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A SUFFICIENT CONDITION FOR TEICHMÜLLER SPACES TO HAVE SMALLEST POSSIBLE INNER RADII

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Abstract. If $\{\mathscr{R}_n\}_{n=1}^{\infty}$ is a sequence of hyperbolic Riemann surfaces such that for any $r>0$ a hyperbolic ball of radius r embeds isometrically into $\mathscr{R}_n$ for all n sufficiently large, then as $n\to\infty$ the inner radii of the associated Teichmüller spaces $i(\mathbf{T}(\mathscr{R}_n))\to 2$. A straightforward consequence of this is that if a Riemann surface $\mathscr{R}$ contains arbitrarily large hyperbolic balls then $i(\mathbf{T}(\mathscr{R}))=2$.

1. Statement of results

Let Γ be a Fuchsian group acting on the upper half plane $\mathscr{H}$, and hence also on the lower half plane $\mathscr{H}^*$, in the complex plane. Let $\mathrm{Q}^{\infty}(\Gamma)$ denote the complex Banach space of bounded quadratic differentials with respect to Γ defined in $\mathscr{H}^*$. The Teichmüller space $\mathbf{T}(\Gamma)$ of Γ is realized via the Bers embedding as a bounded region in $\mathrm{Q}^{\infty}(\Gamma)$.

The *inner radius* $i(\Gamma)$ of $\mathbf{T}(\Gamma)$ is the supremum of radii of balls in $\mathrm{Q}^{\infty}(\Gamma)$ centered at the origin which are contained in $\mathbf{T}(\Gamma)$. It has long been known [2] that if $\mathscr{H}^*$ carries the hyperbolic metric of curvature $\equiv -4$ then

$$i(\Gamma)\geq 2 \qquad \text{whenever} \qquad \mathbf{T}(\Gamma)\neq\{0\}. \tag{1.1}$$

If Γ is an elementary group then we have equality in (1.1), [4]. On the other hand it can be shown, following Gehring and Pommerenke [3], that the inequality in (1.1) is strict for cofinite Γ ([6], [11]). We now state our main theorem:

Theorem 1. *Let $\{\Gamma_n\}_{n=1}^{\infty}$ be a sequence of Fuchsian groups. Assume that for any $r>0$ a hyperbolic ball of radius r embeds isometrically in $\mathscr{H}^*/\Gamma_n$ for all n sufficiently large. Then* $\inf\bigl(i(\Gamma_n)\bigr)=2$.

A straightforward consequence of this theorem is

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Theorem 2. *Let Γ be a Fuchsian group and assume that the Riemann surface $\mathscr{H}^*/\Gamma$ contains arbitrarily large hyperbolic balls, i.e. for any $r > 0$ a hyperbolic ball of radius r embeds isometrically in $\mathscr{H}^*/\Gamma$. Then $i(\Gamma) = 2$.*

This is precisely the condition (O_1) in [13], as will be discussed in the next section. To obtain this result apply Theorem 1 to $\Gamma_n = \Gamma$, $n = 1, 2, \dots$.

Also following immediately is

Theorem 3. *If Γ is a Fuchsian group of the second kind then $i(\Gamma) = 2$.*

2. Preliminaries

Let $\mathbf{H}^2$ denote 2-dimensional hyperbolic space with conformal metric $ds = \varrho\,|dz|$ and curvature -4. Two commonly used conformal models of this are $\mathscr{H}$, with metric $|dz|/2y$, and the unit disc $\mathbf{D}$, with metric $|dz|/(1-|z|^2)$. A holomorphic quadratic differential φ on $\mathbf{H}^2$ is a holomorphic section of $T^{2,0}(\mathbf{H}^2)$, so that if z, w are two conformal coordinates on $\mathbf{H}^2$ and φ_z, φ_w the respective coordinate representations of φ then

$$\varphi_w = \varphi_z\left(\frac{dz}{dw}\right)^2. \tag{2.1}$$

A Beltrami differential μ on $\mathbf{H}^2$ is a section of $L^\infty(\mathbf{H}^2) \otimes T^{0,1}(\mathbf{H}^2) \otimes T^{-1,0}(\mathbf{H}^2)$, so that

$$\mu_w = \mu_z\overline{\left(\frac{dz}{dw}\right)}\left(\frac{dz}{dw}\right)^{-1}. \tag{2.2}$$

There is a natural metric on the Beltrami differentials:

$$|\mu|_\infty = \sup_{x \in \mathbf{H}^2} |\mu(\mathrm{x})|. \tag{2.3}$$

We also have a natural metric on quadratic differentials:

$$\|\varphi\|_\infty = \sup_{x \in \mathbf{H}^2} \varrho(\mathrm{x})^{-2}|\varphi(\mathrm{x})|. \tag{2.4}$$

One readily checks that both of these are coordinate independent, and hence are intrinsic to $\mathbf{H}^2$.

If we realize $\mathbf{H}^2$ conformally by a simply connected domain $\Omega \subset \hat{\mathbf{C}}$, we will denote by $\mathrm{Q}^\infty(\Omega)$ the space of holomorphic quadratic differentials $\{$holomorphic $\varphi \in T^{2,0}(\Omega) : \|\varphi\|_\infty < \infty\}$. We also let $\mathrm{Bel}\,(\Omega) = \{\mu \in L^\infty(\Omega) \otimes T^{0,1}(\Omega) \otimes T^{-1,0}(\Omega) : |\mu|_\infty < 1\}$.

Let Γ denote a discrete subgroup of the orientation preserving isometries of $\mathbf{H}^2$. (If $\mathbf{H}^2$ is realized as the image of $\mathbf{D}$ under a Möbius transformation then Γ

is called a Fuchsian group.) We say the quadratic differential φ is Γ-invariant if it satisfies

$$\varphi(z)=\varphi\big(\gamma(z)\big)\cdot\gamma'(z)^2 \qquad \text{for all} \quad \gamma\in\Gamma,\ z\in\mathbf{H}^2. \tag{2.5}$$

Similarly μ is said to be a Beltrami differential with respect to Γ if

$$\mu(z)=\mu\big(\gamma(z)\big)\overline{\gamma'(z)}\big(\gamma'(z)\big)^{-1} \qquad \text{for all} \quad \gamma\in\Gamma,\ z\in\mathbf{H}^2. \tag{2.6}$$

The above norms are Γ-invariant for such forms. Thus if Ω is as above and Γ acts on Ω we denote by $\mathrm{Q}^\infty(\Omega;\Gamma)$ and $\mathrm{Bel}\,(\Omega;\Gamma)$ the subspaces of $\mathrm{Q}^\infty(\Omega)$ and $\mathrm{Bel}\,(\Omega)$ which are Γ-invariant.

If $\mu\in\mathrm{Bel}\,(\Omega)$ we will henceforth always suppose it to be extended to be 0 on $\hat{\mathbf{C}}\setminus\Omega$, so that $\mu\in\mathrm{Bel}\,(\hat{\mathbf{C}})$. Consider now the PDE

$$\partial_{\bar z} f=\mu\cdot\partial_z f \qquad \text{on} \quad \hat{\mathbf{C}}. \tag{2.7}$$

This equation has a solution f^μ which is a homeomorphism of $\hat{\mathbf{C}}$ and is defined uniquely up to post-composition by Möbius transformations. Now if $\hat{\mathbf{C}}\setminus\Omega$ has interior, f^μ is conformal on this open domain. In particular, letting $1/\mathbf{D}=\hat{\mathbf{C}}\setminus\mathrm{cl}\,(\mathbf{D})$ ($\mathrm{cl}\,(X)$ means the closure of the set X), if $\Omega=\mathscr{H}$ (or $\mathbf{D}$) and the interior of its complement is $\mathscr{H}^*$ (or $1/\mathbf{D}$), then $f^\mu\colon\mathscr{H}^*$ (or $1/\mathbf{D}$) $\to\hat{\mathbf{C}}$ is univalent.

Given that $f\colon\mathbf{H}^2\to\hat{\mathbf{C}}$ is a locally univalent map, and z is a conformal parameter on $\mathbf{H}^2$, the Schwarzian derivative of f, $\mathscr{S}_f$, is defined by

$$\mathscr{S}_f(z)=\left(\left(\frac{f''}{f'}\right)'-\frac{1}{2}\left(\frac{f''}{f'}\right)^2\right)(z). \tag{2.8}$$

One verifies that $\mathscr{S}_f(z)\equiv 0$ if and only if $f(z)$ is Möbius, and that if w is another conformal parameter on $\mathbf{H}^2$ with $z=g(w)$ then

$$\mathscr{S}_{f\circ g}(w)=\mathscr{S}_f(z)\left(\frac{dz}{dw}\right)^2+\mathscr{S}_g(w). \tag{2.9}$$

Hence if γ is a Möbius transformation then $\mathscr{S}_{\gamma\circ f}(z)=\mathscr{S}_f(z)$, and $\mathscr{S}_{f\circ\gamma}(z)=\mathscr{S}_f(\gamma z)\gamma'(z)^2$.

Henceforth we will assume that $\mathbf{H}^2$ is so realized, and our Γ are thus all Fuchsian groups. The latter of these transformation laws tells us that if $\mathbf{H}^2$ is realized as $\mathscr{H}^*$ or $1/\mathbf{D}$ and f^μ is univalent, then $\mathscr{S}_{f^\mu}$ is a quadratic differential on the domain. By a theorem of Kraus [5] we have (in the case of $\mathscr{H}^*$)

$$\|\mathscr{S}_{f^\mu}\|_\infty=\sup_{z\in\mathscr{H}^*}4(\operatorname{Im} z)^2\big|\mathscr{S}_{f^\mu}(z)\big|\le 6. \tag{2.10}$$

The same estimate holds in the case of $1/\mathbf{D}$. Hence $\mu\mapsto\mathscr{S}_{f^\mu}$ maps $\mathrm{Bel}\,(\mathscr{H})$ to $\mathrm{Q}^\infty(\mathscr{H}^*)$. If $\Gamma<\mathrm{PSL}\,(2,\mathbf{R})$ then one checks that in fact $\mathrm{Bel}(\mathscr{H};\Gamma)\to\mathrm{Q}^\infty(\mathscr{H}^*;\Gamma)$, and the same holds in the case of $\mathbf{D}$ and $1/\mathbf{D}$.

Definition. *The Teichmüller space* $\mathbf{T}(\Gamma)$ of Γ is the image of $\mathrm{Bel}\,(\mathscr{H};\Gamma)$ in $\mathrm{Q}^\infty(\mathscr{H}^*;\Gamma)$ under the mapping $\mu \mapsto \mathscr{S}_{f^\mu}$.

The following numbers are called the outer radius and the inner radius, respectively, of $\mathbf{T}(\Gamma)$:

$$\begin{aligned} o(\Gamma) &= \sup\bigl\{\|\varphi\|_\infty : \varphi \in \mathbf{T}(\Gamma)\bigr\},\\ i(\Gamma) &= \inf\bigl\{\|\varphi\|_\infty : \varphi \in \mathrm{Q}^\infty(\mathscr{H}^*;\Gamma) \setminus \mathbf{T}(\Gamma)\bigr\}. \end{aligned}$$

By (2.4) we have $o(\Gamma) \le 6$. As mentioned in the introduction, Ahlfors and Weill showed that $i(\Gamma) \ge 2$. In [13], Nakanishi and Yamamoto showed that $o(\Gamma) = 6$ if and only if one of the following conditions are satisfied:

(O_1) for any $r > 0$, a hyperbolic geodesic ball of radius r embeds isometrically in $\mathbf{H}^2/\Gamma$ (let $B_r(z)$ denote the ball of hyperbolic radius r about $z \in \mathbf{H}^2/\Gamma$), or

(O_2) for any $d > 0$, a collar of width d exists about the axis of some hyperbolic element of Γ.

It follows from work of Gehring and Pommerenke that $i(\Gamma) > 2$ for cofinite Γ, and the authors showed in [12] that (O_2) implies $i(\Gamma) = 2$. Herein we show that (O_1) also implies this fact.

We would like to note that these theorems may be thought of as the analytic equivalent of results of Curt McMullen regarding the geometric limits of quadratic differentials (see the appendix of [9]).

3. Several lemmas

Let $A \in \mathrm{M\ddot{o}b}\,(\hat{\mathbf{C}})$ be a transformation sending $\mathbf{D}$ onto $\mathscr{H}$ and $\mu \in \mathrm{Bel}\,(\mathscr{H};\Gamma)$, where Γ is a Fuchsian group acting on $\mathscr{H}$. Set $\nu(z) = \mu\bigl(A(z)\bigr)\overline{A'(z)}\bigl(A'(z)\bigr)^{-1}$ (and $\nu(z) = 0$ for $z \in 1/\mathbf{D}$). We consider the Beltrami equation

$$\frac{\partial f}{\partial \bar z} = \nu \cdot \frac{\partial f}{\partial z}. \tag{3.1}$$

If f^ν is a solution of (3.1), then $f^\mu = f^\nu \circ A^{-1}$ satisfies $\partial_{\bar z} f^\mu = \mu \cdot \partial_z f^\mu$. For more details of the following description, see [1, Chapter V]. Consider the following two operators from $L^p(\mathbf{C})$ $(p > 2)$ to itself:

$$\begin{aligned} Ph(z) &= \frac{1}{2\pi i}\int\!\!\int \frac{h(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar\zeta\\ Th(z) &= \mathrm{p.v.}\,\frac{1}{2\pi i}\int\!\!\int \frac{h(\zeta)}{(\zeta - z)^2}\, d\zeta \wedge d\bar\zeta. \end{aligned} \tag{3.2}$$

The operator T is also defined on $L^2(\mathbf{C})$, where it is an isometry. For $p > 2$, there exist constants C_p with $\lim_{p\to 2} C_p = 1$ and $\|T\|_p < C_p$. Let $k = \mathrm{ess.\,sup.}|\mu| =$

ess. sup.$|\nu|$ and choose $p > 2$ so that $kC_p < 1$. Let h be a solution in L^p of the equation $h = T\big(\nu(h+1)\big)$. Then the following function f is a solution of (3.1):

$$\begin{aligned} f(z) &= z + P(\nu(h+1))(z) \\ &= z + \frac{1}{2\pi i}\int\!\!\int \nu(\zeta)\big(h(\zeta)+1\big)\Big[\frac{1}{\zeta - z}\Big]\, d\zeta \wedge d\bar\zeta. \end{aligned} \tag{3.3}$$

Remark. The definition of P is slightly different from that in [1]. This is because in our treatment we do not need the normalized solution such that $f(0) = 0$.

Let $z_0 \in \mathbf{D}$ and $z_0^* = \bar z_0^{-1} \in 1/\mathbf{D}$. Letting $\mathbf{D}$ and $1/\mathbf{D}$ have hyperbolic metrics of curvature -4, we denote by $B_r(z_0)$ the hyperbolic disk of radius r and center z_0. Let $\operatorname{Bel}^k(\mathbf{D}) = \big\{\mu \in \operatorname{Bel}(\mathbf{D}) : \|\mu\|_\infty \le k < 1\big\}$. We now have a lemma which will be used in the proof of the ensuing one.

Lemma 3.1. *If $\nu \in \operatorname{Bel}^k(\mathbf{D})$ and if f^ν is the solution of (3.1) satisfying $f^\nu(z) = z + O(1/z)$ at $z = \infty$, then there exist constants $a(k,r)$, depending only on k and r, such that the planar measure of $f^\nu\big(\mathbf{D} \setminus B_r(0)\big)$ is $< a(k,r)$. Moreover, for each fixed k, $a(k,r) \to 0$ as $r \to \infty$.*

Proof. From Koebe's famous $\frac{1}{4}$-theorem (see [7, Sektion 28, Satz 6]) it is trivial that there exists such a constant, as $f^\nu(1/\mathbf{D})$ contains $\big\{z : |z| > 2\big\}$. To show that as $r \to \infty$, $a(k,r) \to 0$, suppose to the contrary that there exist $\varepsilon > 0$ and $\nu_n \in \operatorname{Bel}^k(\mathbf{D})$ such that $f^{\nu_n}\big(\mathbf{D} \setminus B_n(0)\big)$ has planar area $> \varepsilon$. The fact that the f^{ν_n} form a normal family (see [8, Chapter II.5.2, Theorem 5.1]), with every limit f^ν satisfying (3.1) for some $\nu \in \operatorname{Bel}^k(\mathbf{D})$, implies that the limiting f^ν maps the unit circle onto a positive measure set in $\mathbf{C}$. This is impossible as quasiconformal maps are absolutely continuous with respect to Lebesgue measure [1, Chapter II B, Theorem 3].

Lemma 3.2. *There exist constants $b(k,r)$, depending only on k and r, such that if μ, $\nu \in \operatorname{Bel}^k(\mathbf{D})$ with $\mu \equiv \nu$ on $B_r(z_0)$ then*

$$\varrho_{1/\mathbf{D}}(z_0^*)^{-2}\big|\mathscr{S}_{f^\mu}(z_0^*) - \mathscr{S}_{f^\nu}(z_0^*)\big| = \big(|z_0^*|^2 - 1\big)^2\big|\mathscr{S}_{f^\mu}(z_0^*) - \mathscr{S}_{f^\nu}(z_0^*)\big| < b(k,r).$$

Moreover, for each fixed k,

$$b(k,r) \to 0 \qquad \text{as} \qquad r \to \infty.$$

Proof. The existence of bounded $b(k,r)$ is straightforward from the Kraus estimate [5]. To see that we may arrange for $b(k,r) \to 0$ as $r \to \infty$, consider sequences $\big\{\mu_n \in \operatorname{Bel}^k(\mathbf{D})\big\}$ and $\big\{\nu_n \in \operatorname{Bel}^k(\mathbf{D})\big\}$ such that $\mu_n \equiv \nu_n$ on $B_n(z_0)$. It suffices to prove that $\varrho_{1/\mathbf{D}}(z_0^*)^{-2}\big|\mathscr{S}_{f^{\mu_n}}(z_0^*) - \mathscr{S}_{f^{\nu_n}}(z_0^*)\big| \to 0$ as $n \to \infty$.

Precomposing by a Möbius transformation, we may assume that $z_0 = 0$, $z_0^* = \infty$. We will show that

$$\varrho_{1/\mathbf{D}}(\infty)^{-2}\left|\mathscr{S}_{f^{\mu_n}}(\infty) - \mathscr{S}_{f^{\nu_n}}(\infty)\right| = \varrho_{1/\mathbf{D}}(\infty)^{-2}\left|\mathscr{S}_{f^{\mu_n}\circ(f^{\nu_n})^{-1}}(\infty)\right| \to 0$$

as $n \to \infty$. Consider now $f_n = f^{\mu_n} \circ (f^{\nu_n})^{-1}$ on $\hat{\mathbf{C}}$. This is holomorphic except on $f^{\nu_n}\big(\mathbf{D} \setminus B_n(0)\big)$. On this set the quasiconformal deformation is

$$\tilde{\mu}_n = \left(f_z^{\nu_n}(\overline{f_z^{\nu_n}})^{-1}\frac{\mu_n - \nu_n}{1 - \bar{\nu_n}\mu_n}\right) \circ (f^{\nu_n})^{-1}$$

with

$$\begin{aligned}
|\tilde{\mu}_n|_\infty &= \sup_{z\in\mathbf{D}}\left|\left(f_z^{\nu_n}(\overline{f_z^{\nu_n}})^{-1}\frac{\mu_n - \nu_n}{1 - \bar{\nu_n}\mu_n}\right) \circ (f^{\nu_n})^{-1}\right| \\
&= \sup_{z\in\mathbf{D}}\left|\left(\frac{\mu_n - \nu_n}{1 - \bar{\nu_n}\mu_n}\right) \circ (f_n^{\nu})^{-1}\right| \le \frac{2k}{1+k^2} = \kappa.
\end{aligned}$$

Now let $f_n(z)$ have expansion $z + c_{n,1}z^{-1} + c_{n,2}z^{-2} + \cdots$ at $z = \infty$, so that

$$c_{n,1} = \frac{-1}{2\pi i}\iint \tilde{\mu}_n(\zeta)\big(h_n(\zeta) + 1\big)\,d\zeta\,d\bar{\zeta}$$

from (3.3). Thus

$$\varrho_{1/\mathbf{D}}(\infty)^{-2}\left|\mathscr{S}_{f_n}(\infty)\right| = 6|c_{n,1}| = \varrho_{1/\mathbf{D}}(\infty)^{-2}\left|\mathscr{S}_{f^{\mu_n}}(\infty) - \mathscr{S}_{f^{\nu}}(\infty)\right|$$

and, since h_n is an L^p solution to $h_n = T\big(\tilde{\mu}_n(h_n + 1)\big)$,

$$\|h_n\|_p \le C_p(1 - \kappa C_p)^{-1}\|\tilde{\mu}_n\|_p.$$

Hence if $\alpha(n)$ denotes the planar area of $f^{\nu_n}\big(\mathbf{D} \setminus B_n(0)\big)$ we have

$$\begin{aligned}
\varrho_{1/\mathbf{D}}(\infty)^{-2}\left|\mathscr{S}_{f_n}(\infty)\right| &= 6|c_{n,1}| \\
&\le 3\pi^{-1}\big\{\kappa\alpha(n) + \|h_n\|_p\|\tilde{\mu}_n\|_q\big\} \qquad (p^{-1} + q^{-1} = 1) \\
&\le 3\pi^{-1}\big\{\kappa\alpha(n) + C_p(1 - \kappa C_p)^{-1}\|\tilde{\mu}_n\|_p\|\tilde{\mu}_n\|_q\big\} \\
&\le 3\kappa a(k,r)\pi^{-1}\big\{1 + C_p(1 - \kappa C_p)^{-1}\kappa\big\} \\
&= b(k,n),
\end{aligned}$$

where $a(k,n)$ is as in Lemma 3.1. For $r \in [n, n+1)$ let $b(k,r) = b(k,n)$. Evidently $b(k,r) \to 0$ as $r \to \infty$, completing the proof.

A special family in the universal Teichmüller space [4]. Let Γ be the trivial group $\langle 1 \rangle$. The quadratic differentials $\varphi_\alpha(z) = \alpha z^{-2}\,dz^2$ ($\alpha \in \mathbf{C}$) belong to $\mathrm{Q}^\infty(\mathscr{H}^*)$. For either root δ of the equation $2\alpha = 1-\delta^2$ we have $\mu_\alpha(z) = (\delta-1)z/\bar{z}$ is a Beltrami differential on $\mathscr{H}$. In this case the Beltrami equation $\partial_{\bar{z}} f = \mu_\alpha \cdot \partial_z f$ is solved explicitly by $f_\alpha(z) = z\bar{z}^{\delta-1}$ which has a conformal continuation $f_\alpha(z) = z^\delta$ to $\mathscr{H}^*$.

We readily check that $\mathscr{S}_{f_\alpha} = \varphi_\alpha$ in $\mathscr{H}^*$. As $|\delta - 1| < 1$ implies

$$\alpha \in \Lambda = \left\{ \tfrac{1}{2}(1 - re^{2i\theta}) \in \mathbf{C} : r < 4\cos^2\theta, 0 \le \theta \le \pi \right\},$$

we conclude that $\{\varphi_\alpha : \alpha \in \Lambda\} \subset \mathbf{T}(1)$ and (letting $\theta = 0$) observe that $o(\Gamma) = 6$ and $i(\Gamma) = 2$. We remark also that a function f in $\mathscr{H}^*$ with $\mathscr{S}_f = \varphi_\alpha$ is univalent in $\mathscr{H}^*$ if and only if $\alpha \in \mathrm{cl}(\Lambda)$.

Consider the Möbius transformation

$$z \mapsto -it\frac{z-(i/t)}{z-it}$$

which maps $B_{\ln t}(i)$ to $\mathscr{H}$. We pull μ_α back to a Beltrami differential $\mu_{\alpha,t}$ with support on $B_{\ln t}(i)$:

(3.4)

$$\mu_{t,\alpha}(z) = \begin{cases} -(\delta-1)(z^2 - i(t+\frac{1}{t})z - 1)\overline{(z^2 - i(t+\frac{1}{t})z-1)}^{-1}, & z \in B_{\ln t}(i), \\ 0, & z \in \hat{C} \setminus B_{\ln t}(i). \end{cases}$$

The solution f^{μ_α} pulls back to a solution to $\partial_{\bar{z}} f = \mu_{\alpha,t}\partial_z f$ given by

$$f_{t,\alpha}(z) = \Big[-it\frac{z-i/t}{z-it} \Big]^\delta$$

on $\hat{\mathbf{C}} \setminus B_{\ln t}(i)$, etc. Thus on $\mathscr{H}^*$, the Schwarzian derivative of this univalent function is

$$\varphi_{\alpha,t}(z) = -\alpha\frac{(t-1/t)^2}{(z-i/t)^2(z-it)^2}\,dz^2.$$

We see that $f_{t,\alpha}(z) \to z^\delta$ and $\varphi_{t,\alpha} \to \varphi_\alpha$ on any compact subset of $\mathscr{H}^*$ as $t \to \infty$. Furthermore, the following lemma shows that $\|\varphi_{t,\alpha}\|_\infty$ increases monotonically to $\|\varphi_\alpha\|_\infty$ as $t \to \infty$.

Lemma 3.3. *If $\varphi \in \mathrm{Q}^\infty(\mathscr{H}^*)$ and $f\colon \mathscr{H}^* \to \mathscr{H}^*$ is locally univalent, let $f^*\varphi$ be the pullback of φ to $\mathscr{H}^*$ via f, i.e. $f^*\varphi = (\varphi \circ f)(f')^2$. Then $\|f^*\varphi\|_\infty \le \|\varphi\|_\infty$, and if $f(\mathscr{H}^*)$ is precompact in $\mathscr{H}^*$ then the inequality is strict.*

Proof. This is immediate from the Schwarz–Pick lemma, as $\varrho\big(f(z)\big)|f'| \le \varrho(z)$ implies that $\big|f^*\varphi(z)\big|\varrho^{-2}(z) \le \big|\varphi\big(f(z)\big)\big|\varrho^{-2}\big(f(z)\big)$. If $f(\mathscr{H}^*)$ is precompact in $\mathscr{H}^*$ then $\varrho\big(f(z)\big)\big|f'(z)\big| < k\varrho(z)$ for some $k < 1$.

4. Proof of Theorem 1

To prove Theorem 1 in [12], it was shown to be sufficient to prove Proposition 4.1 in that paper. The situation here is similar, and the proof of our Theorem 1 follows, verbatum, from

Proposition 4.1. *Let $\{\Gamma_n\}$ be a sequence of Fuchsian groups satisfying the hypothesis of Theorem 1. Let $\alpha \in \Lambda$. Then there are $\varphi_n \in \mathbf{T}(\Gamma_n)$ such that $\{\varphi_n\}$ contains a subsequence $\{\varphi_{n_j}\}$ with $\varphi_{n_j} \to \varphi_\alpha(z) = \alpha z^{-2}\,dz^2$ uniformly on compact sets in $\mathscr{H}^*$, and $\|\varphi_{n_j}\|_\infty \to 4|\alpha| = \|\varphi_\alpha\|_\infty$ as $j \to \infty$.*

Hence we will show that this proposition is true.

Let $z_n \in \mathscr{H}$ be such that $\gamma_n\bigl(B_{2r_n}(z_n)\bigr) \cap B_{2r_n}(z_n) = \emptyset$ for all $\gamma_n \in \Gamma_n$, $\gamma_n \neq \mathrm{id}$, and $r_n \to \infty$ as $n \to \infty$. By conjugating Γ_n we may assume without loss of generality that $z_n = i$.

Now, fixing $\alpha \in \Lambda$, let $\mu_n = \mu_{r_n,\alpha}$; see (3.4). Let f^{μ_n} be the solutions $\partial_{\bar z} f = \mu_n \cdot \partial_z f$ normalized at $-i$. Then we have seen (following (3.4), where δ satisfies $|\delta - 1| < 1$ and $2\alpha = 1 - \delta^2$) that as $n \to \infty$, $f^{\mu_n} \to z^\delta$ uniformly on compact sets in $\mathscr{H}^*$, and $\|S_{f^{\mu_n}}\|_\infty \leq 4|\alpha|$.

We define $\nu_n(z) = \sum_{\gamma\in\Gamma_n} \mu_n\bigl(\gamma(z)\bigr)\overline{\gamma'(z)}\bigl(\gamma'(z)\bigr)^{-1}$ so that $\nu_n = 0$ on $\mathscr{H}^*$. Then in the Dirichlet fundamental region $\mathscr{F}(i)$ we have $(\nu_n - \mu_n)(z) = 0$ in $B_{r_n}(z)$, so that $\bigl|S_{f^{\nu_n}}(z) - S_{f^{\mu_n}}(z)\bigr|\varrho_{\mathbf{H}^2}^{-2}(z) < a(k,r)$. Letting n be large enough we have that $\|S_{f^{\nu_n}}\|_\infty < 4|\alpha| + \varepsilon$. Also as $\nu_n \to \mu_n$ in measure we get that $f^{\nu_n} \to z^\delta$ uniformly on compact sets, which completes the proof of the lemma and hence the theorem.

Remark. In the above argument we used several properties of quasiconformal mappings. The theory of these maps is found in [9] (in particular Section 5 of Chapter II and Section 4 of Chapter V).

Remark. Proposition 4.1 does not mean that $\|\varphi_{n_j} - \varphi_\alpha\|_\infty \to 0$ (see Section 4 of [11]).

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