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ON THE TANGENT SPACE TO THE UNIVERSAL TEICHMÜLLER SPACE

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Abstract. We find a remarkably simple relationship between the following two models of the tangent space to the universal Teichmüller space:

(1) The real-analytic model consisting of Zygmund class vector fields on the unit circle;

(2) The complex-analytic model comprising 1-parameter families of schlicht functions on the exterior of the unit disc which allow quasiconformal extension.

Indeed, the Fourier coefficients of the vector field in (1) turn out to be essentially the same as (the first variations of) the corresponding power series coefficients in (2).

These identities have many applications; in particular, to conformal welding, to the almost complex structure of Teichmüller space, to study of the Weil–Petersson metric, to variational formulas for period matrices, etc. These utilities are explored.

1. Introduction

Let Δ denote the open unit disc, and $S^1 = \partial\Delta$. Two classic models of the universal Teichmüller space $T(1) = T(\Delta)$ are well-known (see [6], [7]):

(a) the real-analytic model containing all (Möbius-normalised) quasisymmetric homeomorphisms of the unit circle S^1;

(b) the complex-analytic model comprising all (normalised) schlicht functions on the exterior of the disc:

$$\Delta^\star = \bigl\{ z \in \widehat{\mathbf{C}} : \ |z| > 1 \bigr\} = \widehat{\mathbf{C}} - (\Delta \cup S^1)$$

which allow quasiconformal extension to the whole of $\widehat{\mathbf{C}}$ (the Riemann sphere).

The connection between them is via the rather mysterious operation called "conformal welding" (see [5], and below). Nevertheless, at the infinitesimal level, the above models have an amazingly simple relationship that forms the basis for this paper. Indeed, the k^{th} *Fourier coefficient* of the vector field representing a tangent vector in model (a), and the (first variation of) the k^{th} *power series coefficient* representing the *same* tangent vector in model (b), turn out to be just ($\sqrt{-1}$ times) complex conjugates of each other. That relationship can also be formulated as a direct identity relating the vector field on the circle with the

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holomorphic function in the exterior of the disc describing the perturbation in the complex-analytic model. See Theorem 1 below.

It seems that this unexpectedly simple tie-up has several interesting consequences.

1. It allows a description of the tangent space to $T(\Delta)$ by "Zygmund class power series" (Section 3).
2. It provides immediate proof for the remarkable fact that the Hilbert transform on Zygmund class vector fields on S^1 represents the almost complex structure on $T(\Delta)$ (Section 4).
3. It provides a simple explicit formula for the derivative of the conformal welding map (Section 5).
4. The infinite-dimensional Weil–Petersson metric on (the "smooth points" of) $T(\Delta)$ that was found by the present author in [10] Part II gets a new expression (Section 6).
5. We get a formula for the derivative of the infinite-dimensional universal period mapping studied by us in [8], [9] (and in a recent IHES preprint with Dennis Sullivan) in terms of power series variations. This relates to formulas claimed in [4] (Section 7).

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2. Teichmüller theory

The universal Teichmüller space $T(\Delta)$ is a holomorphically homogeneous complex Banach manifold that serves as the universal ambient space where all the Teichmüller spaces (of arbitrary Fuchsian groups) lie holomorphically embedded.

As usual, we set the stage by introducing the chief actor—namely the space of (proper) Beltrami coefficients $L^\infty(\Delta)_1$; it is the open unit ball in the complex Banach space of L^∞ functions on the unit disc Δ. The principal construction is to solve the Beltrami equation

$$w_{\bar z} = \mu w_z \tag{1}$$

for any $\mu \in L^\infty(\Delta)_1$. The two above-mentioned models of Teichmüller space correspond to discussing two pertinent solutions for (1):

(a) w_μ-theory: The quasiconformal homeomorphism of $\mathbf{C}$ which is μ-conformal (i.e. solves (1)) in Δ, fixes ± 1 and $-i$, and keeps Δ and $\Delta^\star$ (= exterior of Δ) both invariant. This w_μ is obtained by applying the existence and uniqueness theorem of Ahlfors–Bers (for (1)) to the Beltrami coefficient which is μ on Δ and extended to $\Delta^\star$ by reflection ($\tilde{\mu}(1/\bar{z}) = \overline{\mu(z)} z^2/\bar{z}^2$ for $z \in \Delta$).

(b) w^μ-theory: The quasiconformal homeomorphism on **C**, fixing $0, 1, \infty$, which is μ-conformal on Δ and conformal on $\Delta^\star$. w^μ is obtained by applying the Ahlfors–Bers theorem to the Beltrami coefficient which is μ on Δ and zero on $\Delta^\star$.

The fact is that w_μ depends only real-analytically on μ, whereas w^μ depends complex-analytically on μ. We therefore obtain two standard models ((**a**) and (**b**) below) of the universal Teichmüller space, $T(\Delta)$.

Define the universal Teichmüller space:

$$T(\Delta) = L^\infty(\Delta)_1 / \sim . \tag{2}$$

Here $\mu \sim \nu$ if and only if $w_\mu = w_\mu$ on $\partial\Delta = S^1$, and that happens if and only if the conformal mappings w^μ and w^ν coincide on $\Delta^\star \cup S^1$.

We let

$$\Phi\colon L^\infty(\Delta)_1 \longrightarrow T(\Delta) \tag{3}$$

denote the quotient ("Bers") projection. $T(\Delta)$ inherits its canonical structure as a complex Banach manifold from the complex structure of $L^\infty(\Delta)_1$; namely, Φ becomes a *holomorphic submersion*.

The derivative of Φ at $\mu = 0$:

$$d_0\Phi\colon L^\infty(\Delta) \longrightarrow T_O T(\Delta) \tag{4}$$

is a complex-linear surjection whose kernel is the space N of "infinitesimally trivial Beltrami coefficients".

$$N = \left\{ \mu \in L^\infty(\Delta) : \iint\limits_{\Delta} \mu\phi = 0 \qquad \text{for all } \phi \in A(\Delta) \right\} \tag{5}$$

where $A(\Delta)$ is the Banach space of integrable (L^1) holomorphic functions on the disc. Thus, the tangent space at the origin ($O = \Phi(0)$) of $T(\Delta)$ is $L^\infty(\Delta)/N$.

See Ahlfors [2], Lehto [6], and Nag [7] for this material and for what follows.

It is now clear that to $\mu \in L^\infty(\Delta)_1$ we can associate the *quasisymmetric homeomorphism*

$$f_\mu = w_\mu \,|_{S^1} \tag{6}$$

as representing the Teichmüller point $[\mu]$ in version (**a**) of $T(\Delta)$. Indeed $T(\Delta)_{(a)}$ is the homogeneous space:

$$\begin{aligned} T(\Delta) &= \mathrm{Homeo}_{\mathrm{q.s.}}(S^1)/\text{Möb}\,(S^1) \\ &= \{\text{quasisymmetric homeomorphisms of } S^1 \text{ fixing } \pm 1 \text{ and } -i\} \end{aligned} \tag{a}$$

Alternatively, $[\mu]$ is represented by the univalent function

$$f^{\mu} = w^{\mu} \,|_{\Delta^{\star}} \tag{7}$$

on $\Delta^{\star}$, in version **(b)** of $T(\Delta)$. A more natural choice of the univalent function representing $[\mu]$ is to use a different normalisation for the solution w^{μ} (since we have the freedom to post-compose by a Möbius transformation). In fact, let

$$W^{\mu} = M^{\mu} o w^{\mu} \tag{8}$$

where M^{μ} is the unique Möbius transformation so that the univalent function (representing $[\mu]$):

$$F^{\mu} = W^{\mu} \,|_{\Delta^{\star}} \tag{9}$$

has the properties:

(i) F^{μ} has a simple pole of residue 1 at ∞

(ii) $\big(F^{\mu}(\zeta) - \zeta\big) \to 0$ as $\zeta \to \infty$.

Namely, the expansion of F^{μ} in $\Delta^{\star}$ is of the form:

$$F^{\mu}(\zeta) = \zeta\Big(1 + \frac{c_2}{\zeta^2} + \frac{c_3}{\zeta^3} + \cdots\Big). \tag{10}$$

Let us note that the original $(0, 1, \infty$ fixing) normalisation gives an expansion of the form:

$$f^{\mu}(\zeta) = \zeta\Big(a + \frac{b_1}{\zeta} + \frac{b_2}{\zeta^2} + \frac{b_3}{\zeta^3} \cdots\Big) \tag{11}$$

and the Möbius transformation M^{μ} must be $M^{\mu}(w) = w/a - b_1/a$. Since $(a, b_1, b_2, \ldots)$ depend holomorphically on μ, we see that $(c_2, c_3, \ldots)$ also depend holomorphically on μ. Thus, our complex-analytic version $T(\Delta)_{(b)}$ of the universal Teichmüller space is:

$$\textbf{(b)} \qquad \begin{aligned} T(\Delta) = \{&\text{univalent functions in } \Delta^{\star} \text{ with power series of the form (10)},\\ &\text{allowing quasiconformal extension to the whole plane}\}. \end{aligned}$$

$T(\Delta)_{(b)}$ is simply a "pre-Schwarzian-derivative" version of the Bers embedding of Teichmüller space.

It is worth remarking here that the criteria that an expansion of the form (10) represents an univalent function, and that it allows quasiconformal extension, can be written down solely in terms of the coefficients c_k, (using the Grunsky inequalities etc.). See Pommerenke [12]. Thus $T(\Delta)_{(b)}$ can be thought of as a certain space of sequences $(c_2, c_3, \ldots)$, and its tangent space will be given the concomitant description below.

Tangent space to the real-analytic model. Since $T(\Delta)$ is a homogeneous space (see version **(a)**) for which the right translation (by any fixed quasisymmetric homeomorphism) acts as a biholomorphic automorphism, it is enough in all that follows to restrict attention to the tangent space at a single point—the origin (O = class of the identity homeomorphism)—of $T(\Delta)$.

Given any $\mu \in L^\infty(\Delta)$, the tangent vector $d_0\Phi(\mu)$ is represented by the real vector field $V[\mu] = \dot{w}[\mu]\partial/\partial z$ on the circle that produces the 1-parameter flow $w_{t\mu}$ of quasisymmetric homeomorphisms:

$$w_{t\mu}(z) = z + t\dot{w}[\mu](z) + o(t). \tag{12}$$

The vector field becomes in the θ-coordinate:

$$V[\mu] = \dot{w}[\mu](z)\frac{\partial}{\partial z} = u(e^{i\theta})\frac{\partial}{\partial \theta},$$

where,

$$u(e^{i\theta}) = \frac{\dot{w}[\mu](e^{i\theta})}{ie^{i\theta}} \,. \tag{13}$$

By our normalisation, u vanishes at $1, -1$ and $-i$.

In Reimann [13], and in Gardiner–Sullivan [3], the precise class of vector fields arising from such quasisymmetric flows is determined as the Zygmund Λ class. They have delineated the theory on the upper half-plane U; we adapt that result to the disc using the Möbius transformation

$$T(z) = \frac{z-i}{z+i}, \qquad T\colon U \longrightarrow \Delta. \tag{14}$$

We point out that $(0, 1, \infty)$ go to $(-1, -i, 1)$ respectively. Notice that the corresponding identification of the real line to S^1 is given by

$$x = -\cot\frac{\theta}{2}, \qquad \text{or,} \qquad e^{i\theta} = \frac{x-i}{x+i}. \tag{15}$$

The continuous vector field $u(e^{i\theta})\partial/\partial\theta$ becomes, on $\mathbf{R}$, $F(x)\partial/\partial x$ with

$$F(x) = \tfrac{1}{2}(x^2+1)u\Big(\frac{x-i}{x+i}\Big). \tag{16}$$

Conversely,

$$u(e^{i\theta}) = \frac{2F(x)}{x^2+1}. \tag{17}$$

Since u vanishes at $(-1,-i,1)$, we see

$$F(0)=F(1)=0 \qquad \text{and} \qquad \frac{F(x)}{x^2+1}\to 0 \ \text{ as } x\to\infty. \tag{18}$$

Introduce (following Zygmund [13]),

$$\begin{aligned}\Lambda(\mathbf{R})=\{F\colon \mathbf{R}\to\mathbf{R}; F &\text{ is continuous, satisfying normalisations (18);}\\ &\text{and, } |F(x+t)+F(x-t)-2F(x)|\le B|t\,| \text{ for some } B,\\ &\text{for all } x \text{ and } t \text{ real.}\}\end{aligned} \tag{19}$$

$\Lambda(\mathbf{R})$ is a (non-separable) Banach space under the Zygmund norm—which is the best constant B for F. Namely,

$$\|F\|=\sup_{x,t}\left|\frac{F(x+t)+F(x-t)-2F(x)}{t}\right|. \tag{20}$$

In [3] it is shown that $\Lambda(\mathbf{R})$ comprises precisely the vector fields for quasisymmetric flows on $\mathbf{R}$. Hence, the tangent space to version **(a)** of $T(\Delta)$ becomes:

$$T_O T(\Delta)_{(a)}=\left\{\begin{array}{ll} u(e^{i\theta})\frac{\partial}{\partial\theta}: & \text{(i) } u\colon S^1\to\mathbf{R} \text{ is continuous,}\\ & \quad\ \text{vanishing at } (1,-1,-i);\\ & \text{(ii) } F_u(x)=\frac{1}{2}(x^2+1)u\dfrac{(x-i)}{(x+i)} \text{ is in } \Lambda(\mathbf{R}).\end{array}\right. \tag{21}$$

Remark. The normalisation by Möbius corresponds to adding an arbitrary $sl(2,\mathbf{R})$ vector field, $(ce^{i\theta}+\bar{c}e^{-i\theta}+b)\partial/\partial\theta$, $(c\in\mathbf{C},\ b\in\mathbf{R})$, to u. On the real line this is exactly adding an arbitrary real quadratic polynomial to $F(x)$. These operations allow us to enforce the 3-point normalization in each description.

We will say a continuous function $u\colon S^1\to\mathbf{R}$ is in *the Zygmund class* $\Lambda(S^1)$ *on the circle*, if, after adding the requisite $(ce^{i\theta}+\bar{c}e^{i\theta}+b)$ to normalise u, the function satisfies (21). [Can we find a characterization of $\Lambda(S^1)$ in terms of the decay properties of the Fourier coefficients?]

Tangent space to the complex analytic model. A tangent vector at O (the identity mapping) to $T(\Delta)_{(b)}$ corresponds to a 1-parameter family F_t of univalent functions (each allowing quasiconformal extension):

$$F_t(\zeta)=\zeta\Big(1+\frac{c_2(t)}{\zeta^2}+\frac{c_3(t)}{\zeta^3}+\cdots\Big), \qquad \text{in} \qquad |\zeta|>1, \tag{22}$$

with $c_k(t)=t\dot{c}_k(0)+o(t)$, $k=2,3,4,\ldots$. The sequences $\{\dot{c}_k(0),\ k\ge 2\}$ arising this way correspond uniquely to the tangent vectors. Theorem 1 will allow us to specify precisely which sequences occur (see Corollary 1).

Again, given the arbitrary Beltrami vector $\mu\in L^\infty(\Delta)$, the tangent vector $d_0\Phi(\mu)$ is therefore represented in this complex-analytic model by the restriction to the exterior of the unit disc of the holomorphic function $\dot{F}[\mu](z)$, where

$$F^{t\mu}(z)=z+t\dot{F}[\mu](z)+o(t), \tag{22$'$}$$

for small real or complex numbers t.

3. The promised relationship

Theorem 1. *The tangent vector to $T(\Delta)$ represented by $\mu \in L^\infty(\Delta)$, corresponds to the vector field on S^1 having the Fourier expansion*

$$u(e^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta} \qquad \text{in version } (\mathbf{a}).$$

The same μ corresponds to the 1-parameter family of schlicht functions in the exterior of the unit disc:

$$F^{t\mu}(\zeta) = \zeta\Big(1 + \frac{t\dot{c}_2(0)}{\zeta^2} + \frac{t\dot{c}_3(0)}{\zeta^3} + \dots\Big) + o(t) = z + t\dot{F}[\mu](z) + o(z)$$

in version **(b)** *(equations (22), (22′)). These are related by the explicit formula:*

$$\dot{F}[\mu](z) = \frac{iz^{-1}}{2\pi}\int_0^{2\pi} \frac{e^{2it}}{1-e^{it}z^{-1}} u(e^{it})\,dt,$$

valid for $|z|>1$.

Equivalently, one has the identities between the Fourier and Laurent coefficients:

$$(*) \qquad \dot{c}_k(0) = ia_{-k} = i\bar{a}_k, \qquad \text{for every} \qquad k \geq 2.$$

Proof I. The principal ingredient in the stew is, of course, the infinitesimal theory for solutions of the Beltrami equation. The most direct proof appears by a comparative study of two applications of the basic perturbation formula.

For any $\nu \in L^\infty(\mathbf{C})$ let $w^{t\nu}$ be the quasiconformal homeomorphism of the plane, fixing $0, 1, \infty$, and having complex dilatation (i.e., Beltrami coefficient) $t\nu$; (t small complex). Then, (see, for example, Ahlfors [2, p. 104]), uniformly on compact ζ-sets we have

$$w^{t\nu}(\zeta) = \zeta + t\dot{f}(\zeta) + o(t), \qquad (t \to 0)$$

where:

$$(23) \qquad \dot{f}(\zeta) = -\frac{\zeta(\zeta-1)}{\pi}\iint_{\mathbf{C}} \frac{\nu(z)}{z(z-1)(z-\zeta)}\,dx\,dy.$$

For version **(b)** considerations, apply this to

$$(24) \qquad \nu = \begin{cases} \mu & \text{on } \Delta, \\ 0 & \text{on } \Delta^\star. \end{cases}$$

We see that

$$\frac{\partial}{\partial t}\Big|_{t=0}\left(f^{t\mu}(\zeta)\right) = -\frac{\zeta(\zeta-1)}{\pi}\iint\limits_{\Delta}\frac{\mu(z)}{z(z-1)(z-\zeta)}\,dx\,dy, \qquad |\zeta|>1, \tag{25}$$

with the univalent functions $f^{t\mu}$ as in (11) above. Expand $(z-\zeta)^{-1}$ in powers of ζ^{-1}, collect terms, and compare with

$$f^{t\mu}(\zeta) = \zeta\Big(a(t) + \frac{b_1(t)}{\zeta} + \frac{b_2(t)}{\zeta^2} + \cdots\Big). \tag{26}$$

One obtains (dot representing $\partial/\partial t$):

$$\dot{a}(0) = \frac{1}{\pi}\iint\limits_{\Delta}\frac{\mu(z)}{z(z-1)}\,dx\,dy \tag{27a}$$

and

$$\dot{b}_k(0) = \frac{1}{\pi}\iint\limits_{\Delta}\mu(z)z^{k-2}\,dx\,dy, \qquad k\geq 1. \tag{27b}$$

The associated normalized univalent functions

$$F^{t\mu}(\zeta) = \zeta\Big(1 + \frac{c_2(t)}{\zeta^2} + \frac{c_3(t)}{\zeta^3} + \cdots\Big), \tag{28}$$

have coefficients $c_k(t) = b_k(t)/a(t)$. Consequently, we derive easily (since $a(0) = 1,\ b_k(0) = 0$):

$$\dot{c}_k(0) = \dot{b}_k(0) = \frac{1}{\pi}\iint\limits_{\Delta}\mu(z)z^{k-2}\,dx\,dy, \qquad k\geq 2. \tag{29}$$

Our aim is to compare these formulas with the Fourier coefficients of the vector field $V[\mu]$ corresponding to the same μ in version **(a)**. Applying (23) to

$$\nu = \begin{cases} \mu & \text{on } \Delta \\ \tilde{\mu} & \text{(obtained by “reflection” of } \mu\text{) on } \Delta^{\star}, \end{cases} \tag{30}$$

and keeping track of the normalisations, one gets (compare p. 134 of [10, Part II]):

$$w_{t\mu}(\zeta) = \zeta + t\dot{w}[\mu](\zeta) + o(t), \qquad t\to 0,$$

$$
\begin{aligned}
\dot{w}[\mu](\zeta) = -\frac{(\zeta-1)(\zeta+1)(\zeta+i)}{\pi}\Bigg\{ &\iint\limits_{\Delta} \frac{\mu(z)}{(z-1)(z+1)(z+i)(z-\zeta)}\,dx\,dy \\
&+ i\iint\limits_{\Delta} \frac{\overline{\mu(z)}}{(\bar z-1)(\bar z+1)(\bar z-i)(1-\zeta\bar z)}\,dx\,dy\Bigg\}.
\end{aligned}
\tag{31}
$$

Now we want to expand in Fourier series the vector field $V[\mu]$:

$$
u(e^{i\theta}) = \frac{\dot{w}[\mu](e^{i\theta})}{ie^{i\theta}} = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}. \tag{32}
$$

Since u is real valued, one knows $a_{-k} = \bar a_k$, $k \geq 1$. Calculating the a_k from (31) one derives, after taking care of some remarkable simplifications, (to which I drew attention in [10] also), that

$$
a_{-k} = -\frac{i}{\pi}\iint\limits_{\Delta} \mu(z) z^{k-2}\,dx\,dy, \qquad k \geq 2. \tag{33}
$$

(The remark after (21) shows that a_0 and a_1 do not matter owing to the $sl(2,\mathbf{R})$ normalisation.)

Compare (29) with (33) to derive the desired $(*)$.

Proof II. The standard Neumann series solution for the Beltrami equation (see, for example, Theorem 4.3 of Lehto [6]) implies that $\dot F[\mu](z) = T\mu$, where T is the integral operator defined by formula

$$
T\mu(z) = \frac{-1}{\pi}\iint\limits_{\mathbf{C}} \frac{\mu(\zeta)}{\zeta - z}\,d\xi\,d\eta.
$$

Comparing the basic perturbation formula for $w_{t\mu}$ as in (1.9) on p. 158 of Ahlfors [1] with the operator T above, we obtain:

$$
\dot{w}[\mu](z) = \dot F[\mu](z) - z^2\overline{\dot F[\mu](1/\bar z)} + M(z)
$$

M being a quadratic polynomial. In view of the special form of $M(z)$, given by (1.8) in [1], we obtain

$$
u(z) = 2\,\mathrm{Re}\big\{\dot F[\mu](z)/iz\big\} + \frac{M(z)}{iz} = 2\,\mathrm{Re}\big\{\dot F[\mu](z)/iz\big\} + \mathrm{Re}\{\alpha z^{-1} + \beta\}, \tag{34}
$$

on S^1. The constants α and β are such that $u(z) = 0$ if z is 1, $-i$, or -1.

To obtain the stated theorem, observe that $\dot{F}[\mu](z)$ is continuous in the extended plane, holomorphic in $\{z : |z| > 1\}$, and vanishes at ∞. Therefore, Herglotz's integral formula for a holomorphic function in terms of its real part on S^1 (adapted to the exterior of the unit disc) can be applied to (34), yielding

$$\frac{2\dot{F}[\mu](z)}{iz} = \frac{1}{2\pi}\int_0^{2\pi} \frac{z+e^{it}}{z-e^{it}} u(e^{it})\,dt - \alpha z^{-1} - \beta$$

for $|z| > 1$. The constants α and β (perhaps different from those in (34)) are now chosen so that the function $\dot{F}[\mu](z)/iz$ has a double zero at ∞. Examination of the Laurent expansion of the integral yields the explicit formula

$$\dot{F}[\mu](z) = \frac{iz^{-1}}{2\pi}\int_0^{2\pi} \frac{e^{2it}}{1-e^{it}z^{-1}} u(e^{it})\,dt,$$

valid for $|z| > 1$, as desired. Clearly, expanding $(1-e^{it}z^{-1})^{-1}$ in powers of z^{-1}, one sees that the above integral identity is equivalent to the claimed relationship between the Fourier and Laurent coefficients.

Proof III. This proof utilises the well-known *harmonic (Bers') Beltrami coefficients*. As is fundamental in Teichmüller theory, we introduce the Banach space of "Nehari-bounded holomorphic quadratic differentials"

$$B(\Delta) = \left\{\phi \in \mathrm{Hol}(\Delta) : \left\|\phi(z)(1-|z|^2)^2\right\|_\infty < \infty\right\}.$$

To every $\phi \in B(\Delta)$ we associate the L^∞ function

$$\nu_\phi = \overline{\phi(z)}(1-|z|^2)^2 \qquad \text{on } \Delta. \tag{35}$$

Foundational results of Teichmüller theory guarantee that the Beltrami coefficients $\{\nu_\phi : \phi \in B(\Delta)\}$ form a complementary subspace to the kernel N (see equation (5)) of the map $d_0\Phi$. Thus, this space of *harmonic Beltrami coefficients*:

$$H = \left\{\nu \in L^\infty(\Delta) : \nu \text{ is of the form (35) for some } \phi \in B(\Delta)\right\} \tag{36}$$

is *isomorphic* to the tangent space $T_O T(\Delta)$. In fact, this remains true for the Teichmüller space $T(G)$ of any Fuchsian group G acting on Δ, simply by replacing $B(\Delta)$ by the subspace $B(\Delta, G)$ comprising those functions which are quadratic differentials for G. See Ahlfors [2, Chapter 6] and Nag [7, Chapter 3] for all this.

Therefore the tangent vector associated to an arbitrary μ is also represented by a unique Beltrami form of harmonic type (35). For harmonic μ we get a beautifully simple formula for the Fourier coefficients, (and hence using $(*)$ also for the power series coefficients), representing that tangent vector $d_0\Phi(\mu)$. Indeed, we have:

Claim. For the harmonic Beltrami form $\mu = \overline{\phi(z)}(1-|z|^2)^2$ on Δ with $\phi \in B(\Delta)$ given by:

$$\phi(z) = h_0 + h_1 z + h_2 z^2 + \cdots, \qquad \text{in } |z| < 1.$$

The relevant Fourier coefficients a_k of the corresponding Zygmund class vector field $V[\mu]$ on S^1 are

$$\bar{a}_{-k} = a_k = \frac{2i}{(k^3 - k)} h_{k-2}, \qquad \text{for } k \geq 2. \tag{37}$$

The proof of the claim is a straightforward computation from (33).

Remark 1. In the presence of a Fuchsian group G, ϕ is a $(2,0)$-form for G and the vector field $V[\mu]$ is also G-invariant. That imposes conditions on the coefficients h_k and a_k respectively, which interact closely with the relationship (37) exhibited above.

Remark 2. The result above can be utilised to analyse why Bers coordinates are geodesic for the Weil–Petersson metric (Section 6). (Vide Ahlfors [1] and later work of Royden and Wolpert.)

To prove Theorem 1, the crucial thing is to verify our formulas (29) and (33). As we have just explained, every Beltrami form μ is uniquely the sum of an infinitesimally trivial one (equation (5)) and a harmonic form (35). Thus the formulas need to be checked only for these two types of forms. Of course, all the relevant quantities are zero for infinitesimally trivial forms. *The gist of the matter is that for harmonic Beltrami forms the Ahlfors–Weill section implies formula* (29), *whereas* (1.21) *of Ahlfors* [1] *implies formula* (33). (As Clifford Earle has remarked to me, those intriguing formulas that Ahlfors exhibited (more than thirty years ago!) in his Annals paper [1], do have a surprising way of cropping up in various contexts in later studies. Note that Ahlfors did not have conformal welding on his mind in [1].)

Ahlfors–Weill and formula (29). Let μ be a harmonic Beltrami form as above. The Ahlfors–Weill theorem tells us explicitly the schlicht function $w^{t\mu}$ on $\Delta^\star$ (for small t) and hence allows us to compute the variation of the power series coefficients, $\dot{c}_k(0)$. We refer to Section 3.8 of Nag [7]—especially 3.8.6—for that result.

In fact, let v_1 and v_2 be linearly independent solutions in the unit disc of the differential equation:

$$v'' = \phi v \tag{38}$$

normalized so that $v_1(0) = v_2'(0) = 1$ and $v_2(0) = v_1'(0) = 0$. Then Ahlfors–Weill tells us that (up to possibly a Möbius transformation)

$$w^{t\mu}(\zeta) = \bar{v}_1(1/\bar{\zeta})/\bar{v}_2(1/\bar{\zeta}) \qquad \text{for } |\zeta| > 1.$$

Solving (38) for v_1 and v_2 by the method of indeterminate coefficients yields:

$$v_1(z) = 1 + t\sum_{k=2}^{\infty} \frac{h_{k-2}}{k(k-1)} z^k + o(t),$$

$$v_2(z) = z + t\sum_{k=2}^{\infty} \frac{h_{k-2}}{k(k+1)} z^{k+1} + o(t).$$

Substituting these into Ahlfors–Weill, we deduce quickly that

$$\dot{c}_k(0) = \frac{2}{(k^3-k)}\overline{h_{k-2}} \qquad \text{for } k \geq 2.$$

For the harmonic form μ the above is exactly formula (29).

Remark. In 3.8.5 of Nag [7] a new proof of the Ahlfors–Weill theorem was given using an idea of Royden. The calculations made there are closely relevant to proving (29) *directly* for harmonic μ without passing to series expansions.

Ahlfors [1] *and formula* (33). Formula (1.21) in Ahlfors [1] in our notation reads:

$$\dot{w}[\mu] = \bar{\Phi}''\left(1-|z|^2\right)^2 + 2\bar{\Phi}'z\left(1-|z|^2\right) + 2\bar{\Phi}z^2 - 2\Phi \tag{39}$$

valid for $|z| \leq 1$, where μ is the harmonic Beltrami form. Here Φ (holomorphic in Δ) is related to ϕ by $\Phi''' = \phi$. See Ahlfors [1] formula (1.20) for this. In order to calculate the Fourier coefficients a_k, defined as in (32) above, we only require (39) on the circle $|z| = 1$. The first two terms of (39) therefore drop off, and a straightforward computation produces:

$$a_k = \frac{2i}{(k^3-k)} h_{k-2} \qquad \text{for } k \geq 2. \tag{40}$$

But this is exactly formula (37), which, as we saw, is nothing but formula (33) for harmonic Beltrami forms. Proof III is complete.

We now state the promised precise description of the complex-analytic tangent space in terms of "Zygmund class power-series":

Corollary 1. *As we saw at the end of Section* 2, *a tangent vector to* $T(\Delta)_{(b)}$ *is determined by a complex sequence* $(\dot{c}_2(0), \dot{c}_3(0), \ldots) = (\gamma_2, \gamma_3, \ldots)$, *say. Precisely those sequences* $(\gamma_2, \gamma_3, \ldots)$ *occur for which the function*

$$u(e^{i\theta}) = i\sum_{k=2}^{\infty} \bar{\gamma}_k e^{ik\theta} - i\sum_{k=2}^{\infty} \gamma_k e^{-ik\theta} \tag{$**$}$$

is in the Zygmund class on S^1.

Explicit family of examples. Here is a computable family of examples for which $w^{t\mu}$ can be explicitly determined, and hence our result can be checked. These examples are a modified form of some that were suggested to me by Clifford Earle.

Look at $\mu \in L^\infty(\Delta)$ given by

$$\mu(z) = -nz^2\bar{z}^{n-1}. \tag{41}$$

Here $n \geq 3$; ($n = 2$ works also, with minor changes).

The vector field $V[\mu]$ on S^1 has Fourier coefficients (from (33)) as exhibited:

$$a_k = \begin{cases} -i & \text{for } k = n-1 \\ i & \text{for } k = 1-n \\ 0 & \text{for any other } k \geq 2 \text{ or } \leq -2. \end{cases} \tag{42}$$

The interesting thing is that we can explicitly describe $w^{t\mu}$ for these μ, for all complex t satisfying $|t| < 1/n$. Indeed, $w^{t\mu}(\zeta) = f^{t\mu}(\zeta)/(1+t)$, where:

$$f^{t\mu}(\zeta) = \begin{cases} \zeta(1+t/\zeta^{n-1})^{-1} & \text{on } |\zeta| \geq 1 \\ (1/\zeta + t\bar{\zeta}^n)^{-1} & \text{on } |\zeta| \leq 1. \end{cases} \tag{43}$$

It is not hard to check that $f^{t\mu}$ is quasiconformal on $\mathbf{C}$, and that its complex dilatation on Δ is $t\mu$. The $\{|\zeta| \geq 1\}$ portion in (43) represents the 1-parameter family of schlicht functions, and we can write down immediately:

$$\dot{c}_k(0) = \begin{cases} -1 & \text{for } k = n-1 \\ 0 & \text{for any other } k \geq 2. \end{cases} \tag{44}$$

This corroborates Theorem 1.

Remark. In constructing these examples, it is, of course, the quasiconformal homeomorphisms (43) that were written down first; (41) and (42) were derived from it.

4. The almost complex structure

Using Theorem 1 we get an immediate proof of the fascinating fact that the almost complex structure of $T(\Delta)$ transmutes to the operation of Hilbert transform on Zygmund class vector fields on S^1. *Namely, we want to prove that the vector field* $V[\mu]$ (equation (13)) *is related to* $V[i\mu]$ *as a pair of conjugate Fourier series.*

But the tangent vector represented by μ in the complex-analytic description $T(\Delta)_{(b)}$ corresponds to a sequence $(\dot{c}_2(0), \dot{c}_3(0), \ldots)$, as explained. Since the c_k are holomorphic in μ, the tangent vector represented by $i\mu$ corresponds to $(i\dot{c}_2(0), i\dot{c}_3(0), \ldots)$. Without further ado, the relation $(*)$ of Theorem 1 shows that the k^{th} Fourier coefficient of $V[i\mu]$ is $-i \cdot \operatorname{sgn}(k)$ times the k^{th} Fourier coefficient of $V[\mu]$. We are through.

Remark 1. The Hilbert transform description of the complex structure on the tangent space of the Teichmüller space was first pointed out by S. Kerckhoff. A proof of this was important for our previous work, and appeared in [10, Part I].

Remark 2. The result above gives an independent proof of the fact that conjugation of Fourier series preserves the Zygmund class $\Lambda(S^1)$. That was an old theorem of Zygmund [15].

5. Conformal welding and its derivative

The Teichmüller point $[\mu]$ in $T(\Delta)_{(b)}$ is the univalent function F^μ [or, equivalently its image quasidisc $F^\mu(\Delta^\star)$]. The same $[\mu]$ appears in $T(\Delta)_{(a)}$ as the quasisymmetric homeomorphism w_μ on S^1. The relating map is the "conformal welding"

$$\mathbf{W}\colon T(\Delta)_{(b)} \longrightarrow T(\Delta)_{(a)}.$$

Namely, given a simply connected Jordan region D on $\widehat{\mathbf{C}}$ one looks at any Riemann mapping ϱ of Δ onto $D^\star$(= exterior of D) and also a Riemann mapping σ of $\Delta^\star$ onto D. Both ϱ and σ extend continuously to the boundaries to provide two homeomorphisms of S^1 onto the Jordan curve ∂D. We define the welding homeomorphism:

$$\mathbf{W}(D) = \varrho^{-1} o \sigma\colon S^1 \to S^1. \tag{45}$$

We can normalize by a Möbius transformation so that $\mathbf{W}(D)$ fixes 1, -1, $-i$.

Clearly, since $\varrho = w^\mu o w_\mu^{-1}$ on Δ, and $\sigma = w^\mu$ on $\Delta^\star$, work as Riemann maps, we see that $T(\Delta)_{(b)}$ and $T(\Delta)_{(a)}$ are indeed related by this fundamental operation of conformal welding.

Theorem 2. *The derivative at the origin of $T(\Delta)$ to the map $\mathbf{W}$ is the linear isomorphism:*

$$d_O\mathbf{W}\colon \big\{(\dot c_2(0), \dot c_3(0), \ldots)\big\} \to \text{Zygmund class } \Lambda(S^1) \tag{46}$$

sending $(\dot c_2(0), \dot c_3(0), \ldots)$ to the vector field $u(e^{i\theta})\partial/\partial\theta$, where

$$u(e^{i\theta}) = i\sum_{k=2}^{\infty} \overline{\dot c_k(0)} e^{ik\theta} - i\sum_{k=2}^{\infty} \dot c_k(0) e^{-ik\theta}. \tag{47}$$

Proof. Follows from Theorem 1 and the remarks above.

Remark. Conformal welding has been studied by many authors even for domains more general than quasidiscs. See, for example, Katznelson–Nag–Sullivan [5]. The derivative formula should now be extended to the larger context.

Remark. By transporting the above formula to arbitrary points of $T(\Delta)$ using right translation automorphisms, one may now develop a differential equation for the conformal welding.

6. Diff(S^1)/Möb(S^1) inside $T(\Delta)$

As usual, let $\mathrm{Diff}(S^1)$ denote the infinite dimensional Lie group of orientation preserving C^∞ diffeomorphisms of S^1. The complex- analytic homogeneous space (see Witten [14])

$$M = \mathrm{Diff}(S^1)/\mathrm{M\ddot{o}b}(S^1) \tag{48}$$

injects holomorphically into $T(\Delta)_{(a)}$. This was proved in [10, Part I]. The submanifold M comprises the "smooth points" of $T(\Delta)$; in fact, in version (b), the points from M are those quasidiscs $F^\mu(\Delta^\star)$ whose boundary curves are C^∞.

M, together with its modular group translates, foliates $T(\Delta)$—and the fundamental Kirillov–Kostant Kähler (sympletic) form (Witten [14]) exists on each leaf of the foliation. Up to an overall scaling this homogeneous Kähler metric gives the following pairing g on the tangent space at the origin of M:

$$g(V,W) = \mathrm{Re}\left[\sum_{k=2}^{\infty} a_k \bar{b}_k (k^3 - k)\right] \tag{49}$$

where

$$V = \sum_2^\infty a_k e^{ik\theta} + \sum_2^\infty \bar{a}_k e^{-ik\theta},$$
$$W = \sum_2^\infty b_k e^{ik\theta} + \sum_2^\infty \bar{b}_k e^{-ik\theta},$$

represent two smooth vector fields on S^1.

The metric g on M was proved by this author [10, Part II] to be *the (regularized) Weil–Petersson metric (WP) of universal Teichmüller space*. Theorem 1 allows us to express the pairing for $g = WP$ in terms of 1-parameter flows of schlicht functions.

Theorem 3. *Let $F_t(\zeta)$ and $G_t(\zeta)$ denote two curves through origin in $T(\Delta)_{(b)}$ of the form (22), representing two tangent vectors, say $\dot{F}$ and $\dot{G}$. Then the Weil–Petersson pairing assigns the inner product*

$$\mathrm{WP}(\dot{F},\dot{G}) = -\,\mathrm{Re}\left[\sum_{k=2}^{\infty} \overline{\dot{c}_k(0)}\,\dot{d}_k(0)(k^3 - k)\right], \tag{50}$$

where $c_k(t)$ and $d_k(t)$ are the power series coefficients for the schlicht functions F_t and G_t respectively, as in (22). The series above converges absolutely whenever the corresponding Zygmund class functions (see equations $(**)$ of Corollary 1 or (47)) are in the Sobolev class $H^{3/2}$.

Proof. Combine Theorem 1 with (49). The convergence statement follows directly from [10, Part II].

7. Variational formula for the period mapping

Recently in [8], [9] the author has studied a generalisation of the classical period mappings to the infinite dimensional context of universal Teichmüller space. Indeed, there is a natural equivariant, holomorphic and Kähler-isometric immersion

$$\prod : M \longrightarrow D_\infty \tag{51}$$

of $M = \mathrm{Diff}(S^1)/\mathrm{M\ddot{o}b}(S^1)$ into the infinite dimensional version, D_∞, of the Siegel disc. D_∞ is a complex manifold comprising certain complex symmetric Hilbert–Schmidt $(Z_+ \times Z_+)$ matrices. $\prod$ qualifies as a generalised period matrix map, and its variation satisfies a Rauch-type formula, see [9]. An extension of $\prod$ to *all* of $T(\Delta)$ has now been established in Nag–Sullivan (IHES preprint [11]).

In the works cited above we proved that for arbitrary μ in $L^\infty(\Delta)$, the $(rs)^{\mathrm{th}}$-entry of the period matrix satisfies $(r, s \geq 1)$

$$\prod\big([t\mu]\big)_{rs} = t\sqrt{-rs}\, a_{-(r+s)} + O(t^2) \tag{52}$$

as $t \to 0$. Here a_k as usual denotes the Fourier coefficients of the vector field represented by μ (equation (32)). (52) is the Rauch variational formula in the universal Teichmüller space context.

By Theorem 1 we see that the formula (52) may be written

$$\prod\big([t\mu]\big)_{rs} = \sqrt{rs}\, c_{r+s}(t) + O(t^2) \tag{53}$$

where $c_k(t)$, as usual, are the power series coefficients appearing in (22) for the schlicht functions $F^{t\mu}$. Equation (53) may be compared with the formula [(30) in their paper] claimed by Hong–Rajeev [4] in this setting.

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