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APPROXIMATION OF THE HERSCH–PFLUGER DISTORTION FUNCTION

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Abstract. This paper aims at giving a method of approximating the distortion function Φ_K by means of elementary functions. As a consequence, new bounds for the function Φ_K are established. Moreover, the n-dimensional counterpart $\Phi_{K,n}$ of Φ_K, $n = 2, 3, \ldots$ is considered.

0. Introduction

Let $\Phi_K(r) = \mu^{-1}\bigl(\mu(r)/K\bigr)$, $0 < r < 1$, $\Phi_K(0) = 0$, $\Phi_K(1) = 1$, $K > 0$, where μ stands for the module of the Grötzsch extremal domain $\mathbf{B}^2 \setminus [0, r]$. Φ_K is called the Hersch–Pfluger distortion function, cf. [HP], and it plays an important role in the theory of plane quasiconformal mappings. In the first section we examine the functions $\varphi_{K,t}$, $\widetilde{\varphi}_{K,t}$, $\psi_{K,t}$, $\widetilde{\psi}_{K,t}$, depending on a real parameter $t \geq 1$, defined by (1.8) and (1.20). We show (Theorem 1.3 and Corollary 1.4) that these sequences are monotonically convergent to the function Φ_K. Putting $t = 2^n$, $n = 0, 1, 2, \ldots$, we are able to approximate Φ_K in a simple way by elementary functions with an arbitrarily preassigned accuracy. In Section 2 we derive (Theorems 2.1 and 2.2) new upper and lower bounds for the function Φ_K, which improve some recent results obtained by Anderson, Vamanamurthy, Vuorinen [AVV1], [AVV3] and Zając [Z1], [Z2]. In the last section of this paper we study the approximation problem for the n-dimensional counterpart $\Phi_{K,n}$, cf. [V], [AVV2], of Φ_K, $n = 2, 3, \ldots$. We establish Theorem 3.1, the n-dimensional counterpart of Theorem 1.3.

Actually, in view of Theorem 1.5 and Corollary 1.6 the sequences $\psi_{K,2^n}$, $\widetilde{\psi}_{K,2^n}$, $n = 0, 1, 2, \ldots$ are convergent to the function Φ_K very fast. So they may be used conveniently for the calculation of the values of Φ_K by a computer. The details will be given in a separate paper [P] in which we shall also estimate the familiar functions μ, μ^{-1} and the distortion function λ, cf. [LV], [L], introduced by Lehto, Virtanen and Väisälä in [LVV], as an application of Theorems 1.3, 1.5 and Corollaries 1.4, 1.6.

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1. Basic approximation theorems

The function Φ_K was studied by many mathematicians. Recently Anderson, Vamanamurthy, Vuorinen, cf. [AVV1], [AVV3], [VV], and Zając, cf. [Z1], [Z2] obtained many interesting results concerning the properties of the function Φ_K and its estimates. We shall now state some inequalities concerning Φ_K which will be used later on.

Lemma 1.1. *) *For any $p > 1$ and $0 < x < 1$*

$$\Phi_K(x^p) < \Phi_K^p(x), \qquad K > 1 \tag{1.1}$$

and

$$\Phi_K(x^p) > \Phi_K^p(x), \qquad 0 < K < 1. \tag{1.2}$$

Proof. We shall first prove by induction that the inequality (1.1) holds for all $p = 1 + 1/n$, $n = 1, 2, \ldots$. To this end we apply the fact shown in [AVV1] that for any $0 < c < 1$ and $K > 1$ the function

$$f(t) = \frac{\Phi_K(ct)}{\Phi_K(t)}, \qquad 0 < t \le 1 \tag{1.3}$$

is strictly increasing. Thus setting $c = x$ and $t_1 = x < t_2 = 1$, we have $f(t_1) < f(t_2)$ for $n = 1$ so

$$\Phi_K(x^{1+1/n}) < \Phi_K^{1+1/n}(x). \tag{1.4}$$

If now $n = 1, 2, \ldots$ is any positive integer and (1.4) holds then setting $c = x^{1/(n+1)}$ and $t_1 = x < t_2 = x^{n/(n+1)}$ we have $f(t_1) < f(t_2)$ from which

$$\Phi_K(x^{1+1/(n+1)}) < \Phi_K(x)\frac{\Phi_K(x)}{\Phi_K(x^{n/(n+1)})}. \tag{1.5}$$

But, in view of the induction assumption we get

$$\Phi_K(x) = \Phi_K\big((x^{n/(n+1)})^{1+1/n}\big) < \Phi_K^{1+1/n}(x^{n/(n+1)}).$$

This and (1.5) yield

$$\Phi_K(x^{1+1/(n+1)}) < \Phi_K^2(x)\Phi_K^{-n/(n+1)}(x) = \Phi_K^{1+1/(n+1)}(x)$$

*) The inequalities (1.1) and (1.2) have been proved independently in a different way in [VV].

so the inequality (1.1) is true for every $p = 1 + 1/n$, $n = 1, 2, \ldots$. If now m is any positive integer and the inequality (1.1) holds for $p = 1 + m/n$, where n is any fixed positive integer, then setting $c = x^{m/n}$ and $t_1 = x^{1+1/n} < t_2 = x$ we have $f(t_1) < f(t_2)$ so

$$\begin{aligned}\Phi_K(x^{1+(m+1)/n}) &< \Phi_K(x^{1+1/n})\frac{\Phi_K(x^{1+m/n})}{\Phi_K(x)} < \Phi_K^{1+1/n}(x)\frac{\Phi_K^{1+m/n}(x)}{\Phi_K(x)}\\ &= \Phi_K^{1+(m+1)/n}(x),\end{aligned}$$

because of (1.4). This way the inequality (1.1) holds for any rational number $p > 1$. It follows from the continuity of the function Φ_K that for every $p > 1$, $\Phi_K(x^p) \le \Phi_K^p(x)$ as $0 < x < 1$. Then there exists a rational number q between 1 and p which allows us to improve the above inequality because

$$\Phi_K(x^p) = \Phi_K\big((x^q)^{p/q}\big) \le \Phi_K^{p/q}(x^q) < \big(\Phi_K^q(x)\big)^{p/q} = \Phi_K^p(x).$$

A substitution $x := \Phi_{1/K}(x)$ in the inequality (1.1) yields the inequality (1.2) so the lemma is proved.

Lemma 1.2. *For any* $p > 1$ *and* $0 < x < 1$

$$\Phi_K(4^{1-p}x^p) > 4^{1-p}\Phi_K^p(x), \qquad K > 1 \tag{1.6}$$

and

$$\Phi_K(4^{1-p}x^p) < 4^{1-p}\Phi_K^p(x), \qquad 0 < K < 1. \tag{1.7}$$

Proof. Applying the inequality $\Phi_K(r) < 4^{1-1/K}r^{1/K}$, $K \ge 1$, cf. [W], [Hü], [LV], [AVV1], and the inequality

$$\Phi_K(ab) \ge \max\{b^{1/K}\Phi_K(a),\ a^{1/K}\Phi_K(b)\}, \qquad 0 \le a,\ b \le 1,$$

showed in [AVV1] we obtain

$$\begin{aligned}\Phi_K(4^{1-p}x^p) &= \Phi_K\Big(\Big(\frac{x}{4}\Big)^{p-1}x\Big) \ge \Big(\frac{x}{4}\Big)^{(p-1)/K}\Phi_K(x)\\ &= 4^{1-p}(4^{1-1/K}x^{1/K})^{p-1}\Phi_K(x) > 4^{1-p}\Phi_K^p(x).\end{aligned}$$

A substitution $x := \Phi_{1/K}(x)$ in the inequality (1.6) yields the inequality (1.7) and this ends the proof.

Now we define the functions $\varphi_{K,t}$, $\psi_{K,t}$, $t \geq 1$, important for our further considerations. We put

$$\varphi_{K,t} = \Phi_t \circ \varphi_{K,1} \circ \Phi_{1/t} \qquad \text{and} \qquad \psi_{K,t} = \Phi_t \circ \psi_{K,1} \circ \Phi_{1/t} \tag{1.8}$$

for any $K > 0$ and $t \geq 1$ where

$$\varphi_{K,1}(x) = x^{1/K} \qquad \text{and} \qquad \psi_{K,1}(x) = \min\{4^{1-1/K}x^{1/K}, 1\}, \qquad 0 \leq x \leq 1.$$

Theorem 1.3. *For any $0 \leq x \leq 1$ and $1 \leq t_1 \leq t_2$*

$$\varphi_{K,t_1}(x) \leq \varphi_{K,t_2}(x) \leq \Phi_K(x) \leq \psi_{K,t_2}(x) \leq \psi_{K,t_1}(x), \qquad K \geq 1 \tag{1.9}$$

and

$$\psi_{K,t_1}(x) \leq \psi_{K,t_2}(x) \leq \Phi_K(x) \leq \varphi_{K,t_2}(x) \leq \varphi_{K,t_1}(x), \qquad 0 < K \leq 1. \tag{1.10}$$

Moreover, for any $0 \leq x \leq 1$ and $K > 0$

$$\lim_{t\to\infty} \varphi_{K,t}(x) = \lim_{t\to\infty} \psi_{K,t}(x) = \Phi_K(x). \tag{1.11}$$

Proof. Let $K \geq 1$ be fixed. Since

$$\varphi_{K,1}(x) \leq \Phi_K(x) \leq \psi_{K,1}(x), \qquad 0 \leq x \leq 1, \tag{1.12}$$

cf. [AVV1], [LV], and Φ_L is an increasing function for any $L > 0$ we obtain for every $t \geq 1$ the following inequality

$$\begin{aligned} \varphi_{K,t}(x) &= \Phi_t \circ \varphi_{K,1} \circ \Phi_{1/t}(x) \leq \Phi_t \circ \Phi_K \circ \Phi_{1/t}(x) \\ &= \Phi_K(x) \leq \Phi_t \circ \psi_{K,1} \circ \Phi_{1/t}(x) = \psi_{K,t}(x). \end{aligned} \tag{1.13}$$

Moreover, setting $r = \Phi_{1/t}^{1/K}(x)$ we get by Lemma 1.1 that

$$\varphi_{K,1}(x) = \Phi_t^{1/K}(r^K) \leq \Phi_t(r) = \Phi_t \circ \varphi_{K,1} \circ \Phi_{1/t}(x) = \varphi_{K,t}(x),\ 0 \leq x \leq 1. \tag{1.14}$$

Let now $r = 4^{1-1/K}\Phi_{1/t}^{1/K}(x)$. If $r \leq 1$ then by Lemma 1.2 we have

$$x = \Phi_t(4^{1-K}r^K) > 4^{1-K}\Phi_t^K(r) = 4^{1-K}\psi_{K,t}^K(x). \tag{1.15}$$

Otherwise $1 < r \leq 4^{1-1/K}x^{1/K}$ so $\psi_{K,1}(x) = 1 \geq \psi_{K,t}(x)$. This and (1.15) lead to the following inequality

$$\psi_{K,1}(x) \geq \psi_{K,t}(x), \qquad 0 \leq x \leq 1. \tag{1.16}$$

Suppose $1 \leq t_1 \leq t_2$ are arbitrary. Then $t = t_2/t_1 \geq 1$ and in view of (1.14)

$$\begin{aligned} \varphi_{K,t_1}(x) &= \Phi_{t_1} \circ \varphi_{K,1} \circ \Phi_{1/t_1}(x) \leq \Phi_{t_1} \circ \varphi_{K,t} \circ \Phi_{1/t_1}(x) \\ &= \Phi_{t_1} \circ \Phi_t \circ \varphi_{K,1} \circ \Phi_{1/t} \circ \Phi_{1/t_1}(x) = \Phi_{t_2} \circ \varphi_{K,1} \circ \Phi_{1/t_2}(x) = \varphi_{K,t_2}(x) \end{aligned}$$

for $0 \leq x \leq 1$, and similarly, it follows from (1.16) that

$$\psi_{K,t_1}(x) \geq \psi_{K,t_2}(x), \qquad 0 \leq x \leq 1.$$

This proves the inequality (1.9).

Let us consider the sequences $\varphi_{K,2^n}$, $\psi_{K,2^n}$, $n = 0,1,2,\dots$. Obviously for any $0 < x \le 1$

$$\frac{\psi_{K,1}(x)}{\varphi_{K,1}(x)} \le 4^{1-1/K}. \tag{1.17}$$

If $n = 0,1,2,\dots$ is arbitrary then by (1.9) and the equality

$$\Phi_2(r) = \frac{2\sqrt{r}}{1+r}, \qquad 0 \le r \le 1 \tag{1.18}$$

we obtain that

(1.19)

$$\begin{aligned} 1 \le \frac{\psi_{K,2^{n+1}}(x)}{\varphi_{K,2^{n+1}}(x)} &= \frac{\Phi_2\big(\psi_{K,2^n}\big(\Phi_{1/2}(x)\big)\big)}{\Phi_2\big(\varphi_{K,2^n}\big(\Phi_{1/2}(x)\big)\big)} \\ &= \left(\frac{\psi_{K,2^n}\big(\Phi_{1/2}(x)\big)}{\varphi_{K,2^n}\big(\Phi_{1/2}(x)\big)}\right)^{1/2} \left(\frac{1+\varphi_{K,2^n}\big(\Phi_{1/2}(x)\big)}{1+\psi_{K,2^n}\big(\Phi_{1/2}(x)\big)}\right) \le \sup_{0<x\le 1} \left(\frac{\psi_{K,2^n}(x)}{\varphi_{K,2^n}(x)}\right)^{1/2}. \end{aligned}$$

Hence and by (1.17) we conclude that

$$\varphi_{K,2^n}(x) \le \psi_{K,2^n}(x) \le 4^{(1-1/K)2^{-n}} \varphi_{K,2^n}(x), \qquad 0 \le x \le 1,$$

and this together with the inequality (1.9) implies (1.11). In a similar way we arrive at the inequality (1.10) and derive (1.11) in the case when $0 < K \le 1$, which ends the proof.

The functions $\varphi_{K,t}$, $\psi_{K,t}$, $t \ge 1$, behave nicely near 0 but not so nicely close to 1. To improve this we shall introduce another couple of functions as follows:

$$\widetilde{\varphi}_{K,t} = h \circ \varphi_{1/K,t} \circ h, \quad \widetilde{\psi}_{K,t} = h \circ \psi_{1/K,t} \circ h, \qquad K > 0,\ t \ge 1 \tag{1.20}$$

where $h(x) = (1-x)/(1+x)$, $0 \le x \le 1$. As shown in [AVV1]

$$\Phi_{1/K} = h \circ \Phi_K \circ h, \qquad K > 0, \tag{1.21}$$

so in view of Theorem 1.3 we immediately obtain the following

Corollary 1.4. *For any $0 \le x \le 1$ and $1 \le t_1 \le t_2$*

$$\widetilde{\varphi}_{K,t_1}(x) \le \widetilde{\varphi}_{K,t_2}(x) \le \Phi_K(x) \le \widetilde{\psi}_{K,t_2}(x) \le \widetilde{\psi}_{K,t_1}(x), \qquad K \ge 1 \tag{1.22}$$

and

$$\widetilde{\psi}_{K,t_1}(x) \le \widetilde{\psi}_{K,t_2}(x) \le \Phi_K(x) \le \widetilde{\varphi}_{K,t_2}(x) \le \widetilde{\varphi}_{K,t_1}(x), \qquad 0 < K \le 1. \tag{1.23}$$

Moreover, for any $0 < x \le 1$ *and* $K > 0$

$$\lim_{t\to\infty} \widetilde{\varphi}_{K,t}(x) = \lim_{t\to\infty} \widetilde{\psi}_{K,t}(x) = \Phi_K(x), \qquad 0 \le x \le 1. \tag{1.24}$$

It follows from Theorem 1.3 that the sequences $\varphi_{K,2^n}$, $\psi_{K,2^n}$, $n = 0,1,2,\dots$ are convergent to Φ_K but computer tests show that $\psi_{K,2^n}$ approaches much faster Φ_K than $\varphi_{K,2^n}$ as $n \to \infty$, $K > 0$. There is a simple justification of this fact. Namely, it can be explained by the asymptotic behaviour of Φ_K near 0 given by

$$\lim_{r\to 0+} \frac{\Phi_K(r)}{r^{1/K}} = 4^{1-1/K} = \lim_{r\to 0+} \frac{\psi_{K,1}(r)}{r^{1/K}}.$$

Actually the sequence $\psi_{K,2^n}$, $n = 0,1,2,\dots$ converges to Φ_K very fast. We will prove the following

Theorem 1.5. *For any* $0 < x < 1$

$$(1 - x^{2^{n+1}/K})\psi_{K,2^n}(x) \le \Phi_K(x) \le \psi_{K,2^n}(x), \qquad K \ge 1,\ n = 2,3,4,\dots \tag{1.25}$$

as well as

$$\psi_{K,2^n}(x) \le \Phi_K(x) \le (1 - x^{2^{n+1}})^{-1/K2^n}\psi_{K,2^n}(x),\ 0 < K \le 1,\ n = 1,2,3,\dots. \tag{1.26}$$

Proof. At first we consider the case when $K \ge 1$. We put

$$r_{K,n}(x) = \frac{\psi_{K,2^{n-1}}(x)}{\psi_{K,2^n}(x)}, \qquad 0 < x \le 1,\ n = 1,2,\dots. \tag{1.27}$$

Similarly as in the proof of Theorem 1.3, see (1.19), we get by the inequality (1.9) and the equality (1.18) that for any $0 < x \le 1$ and $n = 1,2,\dots$

$$\begin{aligned} 1 \le r_{K,n+1}(x) &= \frac{\Phi_2\big(\psi_{K,2^{n-1}}(\Phi_{1/2}(x))\big)}{\Phi_2\big(\psi_{K,2^n}(\Phi_{1/2}(x))\big)} \\ &\le \left(\frac{\psi_{K,2^{n-1}}\big(\Phi_{1/2}(x)\big)}{\psi_{K,2^n}\big(\Phi_{1/2}(x)\big)}\right)^{1/2} = r_{K,n}^{1/2}\big(\Phi_{1/2}(x)\big). \end{aligned} \tag{1.28}$$

Hence for any $0 < x \le 1$ and $n = 1,2,\dots$

$$r_{K,n+1}(x) \le r_{K,1}^{1/2^n}\big(\Phi_{1/2^n}(x)\big). \tag{1.29}$$

It follows from (1.18) that

$$\begin{aligned} r_{K,1}(x) &\le \left(\frac{1+\sqrt{1-x^2}}{2}\right)^{1/K} + 4^{1-1/K}\left(\frac{1-\sqrt{1-x^2}}{2}\right)^{1/K} \\ &\le 1 + 4^{1-1/K}\left(\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right)^{1/K} = 1 + 4^{1-1/K}\left(\frac{x}{1+\sqrt{1-x^2}}\right)^{2/K} \\ &= 1 + 4^{1-1/K}\big(\Phi_{1/2}(x)\big)^{1/K}. \end{aligned}$$

This together with (1.29) and (1.10) gives

$$
\begin{aligned}
r_{K,n+1}(x) &\le \left(1 + 4^{1-1/K}\left(\Phi_{1/2^{n+1}}(x)\right)^{1/K}\right)^{1/2^n} \le 1 + \frac{4^{1-1/K}}{2^n} x^{2^{n+1}/K} \\
&\le 1 + x^{2^{n+1}/K}
\end{aligned}
\tag{1.30}
$$

as $n = 2, 3, \ldots$ and $0 < x \le 1$. By (1.11) we have for any $0 < x < 1$ and $n = 2, 3, \ldots$

$$
\begin{aligned}
\frac{\psi_{K,2^n}(x)}{\Phi_K(x)} &= \lim_{m\to\infty} \frac{\psi_{K,2^n}(x)}{\psi_{K,2^m}(x)} = \lim_{m\to\infty} \prod_{l=n+1}^{m} r_{K,l}(x) = \prod_{l=n+1}^{\infty} r_{K,l}(x) \\
&\le \prod_{l=n+1}^{\infty} (1 + x^{2^l/K}) = \frac{1}{1 - x^{2^{n+1}/K}},
\end{aligned}
$$

because of (1.30). This proves the inequality (1.25).

Let now $0 < K \le 1$. Similarly as in the first case we put

$$
r_{K,n}(x) = \frac{\psi_{K,2^n}(x)}{\psi_{K,2^{n-1}}(x)}, \qquad 0 < x \le 1,\ n = 1, 2, \ldots, \tag{1.31}
$$

and by (1.10) and (1.18) we get the inequalities (1.28) and (1.29) true for $0 < K \le 1$, $0 < x \le 1$ and $n = 1, 2, \ldots$. Applying the equality (1.18) once again we have

$$
\begin{aligned}
r_{K,1}(x) &\le \frac{2^{1/K}}{\left(1 + \sqrt{1-x^2}\,\right)^{1/K} + 4^{1-1/K}\left(1 - \sqrt{1-x^2}\,\right)^{1/K}} \\
&\le \left(\frac{2}{1 + \sqrt{1-x^2}}\right)^{1/K} = \left(1 + \Phi_{1/2}(x)\right)^{1/K}
\end{aligned}
\tag{1.32}
$$

which together with (1.29) and (1.10) leads to

$$
r_{K,n+1}(x) \le \left(1 + \Phi_{1/2^{n+1}}(x)\right)^{1/K2^n} \le (1 + x^{2^{n+1}})^{1/K2^n} \tag{1.33}
$$

as $0 < x \le 1$ and $n = 1, 2, \ldots$. Hence and by (1.11)

$$
\begin{aligned}
\frac{\Phi_K(x)}{\psi_{K,2^n}(x)} &= \lim_{m\to\infty} \frac{\psi_{K,2^m}(x)}{\psi_{K,2^n}(x)} = \lim_{m\to\infty} \prod_{l=n+1}^{m} r_{K,l}(x) \le \left(\lim_{m\to\infty} \prod_{l=n+1}^{m} (1 + x^{2^l})\right)^{1/K2^n} \\
&= \left(\prod_{l=n+1}^{\infty} (1 + x^{2^l})\right)^{1/K2^n} = \frac{1}{(1 - x^{2^{n+1}})^{1/K2^n}},\ 0 < x < 1,\ n = 1, 2, \ldots.
\end{aligned}
$$

This implies the inequality (1.26) and ends the proof.

It follows from the above theorem and the equalities (1.20) and (1.21) that the following counterpart of Theorem 1.5 for $\widetilde{\psi}_{K,t}$, $t \geq 1$, holds.

Corollary 1.6. *For any* $0 < x < 1$

$$0 \leq \widetilde{\psi}_{K,2^n}(x) - \Phi_K(x) \leq 2\big(\big(1 - h(x)^{2^{n+1}}\big)^{-K2^{-n}} - 1\big)h^K(x) \tag{1.34}$$

as $K \geq 1$ *and* $n = 1, 2, 3, \dots$ *as well as*

$$0 \leq \Phi_K(x) - \widetilde{\psi}_{K,2^n}(x) \leq 2\big(\big(1 - h(x)^{K2^{n+1}}\big)^{-1} - 1\big)\min\{4^{1-K}h^K(x), 1\} \tag{1.35}$$

as $0 < K \leq 1$ *and* $n = 2, 3, 4, \dots$.

Proof. Let $K \geq 1$ be any fixed number. By the inequality

$$\big|h(x) - h(y)\big| \leq 2|x - y|, \qquad 0 \leq x, y \leq 1,$$

the equalities (1.20), (1.21) and the inequalities (1.9), (1.10), (1.22) we get for any $0 < x < 1$

$$\begin{aligned} 0 \leq \widetilde{\psi}_{K,2^n}(x) - \Phi_K(x) &\leq 2\big(\Phi_{1/K}\big(h(x)\big) - \psi_{1/K,2^n}(h(x))\big) \\ &\leq 2\left(\frac{\Phi_{1/K}\big(h(x)\big)}{\psi_{1/K,2^n}\big(h(x)\big)} - 1\right)\psi_{1/K,2^n}\big(h(x)\big). \end{aligned}$$

Hence and by the inequalities (1.26) and (1.10) we obtain immediately the inequality (1.34). In a similar way we derive the inequality (1.35) which ends the proof.

The convergence of another couple of sequences $\varphi_{K,2^n}$, $\widetilde{\varphi}_{K,2^n}$, $n = 0, 1, 2, \dots$, $K > 0$, will be considered in a separate paper [P].

2. Applications

By Theorem 1.3 we obtain upper and lower bounds for the function Φ_K of the form

$$\varphi_{K,2^n}(x) \leq \Phi_K(x) \leq \psi_{K,2^n}(x) \qquad \text{as } K \geq 1 \tag{2.1}$$

and

$$\psi_{K,2^n}(x) \leq \Phi_K(x) \leq \varphi_{K,2^n}(x) \qquad \text{as } 0 < K \leq 1 \tag{2.2}$$

for $0 \leq x \leq 1$ and $n = 0, 1, 2, \dots$. In view of (1.18) the estimates (2.1) and (2.2) are expressed in an explicit form by elementary functions. Moreover, Theorem 1.3

says that they are closer step by step to the function Φ_K as n increases to infinity. In particular, if $n=0$, we obtain from (2.1) the classical bounds

$$x^{1/K} \le \Phi_K(x) \le 4^{1-1/K} x^{1/K}, \qquad 0 \le x \le 1,\ K \ge 1,$$

cf. [W], [Hü], [LV], [AVV1]. If $n=1$ then (2.1) yields for any $K \ge 1$ and $0 \le x \le 1$ the estimate

$$\frac{2x^{1/K}}{(1+\sqrt{1-x^2}\,)^{1/K} + (1-\sqrt{1-x^2}\,)^{1/K}} \le \Phi_K(x) \le \frac{2^{2-1/K} x^{1/K}}{(1+\sqrt{1-x^2}\,)^{1/K} + (1-\sqrt{1-x^2}\,)^{1/K}}$$

which coincides with that established in [AVV1]. So setting $n=2$ we obtain a new improvement of the above mentioned results:

Theorem 2.1. *For $K \ge 1$ and $0 \le x \le 1$ the following bounds hold*

$$\text{(2.3)}\quad \Phi_K(x) \ge \frac{2^{3/2} x^{1/K}}{(1+\sqrt{1-x^2}\,)^{1/2K}} \frac{\big((1+\sqrt[4]{1-x^2}\,)^{2/K} + (1-\sqrt[4]{1-x^2}\,)^{2/K}\big)^{1/2}}{\big((1+\sqrt[4]{1-x^2}\,)^{1/K} + (1-\sqrt[4]{1-x^2}\,)^{1/K}\big)^2}$$

and
(2.4)

$$\Phi_K(x) \le \frac{2^{2-1/2K} x^{1/K}}{(1+\sqrt{1-x^2}\,)^{1/2K}} \frac{\big((1+\sqrt[4]{1-x^2}\,)^{2/K} + 4^{1-1/K}(1-\sqrt[4]{1-x^2}\,)^{2/K}\big)^{1/2}}{\big((1+\sqrt[4]{1-x^2}\,)^{1/K} + 2^{1-1/K}(1-\sqrt[4]{1-x^2}\,)^{1/K}\big)^2}$$

as $0 \le x < \Phi_4(4^{1-K})$ and $\Phi_K(x) \le 1$ as $\Phi_4(4^{1-K}) \le x \le 1$.

From the inequalities (2.2), estimates of the function Φ_K for $0 < K \le 1$ corresponding to (2.3) and (2.4) can be derived.

Contrary to Theorem 1.3, Corollary 1.4 yields upper and lower bounds for the function Φ_K, good for x near 1, of the form

$$\text{(2.5)}\qquad \widetilde{\varphi}_{K,2^n}(x) \le \Phi_K(x) \le \widetilde{\psi}_{K,2^n}(x) \qquad \text{as } K \ge 1$$

and

$$\text{(2.6)}\qquad \widetilde{\psi}_{K,2^n}(x) \le \Phi_K(x) \le \widetilde{\varphi}_{K,2^n}(x) \qquad \text{as } 0 < K \le 1$$

for $0 \le x \le 1$ and $n = 0,1,2,\ldots$. Similarly as (2.1) and (2.2), the estimates (2.5) and (2.6) can be expressed in an explicit form by elementary functions and they are, in view of Corollary 1.4, closer step by step to the function Φ_K as n increases to infinity. Setting, for example, $n = 0, 1$ in (2.5) we obtain the following explicit estimates of Φ_K for $K \geq 1$ and $0 \leq x \leq 1$, good for x close to 1, cf. [Z1],

$$\frac{(1+x)^K - (1-x)^K}{(1+x)^K + (1-x)^K} \leq \Phi_K(x) \leq \frac{(1+x)^K - 4^{1-K}(1-x)^K}{(1+x)^K + 4^{1-K}(1-x)^K}$$

and a more precise bound

$$\left(\frac{(1+\sqrt{x}\,)^K - (1-\sqrt{x}\,)^K}{(1+\sqrt{x}\,)^K + (1-\sqrt{x}\,)^K}\right)^2 \leq \Phi_K(x) \leq \left(\frac{(1+\sqrt{x}\,)^K - 2^{1-K}(1-\sqrt{x}\,)^K}{(1+\sqrt{x}\,)^K + 2^{1-K}(1-\sqrt{x}\,)^K}\right)^2,$$

found in [Z2] and [AVV3]. So, if $n = 2$ then the inequalities (2.5) yield a new improvement of the above mentioned results.

Theorem 2.2. *For $K \geq 1$ and $0 \leq x \leq 1$ the following bounds hold*

(2.7)
$$\left(\frac{1-p(x)}{\sqrt{1+p^2(x)} + \sqrt{2p(x)}}\right)^4 \leq \Phi_K(x) \leq \left(\frac{1-2^{1-K}p(x)}{\sqrt{1+4^{1-K}p^2(x)} + \sqrt{2^{2-K}p(x)}}\right)^4$$

where

$$p(x) = \left(\frac{\sqrt{1+x} - \sqrt[4]{4x}}{\sqrt{1+x} + \sqrt[4]{4x}}\right)^K, \qquad 0 \leq x \leq 1.$$

From the inequalities (2.6), estimates of the function Φ_K for $0 < K \leq 1$ corresponding to (2.7) can be derived.

It follows from Theorem 1.3 and Corollary 1.4 that the sequences $\psi_{K,2^n}$, $\widetilde{\psi}_{K,2^n}$, $n = 0, 1, 2, \ldots$ are convergent to the distortion function Φ_K and the convergence is very fast, because of Theorem 1.5 and Corollary 1.6. For example, if $1 \leq K \leq 2$ and $0 \leq x \leq 0.1$ then by virtue of Theorem 1.5

$$0 \leq \psi_{K,2^7}(x) - \Phi_K(x) \leq x^{2^8/K}\psi_{K,2^7}(x) \leq 10^{-128}\psi_{K,2^7}(x) < 10^{-100}$$

and

$$0 \leq \psi_{K,2^{10}}(x) - \Phi_K(x) \leq x^{2^{10}/K}\psi_{K,2^{10}}(x) \leq 10^{-1024}\psi_{K,2^{10}}(x) < 10^{-1000}.$$

This way we obtained a convenient algorithm for computation of values of the function Φ_K with an arbitrary accuracy, which can be easily handled with a computer. Details will be presented in a separate paper [P]. We shall present there some other applications of the results obtained in the first section to the estimates of functions μ, μ^{-1} and λ closely related to Φ_K as well as a method of numerical computation of their values.

3. Generalization to the n-dimensional case

Let $\gamma_n(r)$, $r > 1$ denote the conformal capacity of the Grötzsch extremal ring in $\mathbf{R}^n$, $n = 2, 3, 4, \ldots$, cf. [AVV2]. We define the function $\mu_n(r)$, $0 < r < 1$, by

$$\mu_n(r) = \left(\frac{1}{\omega_{n-1}} \gamma_n\left(\frac{1}{r}\right)\right)^{1/(1-n)}$$

where ω_{n-1} is the $n-1$-dimensional measure of the unit sphere S^{n-1} in $\mathbf{R}^n$. For $K > 0$ the function

$$\Phi_{K,n}(r) = \mu_n^{-1}\big(K^{1/(1-n)}\mu_n(r)\big), \qquad 0 < r < 1, \ \Phi_{K,n}(0) = 0, \ \Phi_{K,n}(1) = 1 \tag{3.1}$$

is called the n-dimensional distortion function. Obviously, the function $\Phi_{K,2}$ coincides with the function Φ_K.

Theorem 3.1. *For every $K > 0$ and $n = 2, 3, 4, \ldots$*

$$\lim_{t\to\infty} \Phi_{t,n} \circ \varphi \circ \Phi_{1/t,n}(x) = \Phi_{K,n}(x), \qquad 0 \le x \le 1 \tag{3.2}$$

where φ: $[0,1] \to [0,1]$ is any function such that

$$\lim_{r\to 0+} \frac{\log \varphi(r)}{\log r} = K^{1/(1-n)}. \tag{3.3}$$

Proof. Let $n \ge 2$ be an arbitrary positive integer and x be any point between 0 and 1. As shown in [G1], [G2], the limit

$$\lim_{r\to 0+} \big(\mu_n(r) - \log r\big) = \log \lambda_n$$

exists and λ_n is called the Grötzsch ring constant. Hence, by (3.3) and the equality

$$\frac{\mu_n\big(\varphi(\tau)\big)}{\mu_n(\tau)} = \frac{\log \tau}{\mu_n(\tau)} \frac{\mu_n\big(\varphi(\tau)\big)}{\log \varphi(\tau)} \frac{\log \varphi(\tau)}{\log \tau}, \qquad 0 < \tau < 1,$$

we get

$$\lim_{\tau\to 0+} \frac{\mu_n\big(\varphi(\tau)\big)}{\mu_n(\tau)} = K^{1/(1-n)}. \tag{3.4}$$

Setting $\tau = \Phi_{1/t,n}(x)$ we derive from (3.1) that $\mu_n(\tau) = t^{1/(n-1)}\mu_n(x)$ and further on

$$\Phi_{t,n} \circ \varphi \circ \Phi_{1/t,n}(x) = \mu_n^{-1}\left(t^{1/(1-n)}\mu_n\big(\varphi(\tau)\big)\right) = \mu_n^{-1}\Big(\frac{\mu_n\big(\varphi(\tau)\big)}{\mu_n(\tau)}\mu_n(x)\Big).$$

Hence, by (3.4) and the continuity of the function μ_n we finally obtain the convergence statement in (3.2), because of $\lim_{t\to\infty} \Phi_{1/t,n}(x) = 0$. This ends the proof.

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