{-1}D^{-1}\rangle$. We claim that the centers of the circumscribing circles of the triangles provide vertices for $D(0)$.

We want to assure that the centers of the circumscribing circles of the triangles follow each other in the same order as the triangles. Consider rotating a ray around 0. Due to Note 3.4, the Euclidean centers of the circumscribing circles of two adjacent triangles lie on the perpendicular of the common edge in the same order as the triangles. Since by symmetry the hyperbolic centers lie on the same ray from 0 as Euclidean, they also follow each other in the correct order.

We then join by geodesics the centers of the circumscribing circles of the triangles with one vertex at 0. The obtained polygon P can be characterized as

$$P = \cap\{z \mid \varrho(z,0) \le \varrho(z, f(0))\}$$

for $f \in E = \{A, B, C, D, AB, BA, ABA^{-1}B^{-1} = CDC^{-1}D^{-1}, CD, DC\}$. P is also the Dirichlet polygon for $G = \langle A, B, C, D; ABA^{-1}B^{-1} = CDC^{-1}D^{-1}\rangle$ since for no $f \neq I$ in $G \setminus E$ is $f(0)$ in the closure of the union of the circumscribing circles of the triangles with a vertex at 0. (This follows from the validity of a face condition for each pair of triangles.) By the construction, the identifications are as in Figure 2, hence G is Fuchsian of genus 2 with only hyperbolic Möbius transformations.

Note 5.3. For a regular 18-gon all L-lengths fulfil

$$L = \sqrt{2\Big(\frac{1}{4\sin^2(\pi/18)} - 1\Big)}.$$

Note 5.4. The condition for changing the pattern is equivalent to a side-pair disappearing from the Dirichlet polygon with maximal number of sides. This is equivalent to an edge in the convex hull disappearing, i.e. an equality of type (1) for the L-lengths. (This equality is equivalent to the Ptolemy equation of [3].) Since this is the same condition as the face condition in [4], the parametrization leads to a cell decomposition. For $g = 2$ we get 9 types of cells of dimension 8. The case of Note 5.3 can be considered the center of a cell. The cells correspond to the 9 patterns in [1]. (In [1] the patterns were considered disregarding orientation, i.e. up to mirror images. One of them, namely number 4, is not symmetric. Here its mirror image is counted separately and listed as $4'$ see Figures 3–11.) The boundaries of the 9 top-dimensional cells correspond to "degenerate" Dirichlet polygons i.e. for $g = 2$, with $8, 10, \dots, 16$ sides.

Note 5.5. Prescribing L-lengths for the edges of the convex hull corresponds to prescribing lengths of the geodesics joining centers of adjacent Dirichlet polygons. In terms of the surface F, we get a triangulation of F into 6 triangles with the base-point as the only vertex, see [3, Theorem 3.4, 3.5]. Hence we get, for each identification pattern, a Fenchel–Nielsen type parametrization for a cell of conformal structures of a closed surface with a distinguished point.

Note 5.6. A similar procedure can be carried out for higher genera as well, see Note 7.3.

6. L-lengths and matrices

The results above have simple interpretations in terms of matrices of Möbius transformations when the distinguished point is fixed to the origin of D.

Lemma 6.1. *Let A be a Möbius transformation with matrix*

$$\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}, \qquad |a|^2 - |b|^2 = 1.$$

Then if

$$L_A = \sqrt{2}\sinh\frac{\varrho\big(0, A(0)\big)}{2},$$

it holds

$$L_A = \sqrt{2}\,|b|. \tag{3}$$

Proof. Since

$$\sinh \frac{\varrho(z, Az)}{2} = \frac{|z - Az|}{\sqrt{1-|z|^2}\sqrt{1-|Az|^2}},$$

where

$$Az = \frac{az+b}{\overline{b}z+\overline{a}},$$

a straightforward calculation yields

$$\sinh \frac{\varrho\big(z, A(z)\big)}{2} = \frac{|\overline{b}z^2 + (\overline{a}-a)z - b|}{1-|z|^2},$$

and substituting $z = 0$ gives the claim.

Lemma 4.1 corresponds to the following:

Lemma 6.2. *Let $T \subset D$ be a hyperbolic triangle with vertices $0, A(0), B(0)$, where the matrices of the Möbius transformations A, B are, respectively*

$$\begin{bmatrix} a_1 & b_1 \\ \overline{b}_1 & \overline{a}_1 \end{bmatrix}, \qquad \begin{bmatrix} a_2 & b_2 \\ \overline{b}_2 & \overline{a}_2 \end{bmatrix}.$$

If R is the hyperbolic radius of the circumscribing circle of T, then R fulfils

$$\frac{2}{\sinh R} = \frac{\sqrt{\big(|b_1|+|b_2|-|b_3|\big)\big(|b_1|+|b_3|-|b_2|\big)\big(|b_2|+|b_3|-|b_1|\big)\big(|b_1|+|b_2|+|b_3|\big)}}{|b_1||b_2||b_3|},$$

where $b_3 = \overline{a}_2 b_1 - \overline{a}_1 b_2$.

Proof. We apply the formula obtained in Lemma 4.1 and substitute $L_A = \sqrt{2}\,|b_1|$, $L_B = \sqrt{2}\,|b_2|$, and use the equality $\varrho\big(A(0), B(0)\big) = \varrho\big(0, B^{-1}A(0)\big)$ to obtain $L_E = \sqrt{2}\,|b_3|$, $b_3 = \overline{a}_2 b_1 - \overline{a}_1 b_2$.

Lemma 6.3. *Let $T \subset D$ be the hyperbolic triangle of Lemma 6.2, P the center of the circumscribing circle of T, and let T be divided into six right-angled triangles with one vertex at P by connecting P to the vertices of T and drawing perpendiculars from P to the sides of T. If γ and φ are the angles of a right-angled triangle, and the side opposite γ has length $\varrho\big(A(0), B(0)\big)/2$, then*

$$\begin{aligned} \cos\gamma &= \frac{\big||b_1|^2+|b_2|^2-|b_3|^2\big|}{2|b_1||b_2|} \\ \sin\varphi &= \frac{\big||b_1|^2+|b_2|^2-|b_3|^2\big|}{2\sqrt{1+|b_3|^2}\,|b_1||b_2|}, \end{aligned} \tag{4}$$

where $b_3 = \overline{a}_2 b_1 - \overline{a}_1 b_2$.

Proof. Let T and P be as above. The formulas for the angles obtained when the triangle is divided into six right-angled triangles with one vertex at P, see Figure 1, were calculated in Lemma 4.2. By substituting as in the proof of Lemma 6.2 we get the formulas (4).

7. A condition for discreteness

We consider the case of genus 2 and continue to use the results of [1].

Theorem 7.1. *Let A,B,C,D be Möbius transformations with relation $ABA^{-1}B^{-1}=CDC^{-1}D^{-1}$, and with matrices*

$$\begin{bmatrix} a_1 & b_1 \\ \overline{b}_1 & \overline{a}_1 \end{bmatrix}, \quad \begin{bmatrix} a_2 & b_2 \\ \overline{b}_2 & \overline{a}_2 \end{bmatrix}, \quad \begin{bmatrix} a_3 & b_3 \\ \overline{b}_3 & \overline{a}_3 \end{bmatrix}, \quad \begin{bmatrix} a_4 & b_4 \\ \overline{b}_4 & \overline{a}_4 \end{bmatrix},$$

respectively, where $|a_i|^2-|b_i|^2=1$, $i=1,\dots,4$. Further, let $b_5,\dots,b_9$ denote the upper right entries in the normalized matrices of $AB, BA, ABA^{-1}B^{-1}=CDC^{-1}D^{-1}$, CD, DC, respectively.

A necessary and sufficient condition for $G=\langle A,B,C,D; ABA^{-1}B^{-1}=CDC^{-1}D^{-1}\rangle$ to have $D(0)$ of pattern 1 (or its reflection) with A,B,C,D,AB, $BA, ABA^{-1}B^{-1}=CDC^{-1}D^{-1}, CD, DC$ as side-pairings (see Figure 3) is that the following conditions are valid:

$$\sum_{\ell=1}^{6}\sum_{I_\ell=(i,j,k)} \overline{\operatorname{arc}}\sin\left(\frac{\bigl||b_i|^2+|b_j|^2-|b_k|^2\bigr|}{2\sqrt{1+|b_k|^2}\,|b_i||b_j|}\right)=\pi \tag{5}$$

with

$$\begin{aligned} &I_1=(5,6,7) \quad I_2=(1,2,5) \quad I_3=(1,2,6)\\ &I_4=(7,8,9) \quad I_5=(3,4,8) \quad I_6=(3,4,9), \end{aligned}$$

where the inner sum is over the three cyclic permutations of I_ℓ, and

$$|b_i||b_j|\bigl(|b_m|^2+|b_n|^2-|b_k|^2\bigr)+|b_m||b_n|\bigl(|b_i|^2+|b_j|^2-|b_k|^2\bigr)>0, \tag{6}$$

where the indices are chosen from the two triples $I_\ell=\{i,j,k\}$ and $I_h=\{m,n,k\}$ having the index k in common. Also, we assume that each pair of triangles fulfilling (6) lie on different sides of the common edge. Since each $k=1,\dots,9$ belongs to exactly two triples, there are nine conditions of type (6).

Proof. Let $D(0)$ have 18 sides and pattern 1. By choosing A,B,C,D from the side-pairings as in Figure 3, we can calculate the other side-pairings in terms of A,B,C,D, see Figure 3 (the 18-gons in Figures 3–11 are drawn to be regular, since these pictures are only to indicate the combinatorics of the patterns). The 18 vertices of $D(0)$ fall into six equivalence-classes, each equivalence-class being associated to three side-pairings. By depicting the images of 0 under the side-pairings, 18 triangles are formed. These are permuted by the side-pairings, and the equivalence-classes thus formed correspond to the six equivalence-classes of vertices. We denote by T_j, $j=1,\ldots,6$ the equivalence-classes of triangles and label the side-pairings $A,B,C,D,AB,BA,ABA^{-1}B^{-1}=CDC^{-1}D^{-1},CD,DC$ with indices $1,\ldots,9$, respectively, to connect with each equivalence-class of triangles T_j a triple I_j, $j=1,\ldots,6$ of indices; the index i labelling f is in I_j if each triangle in the equivalence-class T_j has a side equivalent to the side with vertices $0,f(0)$.

The following list of triples is obtained:

$$\begin{array}{lll} I_1=(5,6,7) & I_2=(1,2,5) & I_3=(1,2,6) \\ I_4=(7,8,9) & I_5=(3,4,8) & I_6=(3,4,9). \end{array}$$

Each index i shared by two triples I_k, I_ℓ means that the equivalence-classes of triangles T_k, T_ℓ share a side equivalent to the side with vertices $0,f(0)$, where f is labelled by the index i. Hence we get 9 conditions of type (2), for example the triangles I_1 and I_2 share the index 5, the remaining indices being (6,7) and (1,2), hence by using the formula (3), we get

$$|b_1||b_2|\big(|b_6|^2+|b_7|^2-|b_5|^2\big)+|b_6||b_7|\big(|b_1|^2+|b_2|^2-|b_5|^2\big)>0.$$

Since the angles of the 18 triangles at 0 add up to 2π, from Lemma 6.3 we obtain the condition

$$\sum_{\ell=1}^{6}\sum_{I_\ell}\overline{\mathrm{arc}}\sin\Big(\frac{\big||b_i|^2+|b_j|^2-|b_k|^2\big|}{2\sqrt{1+|b_k|^2}\,|b_i||b_j|}\Big)=\pi,$$

where the inner sum is over all three cyclic permutations of the indices in $I_\ell=(i,j,k)$.

For sufficiency, we connect in pairs those points in $G(0)$ that are connected in Figure 2 by an edge of the triangulation. By Lemma 6.1 the condition (6) corresponds to the face condition (2) and (5) to the closing condition. As mentioned in Note 3.4, our assumptions imply strict triangle inequalities for the corresponding L-lengths. The triangulation obtained can be continued to a tesselation of the unit disk and it is the same as that constructed in the proof of Theorem 5.2 (with same L-lengths) up to orientation and rotation around 0, the choice of orientation being determined by any two adjacent triangles.

The other patterns for 18-gons can be dealt with in a similar way: we make choices of generators, see Figures 3–11, label them with indices, and derive the corresponding lists of triples. (The figures only indicate the combinatorics of the case. Pattern $4'$ denotes the mirror image of pattern 4). The results are presented in Tables 1 and 2, which are related to the last diagram as explained later.

Table 1.

Pattern	Side-pairings labelled $1, \dots, 9$, and their relation
1	A, B, C, D, AB, BA, $ABA^{-1}B^{-1} = CDC^{-1}D^{-1}$, CD, DC
2	A, B, $\mathbf{C}$, D, AB, BA, $D^{-1}C^{-1}AB = C^{-1}D^{-1}BA$, CD, DC
3	A, B, C, D, $A^{-1}CD$, BA, $D^{-1}C^{-1}AB = C^{-1}D^{-1}BA$, CD, DC
4	A, B, C, D, $A^{-1}CD$, BA, $D^{-1}C^{-1}AB = C^{-1}D^{-1}BA$, CD, $A^{-1}B^{-1}D$
$4'$	A, B, C, D, $A^{-1}CD$, BA, $D^{-1}C^{-1}AB = C^{-1}D^{-1}BA$, $C^{-1}A$, DC
5	A, B, C, D, $A^{-1}CD$, $B^{-1}DC$, $D^{-1}C^{-1}AB = C^{-1}D^{-1}BA$, CD, DC
6	A, B, C, D, $A^{-1}CD$, BA, $D^{-1}C^{-1}AB = C^{-1}D^{-1}BA$, $C^{-1}A$, $A^{-1}B^{-1}D$
7	A, B, C, D, $A^{-1}CD$, BA, $CD^{-1}C^{-1}A = D^{-1}BAB^{-1}$, $C^{-1}A$, $A^{-1}B^{-1}D$
8	A, $D^{-1}C^{-1}ADC$, C, D, $A^{-1}CD$, $B^{-1}DC$, $D^{-1}C^{-1}AB = C^{-1}D^{-1}BA$, CD, DC

Table 2.

Pattern	Triples of indices					
1	(5,6,7)	(1,2,5)	(1,2,6)	(7,8,9)	(3,4,8)	(3,4,9)
2	(5,7,8)	(1,2,5)	(1,2,6)	(6,7,9)	(3,4,8)	(3,4,9)
3	(1,5,8)	(2,5,7)	(1,2,6)	(6,7,9)	(3,4,8)	(3,4,9)
4	(1,5,8)	(2,5,7)	(1,2,6)	(3,7,9)	(3,4,8)	(4,6,9)
$4'$	(4,5,8)	(2,5,7)	(1,2,6)	(6,7,9)	(1,3,8)	(3,4,9)
5	(1,5,8)	(2,5,7)	(2,6,9)	(1,6,7)	(3,4,8)	(3,4,9)
6	(4,5,8)	(2,5,7)	(1,2,6)	(4,6,9)	(1,3,8)	(3,7,9)
7	(4,5,8)	(3,5,7)	(1,2,6)	(4,6,9)	(1,3,8)	(2,7,9)
8	(1,5,8)	(2,6,7)	(2,5,9)	(1,6,7)	(3,4,8)	(3,4,9)

These patterns are obtained, for example starting from pattern 1, by elementary moves, see [1]. The index corresponding to the removed side-pair is indicated in the next diagram. For example $1 \underset{7}{\Longrightarrow} 2$ means that by removing the side-pair connected to index 7 from pattern 1, pattern 2 is obtained. Table 1 corresponds to the following diagram:

$$
\begin{tikzcd}
& & & 4 \arrow[dr, Rightarrow, "8"] & & \\
1 \arrow[r, Rightarrow, "7"'] & 2 \arrow[r, Rightarrow, "5"'] & 3 \arrow[ur, Rightarrow, "9"] \arrow[dr, Rightarrow, "8"'] \arrow[dd, Rightarrow, "6"] & & 6 \arrow[r, Rightarrow, "7"'] & 7 \\
& & & 4' \arrow[ur, Rightarrow, "9"'] & & \\
& & 5 \arrow[r, Rightarrow, "2"'] & 8 & &
\end{tikzcd}
$$

Note 7.2. The Dirichlet polygons with $8, 10, \dots, 16$ sides are obtained with certain side-pairs degenerating to vertices of the polygon, i.e. with equality in (6) for the corresponding indices. This corresponds to the convex hull having as cells n-gons, $3 < n \leq 8$.

Note 7.3. The parametrization done above for genus 2 surfaces generalizes in an analogous manner for a closed surface of higher genus g. Then, the generic Dirichlet polygon has $4g-2$ equivalence-classes of vertices and $6g-3$ side-pairs. Hence, there will be $6g-3$ conditions of type (6). The closing condition of type (5) will have $4g-2$ terms in the outer sum and the triples of indices in the inner sum will be according to the combinatorics of the identification pattern. The parametrization leads to a cell decomposition, with cells of dimension $6g-4$.

The surface is triangulated into $4g-2$ triangles with the only vertex at the distinguished point.

Similar methods are used for groups with elliptic and parabolic elements in [2] and [4].

Figures 3–8.

Figures 9–11.

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