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A PLY INEQUALITY FOR KLEINIAN GROUPS

Carsten Lunde Petersen

Århus Universitet, Matematisk institut
DK-8000 Århus C, Denmark; clp@mi.aau.dk

Abstract. We prove an inequality for the multipliers of loxodromic elements in certain Kleinian groups. The inequality is a generalization of Bers inequality (see L. Bers [B, Theorem 3] and C. McMullen [McM, Theorem 6.4]). It is also an analogue of an inequality by Ch. Pommerenke, G. M. Levin and J.-C. Yoccoz for multipliers of repelling periodic orbits of rational maps [Po], [L], [Y] and [Pe].

1. Introduction

Let Γ be a Kleinian group. Suppose Ω_0 is a simply connected domain of discontinuity for Γ and $\phi\colon \Omega_0 \to \mathbf{D}$ is a Riemann map. Denote the stabilizer of Ω_0 by Γ_0 and denote by $\widehat{\Gamma}_0$ the Fuchsian group conjugate to Γ_0 by ϕ. We will call $\widehat{\Gamma}_0$ a *flat model* of Γ on Ω_0.

Let $\gamma \in \Gamma_0$ and $\widehat{\gamma} \in \widehat{\Gamma}_0$ be conjugate, i.e. $\widehat{\gamma} = \phi \circ \gamma \circ \phi^{-1}$. If γ is not elliptic, the fixed point(s) of γ is (are) accessible boundary points of Ω_0. In fact: (1) If $\widehat{\gamma}$ is hyperbolic, it stabilizes a unique hyperbolic geodesic $\widehat{\alpha}$. The *axis* α of γ in Ω_0 given by $\alpha := \phi^{-1}(\widehat{\alpha})$ defines accesses to the fixed point(s) of γ; (2) If $\widehat{\gamma}$ is parabolic, γ is parabolic and the preimage of the line segment between 0 and the fixed point for $\widehat{\gamma}$ defines an access to the fixed point of γ.

The main theorem of this paper is Theorem C. Theorems A and B are corollaries of Theorem C. They have been singled out to exhibit more clearly the contents of Theorem C. The layout of the paper is to formulate the theorems in the order that their prerequisites are introduced and to leave the proofs to the end.

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2. Theorems

Let $\gamma \in \Gamma$ be loxodromic. Conjugating by a Möbius transform if necessary, we can assume that $\gamma(z) = \lambda z$, with $|\lambda| > 1$, so that the fixed points are 0 and ∞. Suppose there exist k simply connected domains of discontinuity for Γ, $\Omega_1, \dots, \Omega_k$, with γ in the intersection of their stabilizers. Then the two fixed points 0 and ∞ are common boundary points of the k domains. Let $\widehat{\gamma}_1, \dots, \widehat{\gamma}_k$ be the corresponding elements of the flat models of Γ on $\Omega_1, \dots, \Omega_k$. Let further $\varrho_1, \dots, \varrho_k$ be the multipliers of the repelling fixed points for $\widehat{\gamma}_1, \dots, \widehat{\gamma}_k$; then:

Theorem A. *There exists a logarithm L of λ such that*

$$|L| \le \frac{2 \cdot \sin(\theta)}{\sum_{j=1}^{k} (1/\log \varrho_j)},$$

where θ is the angle between L and $2\pi i$.

For quasi-Fuchsian groups this is the Bers inequality.

Let us next relax the assumptions on γ and introduce the combinatorial rotation number. Suppose there exist a simply connected domain of discontinuity Ω_0 and $q \in \mathbf{N}$ such that $\gamma^q \in \Gamma_0$. We will suppose q to be minimal with this property. Let $\widetilde{\gamma}$ be the conjugate $\phi \circ \gamma^q \circ \phi^{-1}$. Let α_0 be the axis of γ^q in Ω_0. The orbit of α_0 under γ consists of q disjoint arcs between 0 and ∞. We define a cyclic order on these arcs in the following way. Let $C(r)$ be the circle with center 0 and radius r. For each arc mark its first intersection with $C(r)$ starting from 0. These intersections have a clockwise cyclic order on $C(r)$, and we assign the same order to the arcs. It is easy to see, using the Jordan curve theorem, that the cyclic order of the arcs does not depend on r. Let $\alpha_0^0, \dots, \alpha_0^{q-1}$ be the q arcs in the orbit of α_0 under γ labelled counter clockwise. Since the cyclic order does not depend on r, there exists a $p \in \{0, \dots, q-1\}$, $(p,q) = 1$ such that

$$\gamma(\alpha_0^j) = \alpha_0^{(j+p) \bmod q}.$$

Definition. *The number p/q is called the combinatorial rotation number for γ.*

An easy application of the Jordan curve theorem shows that if $\gamma^{q'}$ (with q' minimal) is in the stabiliser of another simply connected domain of discontinuity Ω_1, then $q = q'$, and Ω_1 gives rise to the same combinatorial rotation number p/q. Thus the combinatorial rotation number is well defined.

The motivation for introducing the combinatorial rotation number comes from the theory of iteration of polynomials and, more generally, rational maps of the Riemann sphere. More precisely, it comes from the Yoccoz estimate of the size of limbs of the Mandelbrot set (see [Y]).

Theorem B. *Let $\gamma(z) = \lambda z$ be loxodromic. Suppose γ has combinatorial rotation number p/q and γ^q is in the stabilizer of Ω_0. Let $\widetilde{\gamma}$ be the conjugate of γ^q in a flat model of Γ on Ω_0 and let ϱ be the eigenvalue of the repelling fixed point for $\widetilde{\gamma}$. Then there exists a logarithm L of λ such that*

$$\Bigl|L - \frac{p}{q}2\pi i\Bigr| \leq \frac{2\sin(\theta)}{q^2} \cdot \log(\varrho),$$

where θ is the angle between $L - (p/q)2\pi i$ and $2\pi i$.

We next introduce the logarithmic density to improve our inequality. This improvement is motivated by Levin ([L]). Let $\mathrm{Area}(W, \eta)$ denote the area with respect to the conformal metric $\eta := |dz|/|z|$ of some Borel subset W of $\mathbf{C}$. Suppose 0 is a boundary point of the Borel subset $U \subset \mathbf{C}$. Let $r > 0$ be given and suppose the following limit exists:

$$B := \lim_{\delta \to 0} \frac{\mathrm{Area}\left(\left(U \cap A(\delta, r)\right), \eta\right)}{\mathrm{Area}\left(A(\delta, r), \eta\right)},$$

where $A(\delta, r) = \bigl\{z \in \mathbf{C} \bigm| \delta < |z| < r\bigr\}$. Then $B \in [0,1]$ does not depend on r, and we say that the set U has *logarithmic density* B at 0. Note that if $\lambda U = U$ for some complex number $\lambda, |\lambda| > 1$, then U has a logarithmic density at 0.

Let γ, Ω_0 and p/q be as in Theorem B. Suppose further that there exist simply connected domains of discontinuity $\Omega_1, \ldots, \Omega_k$ with γ^q in their stabilizers and that the orbits of $\Omega_0, \ldots, \Omega_k$ under γ are disjoint. Let $\widetilde{\gamma}_0, \ldots, \widetilde{\gamma}_k$ be the elements of the flat model of Γ on $\Omega_0, \ldots, \Omega_k$ corresponding to γ^q. Let further $\varrho_1, \ldots, \varrho_k$ be the multipliers of the repelling fixed points for $\widetilde{\gamma}_0, \ldots, \widetilde{\gamma}_k$; then:

Theorem C. *There exists a logarithm L of λ such that*

$$\Bigl|L - \frac{p}{q}2\pi i\Bigr| \leq B \cdot \frac{2\sin(\theta)}{q^2 \cdot \sum_{j=1}^{k}(1/\log \varrho_j)},$$

where θ is the angle between $L-(p/q)2\pi i$ and $2\pi i$ and $B \in [0,1]$ is the logarithmic density of the set $\cup_{j=0}^{k} \cup_{l=0}^{q-1} \gamma^l(\Omega_j)$ at 0.

Complement to Theorems A, B and C. *Geometrically the inequality in Theorem B indicates that L is contained in the disc $\mathbf{D}(r + (p/q)2\pi i, r)$ and thus*

$$\lambda \in \exp\Bigl(\mathbf{D}\Bigl(r + i2\pi\frac{p}{q}, r\Bigr)\Bigr),$$

where $r = \log \varrho/q^2$ and $\mathbf{D}(r + i2\pi(p/q), r)$ is the disc with center $r + i2\pi(p/q)$ and radius r. A similar interpretation is valid in Theorems A and C.

Figure 1.

The groups with which this paper is concerned do exist. B. Maskit [M1] has used combination theorems to construct Kleinian groups with an element γ of prescribed combinatorial rotation number and a prescribed number of cycles of simply connected domains of discontinuity stabilized by the appropriate iterate of γ.

To get an idea of the capabilities of Theorems A, B and C, let us consider a family $\{\Gamma_\varepsilon\}$ of Kleinian groups which are deformations of some Kleinian group, say Γ_0. Suppose each Γ_ε has a simply connected domain of discontinuity, $\Omega_{\varepsilon,0}$, with stabilizer $\Gamma_{\varepsilon,0}$ (for instance Γ_ε could be a function group, in which case $\Omega_{\varepsilon,0}$ would be invariant). Suppose in addition the $\Gamma_{\varepsilon,0}$ have a common flat model $\widehat{\Gamma}_0$ on $\Omega_{\varepsilon,0}$. Then Theorem A gives uniform estimates for the multipliers of the loxodromic elements of $\Gamma_{\varepsilon,0}$, depending only on the flat model $\widehat{\Gamma}_0$.

We also have an immediate consequence of Theorem B. Let Γ be a Kleinian group with a simply connected domain of discontinuity Ω. Further let $\psi_t\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ be a family of quasiconformal deformations with Γ-invariant Beltrami differentials supported in the orbit of Ω under Γ. Let $\widetilde{\psi}_t\colon \Gamma \to \mathrm{PSL}(2,\mathbf{C})$ be the corresponding family of Kleinian group monomorphisms, i.e. $\widetilde{\psi}_t(\gamma) = \psi_t \circ \gamma \circ \psi_t^{-1}$ for $\gamma \in \Gamma$. Suppose that the conjugate $(\widetilde{\gamma})_t$ of $\widetilde{\psi}_t(\gamma^q)$ in the flat model of $\widetilde{\psi}_t(\Gamma)$ on $\psi_t(\Omega)$ tends to a parabolic element when t tends to 0. Then it follows from Theorem B that the multipliers of $\widetilde{\psi}_t(\gamma)$ tend to $\exp(\pm p/q \cdot 2\pi i)$. Thus, if the coefficients of $\widetilde{\psi}_t(\gamma)$ converge to the coefficients of some element γ_0, i.e. the sequence algebraically converges to γ_0 for t tending to 0 and $p/q \neq 0/1$, then γ_0 is an elliptic element.

It was pointed out to the author by Maskit that this in conjunction with Chuckrow's theorem leads to the corollary below. For completeness we also state Chuckrow's theorem.

Chuckrow's theorem. *Let Γ be a non elementary, finitely generated Kleinian group, and let $\{\psi_m\}$ be a sequence of type-preserving isomorphisms of Γ into $\mathrm{PSL}(2,\mathbf{C})$. Suppose that $\psi_m(\Gamma)$ is Kleinian for each m, and that ψ_m alge-*

braically converges to $\psi\colon \Gamma \to \widetilde{\Gamma}$. *Then* ψ *is an isomorphism.*

The reader will find a proof of Chuckrow's theorem in [M2, p. 97].

Corollary. *Suppose* $\{\widetilde{\psi}_t\colon \Gamma \to \mathrm{PSL}(2,\mathbf{C})\}_{t\in I}$ *is a family of group monomorphisms as described above and* $p/q \neq 0/1$. *Then the family cannot have an algebraic limit when* t *tends to* 0.

Proof. Suppose this is false and that $\widetilde{\psi}$ is an algebraic limit of $\widetilde{\psi}_t$ when t tends to 0. Then it follows from Chuckrow's theorem that $\widetilde{\psi}$ is a group isomorphism. On the other hand, we have shown above that the image by $\widetilde{\psi}$ of the loxodromic element γ (thus of infinite order), is an elliptic element of (finite) order q. This is a contradiction. □

3. Proof of Theorems A, B and C

We need a few basic tools: the modulus of a torus with a non-trivial homotopy class of Jordan curves and a Grötzsch inequality. Let T be a torus isomorphic to $\mathbf{C}/G$, where $G := L\cdot \mathrm{Z} \oplus 2\pi i\cdot \mathrm{Z}$, $\mathrm{Re}(L) > 0$. Furthermore let $\Pi\colon \mathbf{C} \to T$ denote the corresponding universal covering. A flat metric on T is a conformal metric, which makes Π a local isometry, when $\mathbf{C}$ is equipped with a flat metric (proportional to the euclidean). Suppose U is a Borel subset of T and let ϱ be a flat metric on T. Define $\mathrm{Area}(U) := \mathrm{mes}(U,\varrho)$; then the quotient $\mathrm{Area}(U)/\mathrm{Area}(T)$ does not depend on the choice of a flat metric ϱ on T, and we call it the *relative flat area* of U.

Suppose $\kappa\colon [0,1] \to T$, $\kappa(0) = \kappa(1)$ is a non-trivial (i.e. not homotopic to a constant) Jordan curve in the torus T. Let $[\kappa]$ denote the homotopy class of κ in T. A metric ϱ on T is called *admissible* if it is conformal Borel metric and $l_\varrho(\kappa') \geq 1$ for all $\kappa' \in [\kappa]$. The modulus for the pair (T,κ) is a conformal invariant defined by $\mathrm{mod}(T,\kappa) = \inf\{\,\mathrm{mes}(T,\varrho) \mid \varrho \text{ admidssible}\,\}$. The modulus is evidently a conformal invariant. Let us remind the reader that the modulus of an annulus is defined in the same way.

There exists $p, q \in \mathrm{Z}$, p and q relatively prime, such that any lift $\widetilde{\kappa}$ of κ to Π satisfies $\widetilde{\kappa}(1) = \widetilde{\kappa}(0) + q\cdot L - p\cdot 2\pi i$. The number $\sigma := q\cdot L - p\cdot 2\pi i$ does not depend on the choice of generator L, whereas the number p generally does. We call σ the *associated segment* of κ or rather of $[\kappa]$, as it is a homotopy invariant. Changing the orientation of κ if necessary, we can suppose that $q \geq 0$. If $q = 0$, then $p = 1$. Further, if $q > 0$, then a change of L to $L + 2\pi i$ leads to a change of the number p to $p + q$. Thus for a suitable choice of L we have $p \in \{0,\dots,q-1\}$. With this normalization we call p/q the *combinatorial rotation number* of $[\kappa]$.

Grötszch inequality (Annuli in a torus). *Let* T *be a torus and let* κ *be a non-trivial Jordan curve in* T *with combinatorial rotation number* p/q, $0 < q$ *and associated segment* σ. *Let* $\{A_j\}_{j\in J}$ *be any family of disjoint annuli homotopic to* κ *in* T. *Denote by* B *the relative flat area of the family,*

$B := \sum_{j\in J} \mathrm{Area}(A_j)/\,\mathrm{Area}(T)$. *Then:*

$$\sum_{j\in J} \mathrm{mod}(A_j) \le B\cdot \mathrm{mod}(T,\kappa) = B\cdot \frac{2\pi\cdot\sin(\theta)}{q\cdot|\sigma|},$$

where θ is the oriented angle from σ to $2\pi i$. Furthermore, equality is attained if and only if each of the sets $\Pi^{-1}(A_j)$ is a straight strip in $\mathbf{C}$ parallel to σ.

Proof. Let ϱ be the flat metric on T corresponding to $1/|\sigma|$ on $\mathbf{C}$. Then ϱ is admissible for both T and each A_j. Furthermore, the standard length-area argument shows that $\mathrm{mod}(T,\kappa) = \mathrm{mes}(T,\varrho)$. We get

$$\sum_{j\in J} \mathrm{mod}(A_j) \le \sum_{j\in J} \mathrm{mes}(A_j,\varrho) = B\cdot \mathrm{mes}(T,\varrho) = B\cdot \frac{2\pi\cdot\sin(\theta)}{q\cdot|\sigma|}.$$

Equality is attained if and only if the restriction of ϱ to each A_j is the extremal (or flat) metric on A_j (see e.g. [L-H, p. 33]), that is, the last condition of the theorem is satisfied. □

Proof of Theorem C. Let T be the torus generated by the action of γ on $\mathbf{C}^*$. Furthermore, let $\widehat{\Pi}$: $\mathbf{C}^* \to T$ denote the corresponding projection map. Then T is isomorphic to $\mathbf{C}/G$, where $G = \{n\cdot L + m\cdot 2\pi i \mid n,m\in \mathrm{Z}, \exp(L) = \lambda\}$. For $j = 0,\dots,k$ let α_j be the axis of γ^q in Ω_j. Define κ_j as the projection of α_j by $\widehat{\Pi}$. Then the κ_j are disjoint Jordan curves and thus homotopic in T. Further, the combinatorial rotation number of the κ_j equals the combinatorial rotation number p/q for the α_j. Let L be the choice of a logarithm of λ, such that the associated segment of the κ_j is $q\cdot L - p\cdot 2\pi i$. For each $j = 0,\dots,k$ let $A_j := \widehat{\Pi}(\Omega_j)$, let ϕ_j: $\Omega_j \to \mathbf{D}$ be a Riemann map and let $\widetilde{\gamma}_j$ be the conjugate of γ^q by ϕ_j. Then $\widetilde{A}_j := \mathbf{D}/\widetilde{\gamma}_j$ is an annulus of modulus $\pi/\log(\varrho_j)$, where ϱ_j is the multiplier of the repelling fixed point for $\widetilde{\gamma}_j$. The map ϕ_j induces a biholomorphic map Φ_j: $A_j \to \widetilde{A}_j$. We have:

a) A_j is an annulus homotopic to κ_j in T with $\mathrm{mod}(A_j) = \pi/\log(\varrho_j)$.

The preimage of A_j by $\widehat{\Pi}$ equals the orbit of Ω_j under γ; thus

b) the A_j are disjoint.

Furthermore, the conformal metric $1/|z|$ descends by $\widehat{\Pi}$ as a flat metric on T, and thus

c) the logarithmic density B of $\cup_{j=1}^{k}\cup_{l=0}^{q-1}\gamma^l(\Omega_j)$ at 0 equals the relative flat area of $\cup_{j=1}^{k} A_j$ in T.

Using a), b) and c) in the Grötzsch inequality we get

$$\sum_{j=0}^{k} \pi/\log(\varrho_j) = \sum_{j=0}^{k} \mathrm{mod}(A_j) \le B\cdot \frac{2\pi\cdot\sin(\theta)}{q|q\cdot L - p\cdot 2\pi i|},$$

where θ is the angle between L and $2\pi i$. This is equivalent to the inequality in Theorem C. □

Theorem A is obtained from Theorem C by using $B \leq 1$ and $p/q = 0/1$. Theorem B is similarly obtained from Theorem C by using $B \leq 1$ and $k = 0$. □

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