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BILIPSCHITZ MAPPINGS AND STRONG A_∞ WEIGHTS

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Abstract. Given a doubling measure μ on $\mathbf{R}^n$, one can define an associated quasidistance on $\mathbf{R}^n$ by

$$\delta(x,y) = \mu(B_{x,y})^{1/n},$$

where $B_{x,y}$ is the smallest ball which contains the points x and y. This paper is concerned with the resulting geometry which is induced by μ. The main result provides a condition on μ under which $\mathbf{R}^n$ equipped with the quasidistance $\delta(\cdot,\cdot)$ admits a bilipschitz embedding into $\mathbf{R}^N$ with the standard Euclidean structure for some $N < \infty$. This sufficient condition is not so far from being necessary, but of course the reader knows how it is.

This paper is closely related to an earlier one by Guy David and the author [DS1].

1. Introduction

Let μ be a doubling measure on $\mathbf{R}^n$, so that there is a $C > 0$ such that

$$0 < \mu(2B) \le C\mu(B) \tag{1.1}$$

for all balls B. (Here $2B$ denotes the ball with the same center as B but twice the diameter.) We can associate to μ the quasidistance $\delta(x,y)$ on $\mathbf{R}^n$ defined by

$$\delta(x,y) = \mu(B_{x,y})^{1/n}, \tag{1.2}$$

where

$$B_{x,y} \text{ is the smallest ball that contains both } x \text{ and } y. \tag{1.3}$$

To say that $\delta(x,y)$ is a quasidistance means that it is nonnegative and symmetric, that it vanishes exactly when $x = y$, and that it satisfies the following weakened form of the triangle inequality:

$$\delta(x,z) \le C\big(\delta(x,y) + \delta(y,z)\big) \tag{1.4}$$

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for some $C > 0$ and all x, y, $z \in \mathbf{R}^n$. If (1.4) were true with $C = 1$ then $\delta(x, y)$ would be a distance function (as opposed to a quasidistance).

This paper is concerned with the following type of issue. We restrict ourselves to the case where $n > 1$ from now on, because the $n = 1$ case is trivial for the topics that will be considered here.

Question 1.5. Under what conditions on μ is $\delta(\cdot, \cdot)$ bilipschitz equivalent to the standard distance on $\mathbf{R}^n$? In other words, when is it true that there is a $C > 0$ and a mapping $f\colon \mathbf{R}^n \to \mathbf{R}^n$ such that

$$C^{-1}\delta(x, y) \le \bigl|f(x) - f(y)\bigr| \le C\delta(x, y) \tag{1.6}$$

for all x, $y \in \mathbf{R}^n$?

This problem has a natural reformulation in terms of quasiconformal mappings. If a mapping f as above exists, then it must be quasiconformal, and μ will be comparable in size to the pull-back of Lebesgue measure on $\mathbf{R}^n$ by f. Gehring's theorem [Ge] then implies that μ must be absolutely continuous, $\mu = \omega(x)\, dx$, with $\omega(x) \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, and ω must be comparable in size to the Jacobian of f. Conversely, if $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is quasiconformal and $\omega(x)$ is comparable in size to its Jacobian, then $\mu = \omega(x)\, dx$ is a doubling measure and f satisfies (1.6). The problem of characterizing the functions ω that arise in this fashion is notoriously difficult.

Here is a variation of Question 1.5 that is somewhat easier.

Question 1.7. Under what conditions on μ does $\bigl(\mathbf{R}^n, \delta(\cdot, \cdot)\bigr)$ admit a bilipschitz embedding into $\mathbf{R}^n$ for some N? In other words, for which μ do there exist constants C, $N > 0$ and a mapping $f\colon \mathbf{R}^n \to \mathbf{R}^N$ such that (1.6) holds?

The following is a simple necessary condition for there to exist a mapping f as in Questions 1.5 or 1.7.

(1.8) There exists a $C > 0$ and a distance function $\delta'(\cdot, \cdot)$ on $\mathbf{R}^n$ such that

$$C^{-1}\delta(x, y) \le \delta'(x, y) \le C\delta(x, y).$$

The point here is that $\delta'(\cdot, \cdot)$ should satisfy the actual triangle inequality, i.e., the analogue of (1.4) with $C = 1$. The necessity of this condition is trivial; simply set

$$\delta'(x, y) = \bigl|f(x) - f(y)\bigr|.$$

The main result of this paper (Theorem 5.2) is that a "slightly" stronger version of (1.8) is also a sufficient condition for Question 1.7.

Condition (1.8)—in a different formulation—is considered already in [DS1]. It was noted there that Gehring's argument in [Ge] implies that if (1.8) holds, then μ is absolutely continuous, and that its density is an A_∞ weight. (See Proposition 3.4 below.) Let us give a formal name for the class of these weights.

Definition. A positive locally integrable function on $\mathbf{R}^n$ is called a strong A_∞ weight if it is the density of a doubling measure μ which satisfies (1.8).

There is another problem which is closely related to Questions 1.5 and 1.7, that will not be addressed in this paper, but which should be mentioned for conceptual reasons. Let M be a connected hypersurface in $\mathbf{R}^{d+1}$, $d \geq 2$. Assume a priori that M is smooth, and even that $M \cup \{\infty\}$ is a smooth (closed) embedded submanifold of $\mathbf{R}^{d+1} \cup \{\infty\} \cong S^{d+1}$. Let $n(x)$ be a smooth choice of the unit normal to M. (A theorem in algebraic topology implies that M is orientable under these hypotheses.) Let $\|n\|_*$ denote the BMO norm of n on M, i.e.,

$$\|n\|_* = \sup_{\substack{x \in M \\ R>0}} \frac{1}{|B(x,r) \cap M|} \int_{B(x,r)\cap M} |n(y) - n_{x,R}|\, dy,$$

where $n_{x,R}$ denotes the average of n over $B(x,r) \cap M$.

Question 1.9. Do there exist constants ε, $k > 0$, depending only on d, so that if M is as above and $\|n\|_* \leq \varepsilon$, then there exists a mapping g from $\mathbf{R}^d$ onto M such that

$$(1.10) \qquad (1+k)^{-1}|x-y| \leq |g(x) - g(y)| \leq (1+k)|x-y|$$

for all x, $y \in \mathbf{R}^d$?

If the answer is yes, it would be better if we could choose k so that $k \to 0$ as $\|n\|_* \to 0$.

The condition that $\|n\|_*$ be small is quite natural. In particular there are various equivalent conditions concerning the geometry of M and analysis on M given in [S1, 3]. One of these equivalences implies a converse to Question 1.9: if M is the image of a g that satisfies (1.10) with k small, then $\|n\|_*$ is small. Notice that $\|n\|_*$ is unaffected by translations, rotations, or dilations of M.

On the other hand, it follows from [S2, 3] that if $\|n\|_*$ is small enough, then M is homeomorphic to $\mathbf{R}^d$, and that one can even construct parameterizations of M with L^p bounds on their differentials and their inverses, with $p \to \infty$ as $\|n\|_* \to 0$. These bounds do not depend on our a priori smoothness assumptions on M in a quantitative way. It is much more difficult, however, to build a bilipschitz parameterization of M.

When $d = 2$ uniformization can be used to produce a conformal mapping h from $\mathbf{R}^2$ onto M. Using extremal length estimates it can be shown that if $\|n\|_*$ is small enough, then there is a doubling measure μ on $\mathbf{R}^2$ such that h satisfies (1.6) with C and the doubling constant depending only on d. This is proved in [S2], along with related facts, e.g., μ comes from an A_∞ weight ω with the BMO norm of $\log \omega$ tending to zero as $\|n\|_* \to 0$. It follows from this that if Question 1.5 can be answered affirmatively when $n = 2$, then Question 1.9 can also be answered affirmatively when $d = 2$. Conversely, if there is a counterexample for Question 1.5, that might provide insight on how to find a counterexample for Question 1.9.

The organization of this paper is as follows. Some basic facts about A_p weights (including the definitions) will be reviewed in the next section. In Section 3 there are some simple reformulations of (1.8) as well as other useful properties of strong A_∞ weights, and in Section 4 some examples and counterexamples pertaining to strong A_∞ weights are given. The main theorem is stated in Section 5, and the remainder of the paper is devoted to its proof.

Throughout this paper we use the notation

$$\fint_A g \qquad \text{for} \qquad \frac{1}{|A|}\int_A g,$$

where $|A|$ denotes the Lebesgue measure of A.

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2. A review of basic facts about A_p weights

General references include [CF], [Ga] (especially Section 6 of Chapter 6), and [JL] (especially pp. 19–28).

Let w be a nonnegative locally integrable function on $\mathbf{R}^n$ with $w > 0$ a.e.

Definition 2.1: (i) w is an A_∞ weight if there exist C, $\gamma > 0$ such that

$$\frac{\int_A w}{\int_Q w} \le C\Big(\frac{|A|}{|Q|}\Big)^\gamma \tag{2.2}$$

for all cubes Q and all measurable subsets A of Q.

(ii) w is an A_p weight, $1 < p < \infty$, if there exists $C > 0$ so that

$$\Big(\fint_Q w\Big)\Big(\fint_Q w^{-1/(p-1)}\Big)^{p-1} \le C \tag{2.3}$$

for all cubes Q.

(iii) w is an A_1 weight if there is a constant $C > 0$ so that

$$\fint_Q w \le C\big(\operatorname*{ess\,inf}_Q w\big). \tag{2.4}$$

In each case you get an equivalent condition if you use balls instead of cubes.

Facts about A_p weights:

(a) $A_1 \subseteq A_p \subseteq A_q \subseteq A_\infty$ when $1 < p < q < \infty$.

(b) $A_\infty = \cup_{p<\infty} A_p$.

(c) If $w \in A_\infty$, then $\sigma(E) = \int_E w(x)\,dx$ defines a doubling measure on $\mathbf{R}^n$.

(d) If w_1, $w_2 \in A_p$, $1 \le p \le \infty$, and if $\lambda \in [0,1]$, then $w_1^\lambda w_2^{1-\lambda} \in A_p$. (This can easily be derived from Hölder's inequality.)

(e) $w \in A_\infty$ if and only if there exist $C > 0$ and $p > 1$ such that the reverse Hölder inequality

$$\left(\fint_Q w^p\right)^{1/p} \le C \fint_Q w \tag{2.5}$$

is satisfied by all cubes Q.

(f) If $w \in A_p$, then there is an $r > 1$ such that $w^r \in A_p$.

(g) If $w \in A_\infty$ then there is a $C > 0$ so that

$$\exp\left(\fint_Q \log w\right) \le \fint_Q w \le C \exp\left(\fint_Q \log w\right) \tag{2.6}$$

holds for all cubes Q (and similarly for all balls). Indeed, the first inequality follows from Jensen's theorem, while the second can be derived, for example, from (b) and Jensen's theorem. The converse is also true: if w satisfies (2.6), then w is an A_∞ weight. (See [H], and also Proposition 3.5 below.)

(h) Suppose that u and v are A_1 weights, and let t be any positive real number. Then uv^{-t} is an A_∞ weight. Conversely, if w is an A_∞ weight, then there exist u, $v \in A_1$ and $t > 0$ so that $w = uv^{-t}$. This is the factorization theorem of Peter Jones [J].

3. Basic facts about strong A_∞ weights

A large proportion of the content of this section can already be found in [DS1], but the presentation here is somewhat different, and in a few instances it is also more detailed.

We begin with a couple of easy and useful reformulations of (1.8).

Lemma 3.1. (1.8) *holds if and only if*

(3.2) *there is a constant $C > 0$ so that for any finite sequence $x_1, \dots, x_k$ of points in $\mathbf{R}^n$ we have*

$$\delta(x_1, x_k) \le C \sum_{j=1}^{k-1} \delta(x_j, x_{j+1}).$$

It is immediate that (1.8) implies (3.2). To prove the converse define $\delta'(\cdot,\cdot)$ by

$$\delta'(x,y) = \inf\Big\{ \sum_{j=1}^{k-1} \delta\left(x_j, x_{j+1}\right) \Big\},$$

where the infimum is taken over all finite sequences $\{x_j\}_{j=1}^k$ such that $x_1 = x$ and $x_k = y$. Clearly $\delta'(x,y) \leq \delta(x,y)$ for all x, y, and (3.2) implies that $C^{-1}\delta(x,y) \leq \delta'(x,y)$. It is also easy to see that $\delta'(\cdot,\cdot)$ is a distance function.

Lemma 3.3. *Let μ be a doubling measure on $\mathbf{R}^n$. Then (3.2) holds if and only if the same condition is true but with the additional constraint that $x_j \in B\big(x_1, 2|x_k - x_1|\big)$ for all j.*

The "only if" part is automatic, and so we need only concern ourselves with the "if" part. Suppose that the above constrained version of (3.2) is satisfied, and let us show that that implies the unconstrained version. Let $x_1, \ldots, x_k$ be given, and suppose that there is a j, $1 \leq j \leq k$, such that

$$x_j \notin B\big(x_1, 2|x_k - x_1|\big).$$

Let j_0 be the smallest such j. Then

$$x_j \in B\big(x_1, 2|x_{j_0} - x_1|\big)$$

when $1 \leq j \leq j_0$, and so the constrained version of (3.2) can be applied to give

$$\delta(x_1, x_{j_0}) \leq C \sum_{i=1}^{j_0-1} \delta(x_i, x_{i+1}) \leq C \sum_{i=1}^{k-1} \delta(x_i, x_{i+1}).$$

On the other hand we have

$$\delta(x_1, x_k) \leq C\delta(x_1, x_{j_0})$$

because $x_k \in B\big(x_1, |x_{j_0} - x_1|\big)$. This uses also the definition (1.2) of $\delta(\cdot,\cdot)$ and the doubling condition on μ. Combining these inequalities yields

$$\delta(x_1, x_k) \leq C \sum_{i=1}^{k-1} \delta(x_i, x_{i+1}),$$

as desired.

Next we discuss the absolute continuity of the μ's for which (1.8) holds.

Proposition 3.4. *If μ is a doubling measure that satisfies (1.8), then μ is absolutely continuous and it is given by $\omega(x)dx$, where ω is an A_∞ weight.*

Proposition 3.4 is essentially due to Gehring [Ge]. He stated it only in a special case, but his proof works in general. A proof will now be given for the convenience of the reader. One of the key ingredients is the following criterion for membership in A_∞.

Proposition 3.5. *Suppose that* $\omega \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, $\omega > 0$ a.e., *and that there exist constants* $C_1 > 0$ *and* $r \in (0,1)$ *so that*

$$\fint_Q \omega \le C_1 \Big(\fint_Q \omega^r \Big)^{1/r} \tag{3.6}$$

for all cubes Q. *Then there exist* $C_2 > 0$ *and* $p > 1$, *depending only on* C_1, r, *and* n, *so that*

$$\Big(\fint_Q \omega^p \Big)^{1/p} \le C_2 \fint_Q \omega$$

for all cubes Q. *In particular,* ω *is an* A_∞ *weight with constants that depend only on* C_1 *and* r (by Fact (e) of Section 2).

See [Ge] or [Ga] (Theorem 6.9, p. 260) for a proof.

Let us now use this to prove Proposition 3.4. Let $\theta(x)$ be a smooth bump function on $\mathbf{R}^n$, with $\theta \ge 0$, $\operatorname{supp}\theta \subseteq B(0,1)$, and $\int_{\mathbf{R}^n} \theta = 1$. Set $\theta_t(x) = t^{-n}\theta(x/t)$, and define ω_t on $\mathbf{R}^n$ for each $t > 0$ by

$$\omega_t = \theta_t * \mu. \tag{3.7}$$

It is not hard to check that μ will satisfy the conclusions of Proposition 3.4 if we can show that each ω_t is in A_∞ with constants that do not depend on t.

According to Proposition 3.5 it suffices to show that there is a C_1 so that each ω_t satisfies (3.6) with $r = 1/n$. To do this we fix t and consider separately the cases where $\operatorname{diam} Q \le t$ and $\operatorname{diam} Q > t$. The first case is trivial, because the doubling condition on μ ensures that ω_t is roughly constant on the scale of t. To treat the situation where $\operatorname{diam} Q > t$ we use the following estimate.

Lemma 3.8. *Let* μ *be a doubling measure that satisfies* (1.8). *Then there is a* $C > 0$ *so that if* L *is any line segment with length* $> t$ *and endpoints* x *and* y, *then*

$$\delta(x,y) \le C \int_L \omega_t(z)^{1/n}\, dz.$$

The "dz" on the right side denotes arclength measure.

This result is easily derived from Lemma 3.1. Let k be the integer part of $\operatorname{length}(L)/t$, and let $z_0, \dots, z_k$ be points on L such that $z_0 = x$, $z_k = y$, and

$t \leq |z_{j+1} - z_j| \leq 2t$ for each j. Using the doubling condition on μ and (3.2) we get

$$\int\limits_L \omega_t(z)^{1/n} dz = \sum_{j=1}^{k} \int\limits_{[z_{j-1}, z_j]} \omega_t(z)^{1/n} dz \geq \sum_{j=1}^{k} C^{-1} \mu\left(B_{z_{j-1}, z_j}\right)^{1/n}$$
$$= C^{-1} \sum_{j=1}^{k} \delta\left(z_{j-1}, z_j\right) \geq C^{-1} \delta(x, y).$$

This proves Lemma 3.8.

Once you have the lemma it is easy to check that (3.6) holds for $r = 1/n$ and $\omega = \omega_t$ when $\operatorname{diam} Q > t$, using Fubini's theorem, the definition of $\delta(x, y)$, and the doubling condition on μ. When $\operatorname{diam} Q \leq t$ (3.6) is trivial. This finishes the proof of Proposition 3.4.

Next we give a characterization of strong A_∞ weights in terms of lengths of curves. This is in fact how the strong A_∞ condition was defined in [DS1].

By a path we mean a continuous map from an interval into $\mathbf{R}^n$. Given a path γ: $[a, b] \to \mathbf{R}^n$ we define its μ-length as follows. To each partition $\{t_j\}_{j=0}^{l}$ of $[a, b]$ (i.e., $a = t_0 < t_1 < \cdots < t_l = b$) we associate the real number

$$\sum_{j=1}^{l} \delta\big(\gamma(t_{j-1}), \gamma(t_j)\big). \tag{3.9}$$

Given $\eta > 0$ consider the quantity obtained by taking the infimum of (3.9) over all partitions $\{t_j\}$ of $[a, b]$ such that $|t_j - t_{j-1}| \leq \eta$ for all j. This quantity gets larger as η gets smaller, and its limit as $\eta \to 0$ is defined to be the μ-length of γ.

(Note that the "lim sup" in the definition of the ω-length of a path in [DS1] should have been a "lim inf", in order to be consistent with the usual definition of Hausdorff measures. However, it is not hard to show that, in the end, this change does not really matter.)

It is easy to check that this definition of the μ-length of a path has many of the normal properties. For example, if γ_1: $[a, b] \to \mathbf{R}^n$ and γ_2: $[b, c] \to \mathbf{R}^n$ are two paths with $\gamma_1(b) = \gamma_2(b)$, and if γ: $[a, c] \to \mathbf{R}^n$ is obtained by joining γ_1 to γ_2 in the obvious way, then

$$\mu\text{-length}(\gamma) = \mu\text{-length}(\gamma_1) + \mu\text{-length}(\gamma_2). \tag{3.10}$$

The deduction of this fact from the definition of the μ-length is straightforward.

Having defined the μ-length of a path we can define the associated geodesic distance in the obvious way, i.e.,

$$\delta_g(x, y) = \inf\left\{\mu\text{-length}(\gamma) : \gamma \text{ is a path that joins } x \text{ to } y\right\}. \tag{3.11}$$

(Here the subscript "g" standard for "geodesic".)

Proposition 3.12. *Let μ be a doubling measure on $\mathbf{R}^n$.*

(a) *μ satisfies* (1.8) *if and only if there is a constant $C > 0$ such that*

$$C^{-1}\delta(x,y) \le \delta_g(x,y) \le C\delta(x,y) \tag{3.13}$$

for all x, $y \in \mathbf{R}^n$.

(b) *Suppose that μ is absolutely continuous and that its density is an A_∞ weight. Then there is a $C > 0$ so that*

$$\delta_g(x,y) \le C\delta(x,y) \tag{3.14}$$

for all x, $y \in \mathbf{R}^n$.

Let us begin with the "only if" portion of (a). It is easy to check from the definitions that the first inequality in (3.13) holds when μ satisfies (1.8) and hence (3.2). The second inequality in (3.13) follows from (b) and Proposition 3.4.

Conversely, (3.13) imples (1.8), because the geodesic distance always satisfies the triangle inequality.

It remains to prove (b). Let ω denote the density of μ, and fix x, $y \in \mathbf{R}^n$. We need to find a path that joins x to y whose μ-length we can control. Set $z = \frac{1}{2}(x+y)$ and $R = |x-y|$, and let H denote the hyperplane through z that is orthogonal to the line segment that joins x to y. For each $u \in B(z,R)\cap H$ let $\gamma_u\colon [0,R] \to \mathbf{R}^n$ be the path such that $\gamma_u(0) = x$, $\gamma_u(R/2) = u$, $\gamma_u(R) = y$, and $\dot\gamma_u$ is constant on $(0,R/2)$ and on $(R/2,R)$. To prove (3.14) it suffices to show that

$$\mu\text{-length}(\gamma_u) \le C\delta(x,y)$$

for some $u \in B(z,R)\cap H$. To do this we shall estimate the average of the left hand side.

For each $t > 0$ define ω_t by (3.7). It is not hard to check that if $\{s_j\}_{j=0}^{l}$ is a partition of $[0,R]$ with $t \le |s_j - s_{j-1}| \le 2t$ for all j, then

$$\sum_{j=1}^{l} \delta\big(\gamma_u(s_j),\gamma_u(s_{j-1})\big) \le C\sum_{j=1}^{l}\int_{s_{j-1}}^{s_j} \omega_t\big(\gamma_u(s)\big)^{1/n} ds \le C\int_0^R \omega_t\big(\gamma_u(s)\big)^{1/n} ds.$$

(Note that $|\dot\gamma_u| \approx 1$.) Using this and the definition of the μ-length of a path it is easy to see that

$$\mu\text{-length}(\gamma_u) \le C\liminf_{t\to 0}\int_0^R \omega_t\big(\gamma_u(s)\big)^{1/n} ds.$$

Hence

$$\begin{aligned}\delta_g(x,y) &\le C \fint_{B(z,R)\cap H} \Big(\liminf_{t\to 0} \int_0^R \omega_t\big(\gamma_u(s)\big)^{1/n} ds\Big)\, du \\ &\le C \liminf_{t\to 0} \fint_{B(z,R)\cap H} \int_0^R \omega_t\big(\gamma_u(s)\big)^{1/n} ds\, du.\end{aligned} \tag{3.15}$$

using Fatou's lemma. Standard computations permit us to bound this last expression by

$$C \liminf_{t\to 0} \int_{B(z,10R)} \omega_t(q)^{1/n} L(x,y,q)\, dq, \tag{3.16}$$

where $L(x,y,q) = \max\big(|x-q|^{1-n}, |y-q|^{1-n}\big)$. (The iterated integral on the right side of (3.15) can be split into two pieces corresponding to s in $[0,R/2]$ and $[R/2,R]$, each of which is basically an integral in polar co-ordinates centered at x or y.)

To control (3.16) we use Fact (e) in Section 2, that ω satisfies a reverse Hölder inequality like (2.5). Let $p>1$ be the corresponding exponent, as in (2.5), and let r be the exponent conjugate to np, so that $r < n/(n-1)$. Then, using Hölder's inequality, we get that (3.16) is dominated by

$$\begin{aligned} CR^n \liminf_{t\to 0}\Big(\fint_{B(z,10R)} \omega_t^p\Big)^{1/(np)} &\Big(\fint_{B(z,10R)} L(x,y,q)^r\, dq\Big)^{1/r} \\ &\le CR^n \Big(\fint_{B(z,10R)} \omega\Big)^{1/n} R^{1-n} \\ &\le C\mu\big(B(z,10R)\big)^{1/n} \le C\delta(x,y).\end{aligned}$$

Combining this with (3.15) gives (3.14). This proves Proposition 3.12.

Before leaving this section we record the following useful criterion for being strong A_∞.

Lemma 3.17. *Assume that $u \in A_1$, ω is a strong A_∞ weight, and that r is a positive real number. If $u^r\omega \in A_\infty$, then $u^r\omega$ is a strong A_∞ weight.*

Let μ, $\tilde\mu$ be the doubling measures that correspond to ω, $u^r\omega$, respectively, and let $\delta(\cdot,\cdot)$ and $\tilde\delta(\cdot,\cdot)$ be the associated quasidistances. Let $x_1,\dots,x_k \in \mathbf{R}^n$ be given, with $x_j \in B\big(x_1, 2|x_k - x_1|\big)$ for all j. According to Lemma 3.3 it is enough to show that

$$\tilde{\delta}(x_1, x_k) \le C \sum_{j=1}^{k-1} \tilde{\delta}(x_j, x_{j+1}). \tag{3.18}$$

Set $B = B_{x_1, x_k}$ and $B_j = B_{x_j, x_{j+1}}$, and notice that $B_j \subseteq 100B$ for all j. By definitions we have

$$\tilde{\delta}(x_j, x_{j+1}) = \Big(\int_{B_j} \tilde{\omega}\Big)^{1/n} \ge \big(\operatorname*{essinf}_{B_j} u^{r/n}\big)\Big(\int_{B_j} \omega\Big)^{1/n} \ge \big(\operatorname*{essinf}_{100B} u^{r/n}\big)\delta(x_j, x_{j+1}),$$

and therefore

$$\begin{aligned} \sum_{j=1}^{k-1} \tilde{\delta}(x_j, x_{j-1}) &\ge \big(\operatorname*{essinf}_{100B} u^{r/n}\big) \sum_{j=1}^{k-1} \delta(x_j, x_{j+1}) \\ &\ge C^{-1}\big(\operatorname*{essinf}_{100B} u^{r/n}\big)\delta(x_1, x_k) \\ &= C^{-1}\big(\operatorname*{essinf}_{100B} u^{r/n}\big)\Big(\int_B \omega\Big)^{1/n}. \end{aligned} \tag{3.19}$$

In the second inequality the assumption that ω is a strong A_∞ weight is used.

From Jensen and the assumption that u is an A_1 weight we have

$$C \int_B \omega \ge C R^n \fint_B \omega \ge C R^n \exp\Big(\fint_B \log \omega\Big)$$

and

$$C \operatorname*{essinf}_{100B} u \ge \exp\Big(\fint_B \log u\Big).$$

In combination with (3.19) this yields (with $\tilde{\omega} = u^r \omega$)

$$\sum_{j=1}^{k-1} \tilde{\delta}(x_j, x_{j+1}) \ge C^{-1} R \exp\Big(\frac{1}{n}\fint_B \log \tilde{\omega}\Big) \ge C^{-1} R \Big(\fint_B \tilde{\omega}\Big)^{1/n} \ge C^{-1}\tilde{\delta}(x_1, x_k).$$

The second inequality uses Fact (g) from Section 2 applied to $\tilde{\omega}$. This gives (3.18), which is what we wanted.

4. Examples and counterexamples

Roughly speaking, the difference between A_∞ and strong A_∞ weights is that the latter must satisfy some additional (rather subtle) constraints on how they can be small. Here are some simple examples to illustrate this point.

Examples. (a) If ω is an A_1 weight, then it is a strong A_∞ weight. This can be derived easily from Lemma 3.3, or it can be viewed as a special case of Lemma 3.17. The point is that the A_1 condition prevents ω from ever being small.

(b) Suppose that ω is a continuous nonnegative function on $\mathbf{R}^n$. Then ω cannot be a strong A_∞ weight if it vanishes on a rectifiable curve. (Otherwise the curve would have μ-length equal to zero, $\mu = \omega dx$, contrary to Proposition 3.12(a).)

(c) If $\omega(x) = |x_1|^t$, $x = (x_1, \ldots, x_n)$, then ω is an A_∞ weight when $t > -1$, but it is a strong A_∞ weight only when $-1 < t \leq 0$.

(d) If $\omega(x) = |x|^p$, then ω is both A_∞ and strong A_∞ exactly when $p > -n$. Thus a strong A_∞ weight can vanish to large order at a single point.

(e) Let Z be a closed set in $\mathbf{R}^n$ that has measure zero, and let $p > 0$ be given. A sufficient condition for

$$\omega(x) = \operatorname{dist}(x, Z)^p$$

to be an A_∞ weight is that Z should be uniformly thin, which means that there is an $a > 0$ so that for each $x \in Z$ and $R > 0$ there is a $y \in B(x, R)$ such that $B(y, aR) \cap Z = \emptyset$. This condition is not adequate to ensure that ω be a strong A_∞ weight. (Consider $Z =$ a line in $\mathbf{R}^2$.) A sufficient condition for ω to be strong A_∞ is that Z should be uniformly scattered, which means that there is a $b > 0$ so that if x and y lie in Z and if E is a closed connected set that contains them, then there is a ball B centered on E with radius $b|x - y|$ such that $B \cap Z = \emptyset$.

It is not so difficult to build Cantor sets in $\mathbf{R}^n$ that are uniformly scattered and that have Hausdorff dimension as close to n as you like. Thus strong A_∞ weights can vanish on rather large sets, and to large order.

Although the uniformly scattered condition is convenient for producing examples of strong A_∞ weights, it is by no means essential. Strong A_∞ weights can vanish on nontrivial connected sets, as we shall soon see. These connected sets must be pretty crooked, though, because they must not contain a rectifiable arc.

Let us consider now some natural questions about strong A_∞ weights which are motivated by classical results for A_∞ weights.

Question 4.1. If ω is a strong A_∞ weight, is the same true of ω^s, $0 < s < 1$?

Question 4.2. If ω is a strong A_∞ weight, must there be a $p > 1$ such that ω^p is also strong A_∞?

In order to state the remaining questions we need a definition.

Definition 4.3. $\omega = uv$ is called an A_1 factorization of ω if u, $v^{-t} \in A_1$ for some $t \geq 0$.

As mentioned in Fact (h) of Section 2, every A_∞ weight admits an A_1 factorization, and the existence of such a factorization implies that $\omega \in A_\infty$.

Question 4.4. Does every strong A_∞ weight ω admit an A_1 factorization $\omega = uv$ such that v is also a strong A_∞ weight?

Question 4.5. Does every strong A_∞ weight ω admit an A_1 factorization $\omega = uv$ such that there is an $\varepsilon > 0$ so that $u^{1-\varepsilon}v^{1+\varepsilon}$ is also a strong A_∞ weight?

In other words, this last question asks whether every strong A_∞ weight can be made smaller, in a certain way, and still be strong A_∞.

The answers to each of these questions is no. The counterexamples presented below were obtained jointly with Guy David. They have the additional feature that they arise as the Jacobians of quasiconformal mappings in the plane.

Let us begin with the counterexample to Question 4.1. Let λ be a completely singular doubling measure on $\mathbf{R}$, and define $h\colon \mathbf{R} \to \mathbf{R}$ by

$$h(x) = \int_0^x d\lambda.$$

Let H be a Beurling–Ahlfors extension of h to a quasiconformal map of the complex plane onto itself. (See [BA] for an example of such a λ as well as the definition of H.) Let ω_1 be the Jacobian of H. Thus ω_1 is a strong A_∞ weight (because it is the Jacobian of a q.c. mapping), but ω_1^s is not a strong A_∞ weight for any $s \in (0,1)$, as we now show.

From the definition of H it follows that H is C^1 away from $\mathbf{R}$ and that

$$\left|\nabla H(z)\right| \approx \frac{1}{|y|} \int_x^{x+|y|} d\lambda, \qquad z = x + iy,$$

for all $z \in \mathbf{C}$ with $y \neq 0$. Because λ is completely singular we have that

$$\lim_{y \to 0} \left|\nabla H(x+iy)\right| = 0 \quad \text{a.e..} \tag{4.6}$$

Fix $s \in (0,1)$, and let $\mu_{1,s}$ be the doubling measure on $\mathbf{C}$ associated to ω_1^s. To prove that ω_1^s is not strong A_∞ it suffices to show that

$$\lim_{t \to 0} \mu_{1,s}\text{-length}(\gamma_t) = 0, \tag{4.7}$$

where $\gamma_t\colon [0,1] \to \mathbf{C}$ is the path defined by $\gamma_t(u) = u + it$.

Because ω_1 is continuous on $\mathbf{C} \setminus \mathbf{R}$ we have

$$\mu_{1,s}\text{-length}(\gamma_t) = \int_0^1 \omega_1(u+it)^{s/2} du \leq \int_0^1 \left|\nabla H(u+it)\right|^s du$$

when $t \neq 0$. For $0 < |t| \leq 1$ we have

$$\int_0^1 \left|\nabla H(u+it)\right| du \leq C \int_0^2 d\lambda,$$

so that $\left|\nabla H(u+it)\right|^s$ lies in $L^{1/s}\big([0,1]\big)$ as a function of u, with norm that is uniformly bounded in t, $0 < |t| < 1$. Combining this with (4.6) and standard arguments we get that

$$\lim_{t \to 0} \int_0^1 \left|\nabla H(u+it)\right|^s du = 0,$$

which implies (4.7).

The counterexample for Question 4.2 will be obtained through sort of a dual construction. This time the starting point will be a quasisymmetric embedding $f\colon \mathbf{R} \to \mathbf{C}$ whose image is very much not rectifiable (e.g., all nontrivial subarcs will have infinite length). Recall that $f\colon \mathbf{R} \to \mathbf{C}$ is said to be quasisymmetric if there is a $C > 0$ so that

$$\left|f(u) - f(v)\right| \leq C\left|f(u) - f(w)\right|$$

whenever u, v, $w \in \mathbf{R}$ satisfy $|u - v| \leq |u - w|$. Instead of the Beurling–Ahlfors theorem we shall employ the following result of Tukia [T].

Theorem 4.8. *If $f\colon \mathbf{R} \to \mathbf{C}$ is quasisymmetric, then there is a quasiconformal mapping $F\colon \mathbf{C} \to \mathbf{C}$ which extends f, is C^1 on $\mathbf{C} \setminus \mathbf{R}$, and satisfies*

$$C^{-1}\left|\nabla F(x+iy)\right| \leq \frac{\left|f(x+y) - f(x-y)\right|}{|y|} \leq C\left|\nabla F(x+iy)\right| \tag{4.9}$$

for some $C > 0$ and all $x + iy \in \mathbf{C} \setminus \mathbf{R}$.

We want to apply this extension theorem to an f whose image is far from rectifiable, e.g., an f whose image is the Van Koch snowflake. The conditions described in the next lemma will be adequate for our purposes.

Given $f\colon \mathbf{R} \to \mathbf{C}$ and an interval $I \subseteq \mathbf{R}$, $I = [a,b]$, set

$$\Delta_I f = \frac{\bigl|f(a) - f(b)\bigr|}{|a-b|}.$$

Lemma 4.10. *There exists a quasisymmetric embedding $f\colon \mathbf{R} \to \mathbf{C}$ such that $\Delta_I f \to \infty$ uniformly as $|I| \to 0$ and*

$$\Delta_I f \geq C^{-1} \Delta_J f$$

whenever $I \subseteq J$, for some $C > 0$.

It is not difficult to build such an f by standard methods. For instance you can take f to be the parameterization of a self-similar quasicircle with Hausdorff dimension larger than 1. You can take f to be "self-similar" too.

Let f be as in Lemma 4.10, and let F be the extension of f promised by Theorem 4.8. Set $G = F^{-1}$, and let ω_2 be the Jacobian of G, so that ω_2 is a strong A_∞ weight. Fix $p > 1$, and consider ω_2^p. If ω_2^p is not an A_∞ weight, then we do not have to do anything. Suppose that it is an A_∞ weight, and let $\mu_{2,p}$ denote the corresponding doubling measure. For each $t > 0$ define $\alpha_t\colon [0,1] \to \mathbf{C}$ by $\alpha_t(x) = F(x + it)$. If we can show that

$$\lim_{t\to 0} \mu_{2,p}\text{-length}(\alpha_t) = 0, \tag{4.11}$$

then it follows that ω_2^p is not a strong A_∞ weight.

By definitions ω_2 is continuous on $\mathbf{C} \setminus f(\mathbf{R})$ and so

$$\mu_{2,p}\text{-length}(\alpha_t) = \int_0^1 \omega_2\bigl(\alpha_t(u)\bigr)^{p/2} \bigl|\alpha_t'(u)\bigr|\, du$$

when $t > 0$. Set

$$\varepsilon_t = \sup\bigl\{\omega_2\bigl(\alpha_t(u)\bigr) : 0 \leq u \leq 1\bigr\},$$

so that

$$\begin{aligned}
\mu_{2,p}\text{-length}(\alpha_t) &\leq \varepsilon_t^{(p-1)/2} \int_0^1 \omega_2\bigl(\alpha_t(u)\bigr)^{1/2} \bigl|\alpha_t'(u)\bigr|\, du \\
&\leq \varepsilon_t^{(p-1)/2} \int_0^1 \omega_2\bigl(\alpha_t(u)\bigr)^{1/2} \bigl|\nabla F(u + it)\bigr|\, du \\
&\leq C \varepsilon_t^{(p-1)/2}.
\end{aligned}$$

For this last inequality we have used the simple fact that

$$\omega_2(z) \approx \big|\nabla F\big(G(z)\big)\big|^{-2}. \tag{4.12}$$

This fact together with (4.9) and the information that $\Delta_I f \to \infty$ uniformly as $|I| \to 0$ also implies that $\varepsilon_t \to 0$ as $t \to 0$. This gives (4.11).

Notice that ω_2 is continuous on $\mathbf{C}$ and vanishes precisely on the connected set $f(\mathbf{R})$.

The counterexample for Question 4.4 is the same weight ω_1 used to dispatch Question 4.1. Indeed, suppose that ω_1 admits an A_1 factorization $\omega_1 = uv$ with v a strong A_∞ weight. Then this contradicts Lemma 3.17, because v is a strong A_∞ weight, $\omega_1^s = (u^s v^{s-1})v$ is an A_∞ weight, and $u^s v^{s-1}$ can be written as a positive power of an A_1 weight when $s < 1$, but ω_1^s is not a strong A_∞ weight when $0 < s < 1$.

Similarly, the counterexample for Question 4.2 also provides a counterexample to Question 4.5. Let ω_2 be as above, and assume that ω_2 admits an A_1 factorization $\omega_2 = uv$ such that $u^{1-\varepsilon}v^{1+\varepsilon}$ is a strong A_∞ weight. To get a contradiction we will use the following.

Lemma 4.13. *Under these conditions $u^{1-\eta}v^{1+\eta}$ is a strong A_∞ weight for all $\eta \in (0,\varepsilon)$.*

Set $\omega_{2,\eta} = u^{1-\eta}v^{1+\eta}$. Then

$$\omega_{2,\eta} = \omega_{2,0}^{1-\eta/\varepsilon}\omega_{2,\varepsilon}^{\eta/\varepsilon},$$

and so $\omega_{2,\eta}$ is an A_∞ weight when $0 < \eta < \varepsilon$, because of Fact (d) in Section 2. On the other hand,

$$\omega_{2,\eta} = \big(u^{\varepsilon-\eta}v^{\eta-\varepsilon}\big)\omega_{2,\varepsilon},$$

and so Lemma 3.17 implies that $\omega_{2,\eta}$ is a strong A_∞ weight when $0 < \eta < \varepsilon$, because $u^{\varepsilon-\eta}v^{\eta-\varepsilon}$ is a positive power of an A_1 weight in that case. This proves Lemma 4.13.

Choose $\eta \in (0,\varepsilon)$ small enough so that $\omega_2^{1+\eta}$ is an A_∞ weight. This is possible, because of Fact (f) from Section 2. Then Lemmas 3.17 and 4.13 imply that $\omega_2^{1+\eta}$ is also a strong A_∞ weight, because

$$\omega_2^{1+\eta} = u^{1+\eta}v^{1+\eta} = u^{2\eta}(u^{1-\eta}v^{1+\eta}).$$

However, we already know that ω_2^p is not a strong A_∞ weight for any $p > 1$. Therefore ω_2 is a counterexample for Question 4.5.

Remark 4.14. It can be shown that $\omega_2^{-t} \in A_1$ for some $t > 0$. This can be derived from (4.12) and the fact that $|\nabla F|^2 \in A_1$. The latter can itself be proved using (4.9) and Lemma 4.10.

5. The statement of the main result

Definition 5.1. ω is a stronger A_∞ weight if it is a strong A_∞ weight and if it admits an A_1 factorization (see Definition 4.3) $\omega = uv$ such that $u^{1-\varepsilon}v^{1+\varepsilon}$ is a strong A_∞ weight for some $\varepsilon > 0$.

Thus, in order for a strong A_∞ weight to be a stronger A_∞ weight, it has to be possible to make it substantially smaller ("more degenerate") without destroying the strong A_∞ property. The strong A_∞ weights which are not stronger A_∞ weights should be thought of as living at the boundary of the set of all strong A_∞ weights. This is made more precise by Example (d) below. In other words, the set of stronger A_∞ weights can be viewed as being a kind of "interior" of the space of strong A_∞ weights.

Theorem 5.2. *If ω is a stronger A_∞ weight, then there exist $N > n$, $C > 0$, and $f\colon \mathbf{R}^n \to \mathbf{R}^N$ that satisfies* (1.6), *with $\mu = \omega dx$.*

The proof of this theorem is given in Sections 6–10. The proof is somewhat complicated, and the reader might find it helpful to consider the important special case of A_1 weights, which is much simpler.

Although every A_∞ weight admits an A_1 factorization, not every strong A_∞ weight is a stronger A_∞ weight, as we saw in Section 4. (Note that the conclusion of Theorem 5.2 does hold for that example—i.e., ω_2—from Section 4.) Still, there are plenty of stronger A_∞ weights, as the following examples indicate.

Examples.

(a) Every A_1 weight is a stronger A_∞ weight.

(b) The examples of strong A_∞ weights discussed in (d) and (e) of Section 4 are also stronger A_∞ weights. This is not hard to check, using the observation that $\omega^{-t} \in A_1$ for some $t > 0$ when ω is as in (e) and also when ω is as in (d) and $p \geq 0$. (If $p < 0$ then ω is already an A_1 weight.)

(c) Suppose that $\omega^{-t} \in A_1$ for some $t > 0$. Then ω is a stronger A_∞ weight if $\omega^{1+\varepsilon}$ is a strong A_∞ weight. This uses the fact that if $v^{-t} \in A_1$ for some $t > 0$ and v is a strong A_∞ weight, then v^s is a strong A_∞ weight for all $s \in (0,1)$. This simple observation can be derived from Lemma 3.17, for instance.

(d) Every strong A_∞ weight can be approximated by a stronger A_∞ weight, in the following sense. Suppose that ω is a strong A_∞ weight, and that $\omega = uv$ is an A_1 factorization of ω. Consider $\omega_\varepsilon = u^{1/(1-\varepsilon)}v^{1/(1+\varepsilon)}$, $\varepsilon > 0$. This lies in A_∞ when $\varepsilon > 0$ is small enough, by Facts (f) and (h) in Section 2. When ω_ε is an A_∞ weight it is also strong A_∞, because of Lemma 3.17, and in that case it is clearly stronger A_∞ as well because ω is strong A_∞.

There are of course a number of issues left unresolved by Theorem 5.2. One of the most basic questions is whether the strong A_∞ condition alone is sufficient to imply the conclusions of Theorem 5.2. I do not have a strong opinion on this, but I am willing to go out on a limb for the analogous issue for Question 1.5.

Conjecture 5.3. There is a strong A_∞ weight ω on $\mathbf{R}^2$ for which there does not exist a mapping $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ that satisfies (1.6).

Persons of a more reckless (or insightful?) nature might be willing to speculate, in print, about the situation in which the stronger A_∞ (or even the A_1) condition is assumed.

Unfortunately there do not seem to be any good methods available for proving the nonexistence of a mapping f as in Conjecture 5.3. One way to try to generate necessary conditions on ω for the existence of such a mapping f is to look for analytical results on $\mathbf{R}^2$ that would have to have a counterpart for ω if f were to exist. For example, you could take the classical Sobolev and Poincaré inequalities. These do not work, because their counterparts for ω are true for all strong A_∞ weights (in all dimensions), by the results of [DS1].

Another problem which is connected to these issues is the following.

Question 5.4. Is Theorem 5.2 still true if we demand that N depend on n alone?

The proof given below has N depending on ω in a substantial way. This is true even when ω is an A_1 weight; in that case the proof is substantially simpler, but N still depends on ω.

Next we discuss ω-regular mappings and their relationship with the topics in this paper.

A continuous mapping $f\colon \mathbf{R}^n \to \mathbf{R}^N$ is said to be ω-regular for some A_∞ weight ω if there is a $C > 0$ so that:

(5.5) f has locally integrable first (distributional) derivatives, and $|\nabla f(x)| \le C\omega(x)^{1/n}$ a.e.; and

$$\int_{f^{-1}(B)} \omega(x)\,dx \le CR^n \text{ for every ball } B \text{ with radius } R \text{ in } \mathbf{R}^N. \tag{5.6}$$

This class of mappings arose in [D] in connection with the behavior of singular integral operators on their images. As pointed out in [DS1], under the assumption that ω is an A_∞ weight it can be shown that (5.5) is equivalent to the requirement that there be a $C > 0$ so that

$$|f(x) - f(y)| \le C\delta(x,y) \qquad \text{for all } \mathrm{x,y} \in \mathbf{R}^{\mathrm{n}}. \tag{5.7}$$

The proof of this equivalence is pretty straightforward, although perhaps a little bit technical. For passing from (5.5) to (5.7) it is useful to remember Proposition 3.12(b), for instance.

This reformulation of (5.7) makes it easy to see that $f\colon \mathbf{R}^n \to \mathbf{R}^N$ is ω-regular if it satisfies (1.6) with $\mu = \omega\,dx$. The converse is not true, because ω-regular mappings need not even be injective. However, if f is ω-regular, then $f^{-1}(p)$ has only a bounded number of elements for all $p \in \mathbf{R}^N$. (This is not hard to check, by estimating the ω-mass of $f^{-1}(B(p,r))$ as $r \to 0$.)

It is natural to hope that it might be easier to characterize the A_∞ weights ω on $\mathbf{R}^n$ for which there is an ω-regular mapping $f\colon \mathbf{R}^n \to \mathbf{R}^N$ for some N than to answer Question 1.7 completely, but so far nothing has occurred to reinforce this hope. A necessary condition for the existence of an ω-regular mapping is that ω be strongly A_∞ (see [DS1]), but there is not a better sufficient condition known than the one provided by Theorem 5.2.

The case of A_1 weights is of particular interest in the context of ω-regular mappings, because of the following result from [DS2]. Suppose that $f\colon \mathbf{R}^n \to \mathbf{R}^N$ is an ω-regular mapping for some A_∞ weight ω. Then there is an A_1 weight ω' and an ω'-regular mapping $g\colon \mathbf{R}^n \to \mathbf{R}^{N+1}$ such that

$$f(\mathbf{R}^n) \subseteq g(\mathbf{R}^n).$$

For this we identify $\mathbf{R}^N$ with a subset of $\mathbf{R}^{N+1}$ in the obvious way.

6. Conceptual preliminaries

Lemma 6.1. *Let μ be a doubling measure on $\mathbf{R}^n$. Then μ satisfies (1.8) if and only if there is a constant $C > 0$ so that for each x, $y \in \mathbf{R}^n$ there is a function $f\colon \mathbf{R}^n \to \mathbf{R}$ with the property that*

$$\big|f(z) - f(w)\big| \le C\delta(z,w) \qquad \text{for all } z,\ w \in \mathbf{R}^n, \text{ and} \tag{6.2}$$

$$\big|f(x) - f(y)\big| \ge C^{-1}\delta(x,y). \tag{6.3}$$

It is important to remember here that f is allowed to depend on x and y. The conclusion of Theorem 5.2 concerns the existence of a single $\mathbf{R}^N$-valued function f that works for all pairs of points x, y in $\mathbf{R}^n$.

The proof of Lemma 6.1 is easy. Suppose first that (1.8) holds. Let $\delta'(\cdot,\cdot)$ be as in (1.8), and let x, $y \in \mathbf{R}^n$ be given. Define $f\colon \mathbf{R}^n \to \mathbf{R}$ by $f(z) = \delta'(x,z)$. Then f satisfies (6.2) and (6.3) because of (1.8) and the fact that $\delta'(\cdot,\cdot)$ satisfies the triangle inequality.

Conversely, suppose that for each x, $y \in \mathbf{R}^n$ we can find f as above. Define $\delta'(\cdot,\cdot)$ by

$$\begin{aligned}\delta'(z,w) = \sup\big\{\big|g(z) - g(w)\big| :&\, g \text{ is a real-valued function on } \mathbf{R}^n \text{such that} \\ &\big|g(a) - g(b)\big| \le \delta(a,b) \text{ for all } a,\ b \in \mathbf{R}^n\big\}.\end{aligned}$$

Then $\delta'(\cdot,\cdot)$ is nonnegative, symmetric, and satisfies the triangle inequality and also $\delta'(z,w) \le \delta(z,w)$ for all z and w. The existence of the good f's implies that

$$\delta(x,y) \le C\delta'(x,y)$$

for all x, $y \in \mathbf{R}^n$ as well, so that (1.8) is true. This proves the lemma.

It is not hard to improve Lemma 6.1 by showing that if μ satisfies (1.8), then there exist C, $m > 0$ so that for each $t > 0$ there is a mapping $f\colon \mathbf{R}^n \to \mathbf{R}^m$ so that (6.2) holds, and also (6.3) for all x, $y \in \mathbf{R}^n$ such that $t < |x-y| < 2t$. In other words, instead of simply separating well a given pair of points, we can separate any pair of points at a given scale.

This is still not good enough for Theorem 5.2; for that we must deal with all scales at the same time. There does not seem to be any simple way to combine the mappings that work for the various individual scales. Similar difficulties arise when you try to get L^∞ estimates for $\overline{\partial}$ problems, and Peter Jones' work in that subject provided part of the inspiration for the methods employed in the next sections.

There is a variant of Lemma 6.1 that will be useful later for constructing partitions of unity that are adapted to a given strong A_∞ weight, and which we record now.

Lemma 6.4. *Suppose that μ satisfies (1.8). Then there is a $C > 0$ so that for each cube Q in $\mathbf{R}^n$ there is a function $\theta\colon \mathbf{R}^n \to \mathbf{R}$ such that $0 \le \theta \le 1$, $\theta = 1$ on $2Q$, $\theta = 0$ on $\mathbf{R}^n \setminus 3Q$, and*

$$\bigl|\theta(x) - \theta(y)\bigr| \le C\delta(x,y)\mu(Q)^{-1/n} \qquad \text{for all } x,\ y \in \mathbf{R}^n.$$

Let $\delta'(\cdot,\cdot)$ be as in (1.8), and consider

$$\delta'(x, \mathbf{R}^n \setminus 3Q) = \inf\bigl\{\delta'(x,y) : y \in \mathbf{R}^n \setminus 3Q\bigr\}.$$

Standard computations imply that this function satisfies (6.2), because $\delta'(x,y)$ satisfies (6.2) as a function of x for each $y \in \mathbf{R}^n$. Define θ by

$$\theta(x) = \min\bigl(1, \bigl\{\delta'(2Q, \mathbf{R}^n \setminus 3Q)^{-1}\delta'(x, \mathbf{R}^n \setminus 3Q)\bigr\}\bigr),$$

where, as usual,

$$\delta'\,(2Q, \mathbf{R}^n \setminus 3Q) = \inf\{\delta'(x, \mathbf{R}^n \setminus 3Q) : x \in 2Q\}.$$

Then it is easy to verify that θ has the required properties.

7. Technical preliminaries

We shall use the following notations from now on.

Let ω be the given stronger A_∞ weight on $\mathbf{R}^n$, and let $\omega = uv$ be as in Definition 5.1. Set

$$\tilde{\omega} = u^{1-\varepsilon}v^{1+\varepsilon}, \tag{7.1}$$

where $\varepsilon > 0$ is as in Definition 5.1, so that $\tilde{\omega}$ is a strong A_∞ weight. (If ω is simply an A_1 weight, then we can take $\tilde{\omega} \equiv 1$.) As in Lemma 4.13 we can shrink ε in (7.1) and still have a strong A_∞ weight, and so we can (and do) assume that $\varepsilon > 0$ is small enough so that

$$\alpha = \omega/\tilde{\omega} = u^\varepsilon v^{-\varepsilon}$$

is an A_1 weight. Let μ and $\tilde{\mu}$ be the measures on $\mathbf{R}^n$ corresponding to the weights ω and $\tilde{\omega}$, and let $\delta(\cdot,\cdot)$ and $\tilde{\delta}(\cdot,\cdot)$ be as in (1.2).

Let $L > 0$ be large, to be chosen later, depending on ε, n, and the relevant constants that arise in the various conditions on ω, $\tilde{\omega}$, u, v, and α. In what follows all of the constants "C" that occur are permitted to depend on any of the constants just mentioned except L; the presence of any dependence on L will always be made clear.

For each $j \in \mathbf{Z}$ set

$$\Omega_j = \{x \in \mathbf{R}^n : \alpha^*(x) > L^{j\varepsilon n}\}, \tag{7.2}$$

where α^* denotes the Hardy–Littlewood maximal function of α, i.e.,

$$\alpha^*(x) = \sup_{B \ni x} \fint_B \alpha,$$

where the supremum is taken over all balls that contain x. In order to build the mapping f promised in Theorem 5.2 we shall first define some auxiliary mappings in the next section which, roughly speaking, do a good job of separating pairs of points that lie in a given Whitney cube of some Ω_j. In this section we derive some estimates concerning the various weights and the Ω_j's.

Define j_0 by

$$j_0 = \inf\{j : \Omega_{j+1} \neq \mathbf{R}^n\} = \sup\{j : \Omega_j = \mathbf{R}^n\}.$$

Clearly, $j_0 < +\infty$, but it could happen that $j_0 = -\infty$. The j's for which $j < j_0$ will not be playing a role in what we do.

Set

$$d_j(x) = \operatorname{dist}(x, \mathbf{R}^n \setminus \Omega_j), \qquad x \in \mathbf{R}^n, \tag{7.3}$$

for $j > j_0$. Here "dist" refers to the Euclidian distance. Let $\delta_j(x)$ and $\tilde{\delta}_j(x)$ be the counterparts of (7.3) associated to $\delta(\cdot,\cdot)$ and $\tilde{\delta}(\cdot,\cdot)$, i.e.,

$$\delta_j(x) = \delta(x, \mathbf{R}^n \setminus \Omega_j) = \inf\{\delta(x,y) : y \in \mathbf{R}^n \setminus \Omega_j\} \tag{7.4}$$

and

$$\tilde{\delta}_j(x) = \tilde{\delta}(x, \mathbf{R}^n \setminus \Omega_j) \tag{7.5}$$

for $x \in \mathbf{R}^n$ and $j > j_0$. When $j \le j_0$ it is convenient to define d_j, δ_j, and $\tilde{\delta}_j$ to all be identically equal to $+\infty$.

One of the important features of the Ω_j's is that Ω_{j+1} is sparse inside of Ω_j when $j > j_0$ and L is large. This is made precise by the following.

Lemma 7.6. *There exist C, $\eta > 0$ so that*

$$d_{j+1}(x) \le C L^{-\varepsilon} d_j(x), \tag{7.7}$$

$$\delta_{j+1}(x) \le C L^{-\eta} \delta_j(x), \tag{7.8}$$

and

$$\tilde{\delta}_{j+1}(x) \le C L^{-\eta} \tilde{\delta}_j(x) \tag{7.9}$$

for all $x \in \mathbf{R}^n$ and $j > j_0$.

The first inequality can be derived from the definitions as follows. Because α is an A_1 weight we have

$$\alpha \ge C^{-1} \alpha^* \ge C^{-1} L^{j\varepsilon n} \quad \text{a.e. on } \Omega^j. \tag{7.10}$$

On the other hand

$$\fint_{B(x,R)} \alpha \le L^{j\varepsilon n} \qquad \text{when } R \ge d_j(x) \tag{7.11}$$

by definition of Ω_j and α^*. Hence

$$C^{-1} L^{(j+1)\varepsilon n} d_{j+1}(x)^n \le \int_{B(x,d_{j+1}(x))} \alpha \le \int_{B(x,d_j(x))} \alpha \le C L^{j\varepsilon n} d_j(x)^n,$$

(by (7.11)). This implies (7.7).

The other two estimates—(7.8) and (7.9)—follow from (7.7) and the observation that if σ is a doubling measure and B_1, B_2 are two balls with $B_1 \subseteq B_2$, then

$$\frac{\sigma(B_1)}{\sigma(B_2)} \le C \Big(\frac{|B_1|}{|B_2|}\Big)^a$$

for some C, $a > 0$. This is easily checked. (Notice that we are only asking that this inequality hold for balls. That is why you only need a doubling condition on σ, and not an A_∞ condition.)

Next we record a lemma on the relationship between μ and $\tilde{\mu}$ in terms of the Ω_j's.

Lemma 7.12. *Let B be a ball in $\mathbf{R}^n$.*

(a) *If $B \subseteq \Omega_j$, then*

$$\mu(B) \geq C^{-1} L^{j\varepsilon n} \tilde{\mu}(B).$$

(b) *If B intersects $\mathbf{R}^n \setminus \Omega_j$, then*

$$\mu(B) \leq C L^{j\varepsilon n} \tilde{\mu}(B).$$

Part (a) follows immediately from (7.10) and the various definitions. To prove (b) we use Fact (g) in Section 2 to get that

$$\fint_B \omega \leq C \Big(\fint_B \tilde{\omega} \Big) \Big(\fint_B \alpha \Big).$$

The definition of Ω_j implies that

$$\fint_B \alpha \leq L^{j\varepsilon n} \qquad \text{when } B \cap (\mathbf{R}^n \setminus \Omega_j) \neq \emptyset, \tag{7.13}$$

and from here Part (b) follows directly.

The last topic that we take up in this section deals, roughly speaking, with controlling ω on $\Omega_j \setminus \Omega_{j+1}$.

Lemma 7.14. *Let B be a ball and Q be a cube. Assume that*

$$Q \subseteq \Omega_j \qquad \textit{and} \qquad Q \setminus \Omega_{j+1} \neq \emptyset \tag{7.15}$$

and that

$$B \subseteq 100Q \qquad \textit{and} \qquad B \textit{ intersects } Q \setminus \Omega_{j+1}. \tag{7.16}$$

Then

$$\Big| \fint_B \log \omega - \fint_Q \log \omega \Big| \leq \log L^n + C. \tag{7.17}$$

The point here is that B could be much smaller than Q, so that (7.17) would not follow from the doubling condition on μ. Notice in particular that (7.17) implies that

$$\Big| \log \omega(x) - \fint_Q \log \omega \Big| \leq \log L^n + C \quad \text{a.e. on } Q \setminus \Omega_{j+1}. \tag{7.18}$$

We assume for the rest of this section that B and Q are as in Lemma 7.14. Observe first that

$$\alpha \geq C^{-1} L^{j\varepsilon n} \quad \text{a.e. on } Q \tag{7.19}$$

and

$$\fint_Q \alpha \leq C L^{(j+1)\varepsilon n}. \tag{7.20}$$

These inequalities follow from (7.15) and the definition of Ω_j. With the aid of Jensen they imply that

$$\log L^{j\varepsilon n} - C \leq \fint_Q \log \alpha \leq \log L^{(j+1)\varepsilon n} + C. \tag{7.21}$$

Similarly, (7.16) implies

$$\fint_B \log \alpha \leq \log\Big(\fint_B \alpha\Big) \leq L^{(j+1)\varepsilon n} + C. \tag{7.22}$$

Next we consider u^ε and $v^{-\varepsilon}$. Because they are both A_1 weights we have that

$$\begin{aligned} &\text{(a)} \qquad \fint_Q \log u^\varepsilon \leq \operatorname*{essinf}_{100Q} (\log u^\varepsilon) + C, \\ &\text{(b)} \qquad \fint_Q \log v^{-\varepsilon} \leq \operatorname*{essinf}_{100Q} \big(\log v^{-\varepsilon}\big) + C. \end{aligned} \tag{7.23}$$

Hence, by (7.21)–(7.23),

$$\begin{aligned} \fint_B \log u^\varepsilon + \fint_B \log v^{-\varepsilon} &= \fint_B \log \alpha \leq \log L^{(j+1)\varepsilon n} + C \\ &\leq \log L^{\varepsilon n} + \fint_Q \log \alpha + C \\ &= \log L^{\varepsilon n} + \fint_Q \log u^\varepsilon + \fint_Q \log v^{-\varepsilon} + C \\ &\leq \log L^{\varepsilon n} + \operatorname*{essinf}_{100Q} (\log u^\varepsilon) + \operatorname*{essinf}_{100Q} \big(\log v^{-\varepsilon}\big) + C. \end{aligned}$$

Because $B \subseteq 100Q$ we get from here that

$$(7.24)\qquad \begin{aligned} &\text{(a)} \qquad \fint_B \log u^\varepsilon \le \log L^{\varepsilon n} + \operatorname*{essinf}_{100Q} (\log u^\varepsilon) + C \text{ and} \\ &\text{(b)} \qquad \fint_B \log v^{-\varepsilon} \le \log L^{\varepsilon n} + \operatorname*{essinf}_{100Q} \left(\log v^{-\varepsilon}\right) + C. \end{aligned}$$

Now let us look at ω. We have, by (7.24)(a) and (7.23)(b),

$$\begin{aligned} \fint_B \log \omega &= \fint_B \log u + \fint_B \log v \\ &\le \log L^n + \operatorname*{essinf}_{100Q} \log u + \operatorname*{esssup}_{100Q} \log v + C \\ &\le \log L^n + \fint_Q \log u + \fint_Q \log v + C \\ &\le \log L^n + \fint_Q \log \omega + C. \end{aligned}$$

Similarly,

$$\fint_B \log \omega \ge -\log L^n + \fint_Q \log \omega - C,$$

using (7.24)(b) and (7.23)(a). These estimates imply (7.17), as desired. This proves Lemma 7.14.

8. Building blocks for defining f

In this section we construct the building blocks that will be used to define f (in the next section). To do that we must first set up some Whitney decompositions and partitions of unity.

For each $j > j_0$ let $\{Q_{j,k}\}$ be an enumeration of the maximal dyadic cubes Q contained in Ω_j that satisfy

$$(8.1)\qquad 3 \operatorname{diam} Q \le (10n)^{-10n} \inf_{3Q} d_j(x).$$

Thus $\cup_k \overline{Q}_{j,k} = \Omega_j$, $Q_{j,k}$ and $Q_{j,l}$ have disjoint interiors unless $k = l$,

(8.2) $3 \operatorname{diam} Q_{j,k} \le (10n)^{-10n} \operatorname{dist}(3Q_{j,k}, \mathbf{R}^n \setminus \Omega_j)$,

(8.3) $\operatorname{diam} Q_{j,k} \ge C^{-1} \operatorname{dist}(3Q_{j,k}, \mathbf{R}^n \setminus \Omega_j)$,

and

(8.4) for each j, $\{10Q_{j,k}\}_k$ is a sequence of cubes with bounded overlap.

We can associate partitions of unity to these Whitney decompositions in the usual way, except that we want the bump functions to be Lipschitz relative to $\tilde{\delta}(\cdot,\cdot)$. For each j, k let $\theta_{j,k}$ be the function promised to $Q = Q_{j,k}$ by Lemma 6.4, except that μ and $\delta(\cdot,\cdot)$ are replaced by $\tilde{\mu}$ and $\tilde{\delta}(x,y)$. Define $\varphi_{j,k}\colon \mathbf{R}^n \to \mathbf{R}$ by

$$\varphi_{j,k}(x) = 0 \qquad \text{when } x \notin \Omega_j,$$

and

$$\varphi_{j,k} = \theta_{j,k}\Big(\sum_l \theta_{j,l}\Big)^{-1} \qquad \text{on } \Omega_j.$$

Then each $\varphi_{j,k}$ satisfies

$$\operatorname{supp}\varphi_{j,k} \subseteq 3Q_{j,k} \subseteq \Omega_j, \tag{8.5}$$

$$0 \le \varphi_{j,k} \le 1, \qquad \sum_k \varphi_{j,k} = \chi_{\Omega_j}, \tag{8.6}$$

and

$$\big|\varphi_{j,k}(x) - \varphi_{j,k}(y)\big| \le C\tilde{\delta}(x,y)\tilde{\mu}(Q_{j,k})^{-1/n} \tag{8.7}$$

for all x, $y \in \mathbf{R}^n$. Note that

$$\tilde{\mu}(Q_{j,k})^{-1/n} \approx \tilde{\delta}_j(x) \qquad \text{for all } x \in Q_{j,k}, \tag{8.8}$$

because of (8.2), (8.3), and the fact that $\tilde{\mu}$ is a doubling measure.

It is very important that the Lipschitz condition (8.7) be in terms of $\tilde{\delta}(\cdot,\cdot)$ instead of $\delta(\cdot,\cdot)$. This is the sort of thing that will allow us to sum in j without losing control, as we shall have to do. It is for this purpose that we need ω to be a stronger A_∞ weight, rather than merely strong A_∞.

Set

$$M_0 = \inf\Big\{\Big(\fint_B \omega\Big)^{1/n} : B \cap (\mathbf{R}^n \setminus \Omega_{j_0+1}) \ne \emptyset\Big\} \tag{8.9}$$

and

$$M_{j,k} = \Big(\fint_{Q_{j\,k}} \omega\Big)^{1/n}. \tag{8.10}$$

Notice that $M_{j,k}$ is comparable to the ratio of the $\delta(\cdot,\cdot)$-diameter of Q to its Euclidean diameter. Also, $M_0 > 0$, as one can show using Lemma 7.14. (See Section 10 for more details, especially the estimates between (10.13) and (10.14).)

The next 3 lemmas concern the previously-heralded building blocks. Let $q_{j,k}$ denote the center of $Q_{j,k}$.

Lemma 8.11. *Assume that $j_0 > -\infty$. Then there is a continuous mapping $f_0\colon \mathbf{R}^n \to \mathbf{R}^n$ such that*

$$|f_0(x) - M_0 x| \le M_0 (10n)^{-10n} d_{j_0+1}(x), \tag{8.12}$$

and

$$|f_0(x) - f_0(y)| \le C L^{(j_0+1)\varepsilon} \tilde{\delta}(x,y) \qquad \text{for all } x,\ y \in \mathbf{R}^n. \tag{8.13}$$

Lemma 8.14. *For each $j > j_0$ and each k there is a continuous mapping $h_{j,k}\colon 3Q_{j,k} \to \mathbf{R}^n$ such that*

$$|h_{j,k}(x) - M_{j,k}(x - q_{j,k})| \le M_{j,k}(10n)^{-10n} d_{j+1}(x) \tag{8.15}$$

and

$$|h_{j,k}(x) - h_{j,k}(y)| \le C L^{(j+1)\varepsilon+1} \tilde{\delta}(x,y) \tag{8.16}$$

for all x, $y \in \mathbf{R}^n$.

Lemma 8.17. *Let $h_{j,k} : 3Q_{j,k} \to \mathbf{R}^n$ be as in Lemma 8.14, and define $f_{j,k}\colon \mathbf{R}^n \to \mathbf{R}^n$ by*

$$\begin{aligned} f_{j,k}(x) &= 0 \qquad && \text{when } x \in \mathbf{R}^n \setminus 3Q_{j,k}, \\ f_{j,k} &= \theta_{j,k} h_{j,k} && \text{on } 3Q_{j,k}. \end{aligned}$$

Then

$$|f_{j,k}(x)| \le C \delta_j(x) \tag{8.18}$$

and

$$|f_{j,k}(x) - f_{j,k}(y)| \le C L^{(j+1)\varepsilon+1} \tilde{\delta}(x,y) \qquad \text{for all } x, y \in \mathbf{R}^n. \tag{8.19}$$

Again, for the purposes of a future summation in j it is important that the Lipschitz conditions in (8.13) and (8.19) be in terms of $\tilde{\delta}(\cdot,\cdot)$ rather than $\delta(\cdot,\cdot)$.

The rest of this section will be devoted to the proofs of these lemmata.

Let us begin with Lemma 8.11. Suppose that $j_0 > -\infty$, and define $f_0\colon \mathbf{R}^n \to \mathbf{R}^n$ by

$$\begin{aligned} f_0(x) &= M_0 x \qquad && \text{when } x \in \mathbf{R}^n \setminus \Omega_{j_0+1} \qquad \text{and} \\ f_0 &= M_0 \sum_k \varphi_{j_0+1,k}\, q_{j_0+1,k} && \text{on } \Omega_{j_0+1}. \end{aligned} \tag{8.20}$$

It is easy to see that f_0 is continuous. We must verify (8.12) and (8.13).

Of course (8.12) is automatic when $x \in \mathbf{R}^n \setminus \Omega_{j_0+1}$, and so we may as well assume that $x \in \Omega_{j_0+1}$. Because $\sum_k \varphi_{j_0+1,k}(x) = 1$ we have

$$\big|f_0(x) - M_0 x\big| \le M_0 \sum_k \varphi_{j_0+1,k}(x)|x - q_{j_0+1,k}|.$$

If $\varphi_{j_0+1,k}(x) \neq 0$, then $x \in 3Q_{j_0+1,k}$, and so

$$\begin{aligned}\big|f_0(x) - M_0 x\big| &\le M_0 \sum_k \varphi_{j_0+1,k}(x)\,(3\operatorname{diam} Q_{j_0+1,k}) \\ &\le M_0 \sum_k \varphi_{j_0+1,k}(x)(10n)^{-10n} d_{j_0+1}(x) \\ &\le M_0 (10n)^{-10n} d_{j_0+1}(x).\end{aligned}$$

In the second inequality we are using the fact that each $Q_{j,k}$ satisfies (8.2). This proves (8.12).

Consider now (8.13). Suppose first that

$$|x - y| \ge \frac{1}{10} d_{j_0+1}(x). \tag{8.21}$$

Then

$$\begin{aligned}\big|f_0(x) - f_0(y)\big| &\le \big|f_0(x) - M_0 x\big| + M_0|x-y| + \big|f_0(y) - M_0 y\big| \\ &\le M_0\big(d_{j_0+1}(x) + |x-y| + d_{j_0+1}(y)\big) \\ &\le 2M_0\big(d_{j_0+1}(x) + |x-y|\big) \le C M_0 |x-y|.\end{aligned} \tag{8.22}$$

In the third inequality we have used the fact that

$$d_{j_0+1}(y) \le d_{j_0+1}(x) + |x-y|. \tag{8.23}$$

On the other hand, (8.21) implies that $20B_{x,y}$ intersects $\mathbf{R}^n \setminus \Omega_{j_0+1}$, and so from (8.9) and Lemma 7.12(b) we get

$$\begin{aligned}M_0|x-y| &\le \Big(\fint_{20B_{x,y}} \omega\Big)^{1/n} |x-y| \le C\Big(\int_{20B_{x,y}} \omega\Big)^{1/n} \\ &\le C L^{(j_0+1)\varepsilon} \tilde{\mu}\,(20B_{x,y})^{1/n} \le C L^{(j_0+1)\varepsilon} \tilde{\delta}(x,y).\end{aligned} \tag{8.24}$$

Combining this with (8.22) gives (8.13) when (8.21) holds. Suppose now that

$$|x - y| < \frac{1}{10} d_{j_0+1}(x). \tag{8.25}$$

In particular both x and y lie in Ω_{j_0+1}. Using (8.20) we get

$$\begin{aligned}\big|f_0(x)-f_0(y)\big| &= M_0\Big|\sum_k \big(\varphi_{j_0+1,k}(x)-\varphi_{j_0+1,k}(y)\big)q_{j_0+1,k}\Big| \\ &= M_0\Big|\sum_k \big(\varphi_{j_0+1,k}(x)-\varphi_{j_0+1,k}(y)\big)\left(q_{j_0+1,k}-x\right)\Big|.\end{aligned} \tag{8.26}$$

In order for a term in the sum not to vanish either x or y must lie in $3Q_{j_0+1,k}$. In either case we must have

$$3Q_{j_0+1,k}\cap B\Big(x,\frac{1}{10}d_{j_0+1}(x)\Big)\neq\emptyset, \tag{8.27}$$

by (8.25). This can occur for at most a bounded number of k's (for a given x). For each k for which (8.27) is true we have that

$$|x-q_{j_0+1,k}|\le C d_{j_0+1}(x)$$

and

$$\big|\varphi_{j_0+1,k}(x)-\varphi_{j_0+1,k}(y)\big|\le C\tilde\delta(x,y)\tilde\mu\left(Q_{j_0+1,k}\right)^{-1/n},$$

by (8.7). Before inserting these estimates into (8.26) it is convenient to set

$$B_x=B\big(x,2d_{j_0+1}(x)\big),$$

so that

$$C\tilde\mu\left(Q_{j_0+1,k}\right)\ge\tilde\mu(B_x).$$

Now combining with (8.26) we get

$$\big|f_0(x)-f_0(y)\big|\le CM_0\tilde\delta(x,y)\tilde\mu\left(B_x\right)^{-1/n}d_{j_0+1}(x). \tag{8.28}$$

Because B_x intersects $\mathbf{R}^n\setminus\Omega_{j_0+1}$ we have from (8.9) and Lemma 7.12(b) that

$$M_0 d_{j_0+1}(x)\le\mu\left(B_x\right)^{1/n}\le CL^{(j_0+1)\varepsilon}\tilde\mu\left(B_x\right)^{1/n}. \tag{8.29}$$

This together with (8.28) gives (8.13) when (8.25) holds. This completes the proof of Lemma 8.11.

The proof of Lemma 8.14 is quite similar. Fix $j>j_0$. Define $h_{j,k}\colon 3Q_{j,k}\to\mathbf{R}^n$ by

$$h_{j,k}(x)=M_{j,k}(x-q_{j,k})\qquad\text{when } x\in 3Q_{j,k}\setminus\Omega_{j+1}$$

and

$$h_{j,k}=M_{j,k}\sum_l\varphi_{j+1,l}\left(q_{j+1,l}-q_{j,k}\right)\qquad\text{on } 3Q_{j,k}\cap\Omega_{j+1}. \tag{8.30}$$

It is easy to check that $h_{j,k}$ is continuous.

The proof of (8.15) is almost identical to that of (8.12), and so we omit it. The proof of (8.16) closely resembles that of (8.13), but there are a couple of changes, and so we sketch the argument.

Suppose that $x,\ y \in 3Q_{j,k}$ satisfy

$$|x-y| \geq \frac{1}{10} d_{j+1}(x). \tag{8.31}$$

Analogous to (8.22) we have

$$\left|h_{j,k}(x) - h_{j,k}(y)\right| \leq C M_{j,k} |x-y|. \tag{8.32}$$

On the other hand

$$M_{j,k} \leq C L^{(j+1)\varepsilon+1} \Big(\fint_{20B_{x,y}} \tilde{\omega} \Big)^{1/n}. \tag{8.33}$$

To see this we first observe that

$$M_{j,k} = \Big(\fint_{Q_{j,k}} \omega \Big)^{1/n} \leq C L \Big(\fint_{20B_{x,y}} \omega \Big)^{1/n}. \tag{8.34}$$

This follows from the doubling condition on μ if $20B_{x,y}$ is not contained in $100Q_{j,k}$ (so that $B_{x,y}$ is not too small compared to $Q_{j,k}$), and from Lemma 7.14 and Fact (g) in Section 2 when $20B_{x,y} \subseteq 100Q_{j,k}$. Clearly (8.33) follows from (8.34), Lemma 7.12(b), and the fact that $20B_{x,y}$ intersects $\mathbf{R}^n \setminus \Omega_{j+1}$. Combining (8.32) with (8.33) gives (8.16) when (8.31) holds.

Assume now that $x,\ y \in 3Q_{j,k}$ satisfy

$$|x-y| < \frac{1}{10} d_{j+1}(x),$$

so that $x,\ y \in \Omega_{j+1}$ in particular. Just as in (8.28) we have

$$\left|h_{j,k}(x) - h_{j,k}(y)\right| \leq C M_{j,k} \tilde{\delta}(x,y) \tilde{\mu}(B_x)^{-1/n} d_{j+1}(x),$$

where now

$$B_x = B\left(x, 2d_{j+1}(x)\right).$$

We also have

$$M_{j,k} \leq C L^{(j+1)\varepsilon+1} \Big(\fint_{B_x} \tilde{\omega} \Big)^{1/n}$$

for essentially the same reasons as for (8.33). These inequalities combine to give (8.16) in this case as well. This proves Lemma 8.14.

It remains to prove Lemma 8.17. Observe first that

$$\big|h_{j,k}(x)\big| \le C\mu(Q_{j,k})^{1/n} \qquad \text{when } x \in 3Q_{j,k}. \tag{8.35}$$

This is easily derived from the definition of $h_{j,k}$ and $M_{j,k}$. With this in hand (8.18) follows directly. To get (8.19) you use Lemma 6.4, (8.16), (8.35), and also the second inequality in the next lemma.

Lemma 8.36. *There is a $C > 0$ so that*

$$C^{-1}L^{j\varepsilon n}\tilde{\mu}(Q_{j,k}) \le \mu(Q_{j,k}) \le CL^{j\varepsilon n}\tilde{\mu}(Q_{j,k})$$

for all j, k.

This is an easy consequence of Lemma 7.12, and (8.3), and the doubling condition on $\tilde{\mu}$.

9. The definition of f

Basically f is going to be defined by combining the $f_{j,k}$'s from Section 8. This is slightly tricky, because we are only allowing f to take values in a finite dimensional space, and so we have to arrange the $f_{j,k}$'s into a finite set of piles. We have to be careful about how we do this, in order to make certain that a pair of $f_{j,k}$'s from the same pile do not interact too much.

Set $N_2 = N_0 N_1$, where N_0 is a large constant to be chosen later (independently of L), and N_1 is a large constant that will be selected very soon (depending only on n).

Lemma 9.1. *If N_1 is large enough then we can define a mapping*

$$Q_{j,k} \mapsto a_{j,k}, \qquad a_{j,k} \in \{1, 2, \ldots, N_2\} \tag{9.2}$$

in such a way that $a_{j,k} = a_{j',k'}$ implies:

(i) $j \equiv j' \bmod N_0$*;*

(ii) $k = k'$ *if also* $j = j'$ *and*

$$\operatorname{dist}(3Q_{j,k}, 3Q_{j',k'}) < \tfrac{1}{2}\operatorname{dist}(Q_{j,k} \cup Q_{j',k'}, \mathbf{R}^n \setminus \Omega_j). \tag{9.3}$$

The proof of this requires just a small coding argument. Choose N_1 so that for each j and k the set

$$\{k' :\ (9.3) \text{ holds with } j' = j\}$$

has less than N_1 elements. Then the correspondence (9.2) can be defined one cube $Q_{j,k}$ at a time (in any order), and our condition on N_1 ensures that at each stage there is a choice for the value of $a_{j,k}$ that is consistent with the previous choices and the requirements listed in the lemma.

We are going to take f to be a mapping from $\mathbf{R}^n$ into $\mathbf{R}^N$ with $N = (N_2+1)n + N_2$ and with f of the form

$$f = (f_0, f_1, \dots, f_{N_2}, g_1, \dots, g_{N_2}),$$

where the f_i's take values in $\mathbf{R}^n$ and the g_i's take values in $\mathbf{R}$. Here f_0 is as in Lemma 8.11 when $j_0 > -\infty$, and we take $f_0 \equiv 0$ otherwise. The remaining f_i's and g_i's are defined as follows.

Given i, $1 \le i \le N_2$, set

$$f_i = \sum_{\{(j,k): a_{j,k}=i\}} f_{j,k} \tag{9.4}$$

and

$$g_i = \sum_{\{(j,k): a_{j,k}=i\}} g_{j,k}, \tag{9.5}$$

where the $f_{j,k}$'s are as in Lemma 8.17, and where $g_{j,k}$: $\mathbf{R}^n \to \mathbf{R}$ is defined by

$$g_{j,k}(x) = L^{j\varepsilon} \tilde{\delta}'(x, \mathbf{R}^n \setminus 2Q_{j,k}). \tag{9.6}$$

Here $\tilde{\delta}'(\cdot,\cdot)$ is a distance function on $\mathbf{R}^n$ that is comparable to $\tilde{\delta}(\cdot,\cdot)$, as in (1.8).

In the next section we shall prove that this choice of f works. Before leaving this section we record some estimates on the $g_{j,k}$'s.

Lemma 9.7. *Each $g_{j,k}$ satisfies*

$$g_{j,k}(x) \le C\delta_j(x) \tag{9.8}$$

and

$$\left|g_{j,k}(x) - g_{j,k}(y)\right| \le C L^{j\varepsilon} \tilde{\delta}(x,y) \qquad \text{for all } x,\ y \in \mathbf{R}^n. \tag{9.9}$$

Also,

$$g_{j,k}(x) \ge C^{-1}\delta_j(x) \qquad \text{when } x \in \overline{Q_{j,k}}. \tag{9.10}$$

It is easy to derive (9.8) and (9.10) from (9.6), Lemma 8.36, and the doubling condition on $\tilde{\mu}$. As usual, (9.9) follows from (1.8) and the fact that $\tilde{\delta}'(\cdot,\cdot)$ satisfies the triangle inequality.

10. The end of the proof of Theorem 5.2

Lemma 10.1. *If L is large enough, then there is a $C(L) > 0$ so that*

$$|f(x) - f(y)| \le C(L)\delta(x,y) \tag{10.2}$$

whenever x, $y \in \mathbf{R}^n$.

Consider first f_0. We may as well assume that $j_0 > -\infty$. Thus $\Omega_{j_0} = \mathbf{R}^n$, and so Lemma 7.12(a) implies that

$$L^{j_0\varepsilon}\tilde{\delta}(x,y) \le C\delta(x,y)$$

for all x, $y \in \mathbf{R}^n$, which in turn implies that the contribution of f_0 to (10.2) is fine, because of (8.13).

Next we look at the f_i's and g_i's, $1 \le i \le N_2$. We begin by deriving estimates for

$$\sum_{j_0 < j \le j_1} \sum_k \left(|f_{j,k}(x) - f_{j,k}(y)| + |g_{j,k}(x) - g_{j,k}(y)|\right) \tag{10.3}$$

and

$$\sum_{j \ge j_1} \sum_k \left(|f_{j,k}(x)| + |g_{j,k}(x)|\right), \tag{10.4}$$

where j_1 is any integer such that $j_1 > j_0$.

Each term in the sum in (10.3) is bounded by $CL^{(j+1)\varepsilon+1}\tilde{\delta}(x,y)$, by (8.19) and (9.9). Also, for each j there are at most a bounded number of terms in the k sum in (10.3) which are nonzero, because $f_{j,k}$ and $g_{j,k}$ are both supported in $3Q_{j,k}$. Hence

$$(10.3) \le \sum_{j_0 < j \le j_1} CL^{(j+1)\varepsilon+1}\tilde{\delta}(x,y) \le CL^{(j_1+1)\varepsilon+1}\left(1 - L^{-\varepsilon}\right)^{-1}\tilde{\delta}(x,y).$$

We may as well require L to be so large that

$$L^{-\varepsilon} \le \tfrac{1}{2}, \tag{10.5}$$

whence

$$(10.3) \le CL^{(j_1+1)\varepsilon+1}\tilde{\delta}(x,y). \tag{10.6}$$

For (10.4) we notice that each term in the sum is bounded by $C\delta_j(x)$, by (8.18) and (9.8), and that for each j there are at most a bounded number of terms in the k sum that do not vanish, because of the conditions on the supports of $f_{j,k}$ and $g_{j,k}$. Hence

$$(10.4) \ \leq \sum_{j \geq j_1} C\delta_j(x).$$

From (7.8) it follows that

$$(10.4) \ \leq C\delta_{j_1}(x), \tag{10.7}$$

at least if L is large enough so that $CL^{-\eta} \leq \frac{1}{2}$.

To finish the proof of Lemma 10.1 it suffices to show that

$$\sum_{j > j_0} \sum_k \left(\left| f_{j,k}(x) - f_{j,k}(y) \right| + \left| g_{j,k}(x) - g_{j,k}(y) \right| \right) \leq C(L)\delta(x,y) \tag{10.8}$$

for all x, $y \in \mathbf{R}^n$. Fix x, $y \in \mathbf{R}^n$. Define $j(x,y)$ to be the largest integer j such that

$$x, y \in \Omega_j \qquad \text{and} \qquad B\big(x, 10|x-y|\big) \subseteq \Omega_j. \tag{10.9}$$

Notice that $j(x,y) \geq j_0$ and $j(x,y) > -\infty$, but that $j(x,y) = j_0$ is possible when $j_0 > -\infty$. To prove (10.8) we split the j sum into two pieces, according to whether or not $j \leq j(x,y)$.

Using (10.6) with $j_1 = j(x,y)$ we get

$$\sum_{j_0 < j \leq j(x,y)} \sum_k \left(\left| f_{j,k}(x) - f_{j,k}(y) \right| + \left| g_{j,k}(x) - g_{j,k}(y) \right| \right)$$
$$\leq CL^{(j(x,y)+1)\varepsilon + 1} \tilde{\delta}(x,y) \leq CL^{1+\varepsilon}\delta(x,y).$$

The second inequality comes from Lemma 7.12(a) and the fact that $B_{x,y} \subseteq \Omega_{j(x,y)}$.

From (10.7) we have

$$\sum_{j > j(x,y)} \sum_k \left(\left| f_{j,k}(x) - f_{j,k}(y) \right| + \left| g_{j,k}(x) - g_{j,k}(y) \right| \right)$$
$$\leq C\big(\delta_{j(x,y)+1}(x) + \delta_{j(x,y)+1}(y)\big) \leq C\delta(x,y).$$

The last inequality uses the fact that (10.9) fails for $j = j(x,y)+1$, as well as the doubling condition on μ.

Combining these estimates gives (10.8). This proves Lemma 10.1.

Lemma 10.10. *If N_0 and L are large enough, then there is a $C(L) > 0$ so that*

$$\left| f(x) - f(y) \right| \geq C(L)^{-1}\delta(x,y) \tag{10.11}$$

for all x, $y \in \mathbf{R}^n$.

Fix x, $y \in \mathbf{R}^n$, and let $j(x,y)$ be as above. In particular

$$|x-y| > \frac{1}{10} d_{j(x,y)+1}(x). \tag{10.12}$$

Suppose first that $j(x,y) = j_0$ (and hence $j_0 > -\infty$). Then, by (8.12),

$$\begin{aligned} \big|f(x)-f(y)\big| &\ge \big|f_0(x)-f_0(y)\big| \\ &\ge M_0|x-y| - \big|f_0(x)-M_0x\big| - \big|f_0(y)-M_0y\big| \\ &\ge M_0\big\{|x-y| - (10n)^{-10n}\big(d_{j_0+1}(x)+d_{j_0+1}(y)\big)\big\} \ge \tfrac{1}{2}M_0|x-y|. \end{aligned}$$

For the last inequality we have used (10.12) and

$$d_{j_0+1}(y) \le d_{j_0+1}(x) + |x-y| \le 11|x-y|. \tag{10.13}$$

We need to show that $M_0|x-y| \ge C(L)^{-1}\delta(x,y)$.

Let $B_{x,y}$ be as in (1.3), as usual. From (10.12) we get that $20B_{x,y}$ intersects $\mathbf{R}^n \setminus \Omega_{j_0+1}$, and so

$$\Big(\fint_B \omega\Big)^{1/n} \ge C^{-1}L^{-2}\Big(\fint_{20B_{x,y}} \omega\Big)^{1/n}$$

for any other ball B that intersects $\mathbf{R}^n \setminus \Omega_{j_0+1}$, because of Lemma 7.14 and Fact (g) from Section 2. (In applying Lemma 7.14 we may take Q to be any cube that contains B and $20B_{x,y}$, because $\Omega_{j_0} = \mathbf{R}^n$.) Using this inequality and the definition (8.9) of M_0 we get

$$M_0 \ge C^{-1}L^{-2}\Big(\fint_{20B_{x,y}} \omega\Big)^{1/n} \ge C^{-1}L^{-2}\delta(x,y)|x-y|^{-1}.$$

Combining this with our previous estimates yields (10.11) when $j(x,y) = j_0$.

Now suppose that $j(x,y) > j_0$. Set $J = j(x,y)$, and choose K so that $\overline{Q_{J,K}}$ contains x. (We can do this because $x \in \Omega_J$.) Set $i = a_{J,K}$. We have to distinguish between the cases where y does or does not belong to $2Q_{J,K}$.

Suppose first that $y \in 2Q_{J,K}$. Then

$$\begin{aligned} \big|f(x)-f(y)\big| &\ge \big|f_i(x)-f_i(y)\big| \\ &\ge \big|f_{J,K}(x)-f_{J,K}(y)\big| - \sum_{(j,k)\in Z} \big|f_{j,k}(x)-f_{j,k}(y)\big|, \end{aligned} \tag{10.14}$$

where $Z = \big\{(j,k) : a_{j,k} = i \text{ but } (j,k) \ne (J,K)\big\}$. We are going to show that the first term on the right side of (10.14) is large, by construction, while the sum is comparitively small if N_0 is large enough.

Because x, $y \in 2Q_{J,K}$ we have $\theta_{J,K}(x) = \theta_{J,K}(y) = 1$, and so

$$\begin{aligned} \big|f_{J,K}(x) - f_{J,K}(y)\big| &= \big|h_{J,K}(x) - h_{J,K}(y)\big| \\ &\geq M_{J,K}|x-y| - \big|h_{J,K}(x) - M_{J,K}(x - q_{J,K})\big| \\ &\quad - \big|h_{J,K}(y) - M_{J,K}(y - q_{J,K})\big| \\ &\geq M_{J,K}\big\{|x-y| - (10n)^{-10n}\big(d_{J+1}(x) + d_{J+1}(y)\big)\big\} \\ &\geq \tfrac{1}{2}M_{J,K}|x-y|, \end{aligned} \tag{10.15}$$

by (8.15). For the last inequality we used (10.12) and the analogue of (10.13) in this situation. To control this from below by $\delta(x,y)$ we use

$$M_{J,K} = \Big(\fint_{Q_{J,K}} \omega\Big)^{1/n} \geq C^{-1}L^{-1}\Big(\fint_{20B_{x,y}} \omega\Big)^{1/n}. \tag{10.16}$$

This follows from the doubling condition on μ when $20B_{x,y}$ is not contained in $100Q_{J,K}$ (so that $B_{x,y}$ is not too small compared to $Q_{J,K}$), and from Lemma 7.14 and Fact (g) in Section 2 when $20B_{x,y} \subseteq 100Q_{J,K}$. (To apply Lemma 7.14 we need to know that $20B_{x,y}$ intersects $\mathbf{R}^n \setminus \Omega_{J+1}$, which is true, by (10.12).) Combining (10.15) with (10.16) gives

$$\big|f_{J,K}(x) - f_{J,K}(y)\big| \geq C^{-1}L^{-1}\delta(x,y). \tag{10.17}$$

Let E denote the sum on the right side of (10.14), which we want to be small compared to the right side of (10.17). Observe that

$$E = \sum_{(j,k)\in W} \big|f_{j,k}(x) - f_{j,k}(y)\big|, \tag{10.18}$$

where $W = \big\{(j,k) : a_{j,k} = i \text{ and } j \neq J\big\}$. Indeed, $W \subseteq Z$, and $(j,k) \in Z \setminus W$ only when $j = J$, $a_{j,k} = i$, but $k \neq K$. Under these circumstances Lemma 9.1 implies that $Q_{j,k}$ is far enough way from $Q_{J,K}$ that $f_{j,k}(x) - f_{j,k}(y) = 0$, since x, $y \in 2Q_{J,K}$. Hence such a (j,k) does not contribute to E, and we have (10.18).

Using Lemma 9.1 again we have

$$E \leq E_+ + E_-,$$

where

$$E_+ = \sum_{j \geq J+N_0} \sum_k \big|f_{j,k}(x) - f_{j,k}(y)\big|$$

and

$$E_- = \sum_{j_0 < j \leq J-N_0} \sum_k \big|f_{j,k}(x) - f_{j,k}(y)\big|.$$

Applying (10.7) and (10.6) we get

$$E_+ \leq C\big(\delta_{J+N_0}(x) + \delta_{J+N_0}(y)\big)$$

and

$$E_- \leq C L^{(J-N_0+1)\varepsilon+1}\tilde{\delta}(x,y).$$

We can use Lemma 7.6 to obtain

$$E_+ \leq C(CL^{-\eta})^{N_0-1}\big(\delta_{J+1}(x) + \delta_{J+1}(y)\big) \leq C(CL^{-\eta})^{N_0-1}\delta(x,y).$$

The second inequality is a consequence of the failure of (10.9) when $j = J+1$. On the other hand,

$$E_- \leq CL^{1-(N_0-1)\varepsilon}\delta(x,y),$$

because of Lemma 7.12(a) and the fact that $B_{x,y} \subseteq \Omega_J$ (since (10.9) holds with $j = J$). Altogether we have

$$E \leq C\big[(CL^{-\eta})^{N_0-1} + L^{1-(N_0-1)\varepsilon}\big]\delta(x,y). \tag{10.19}$$

Thus if N_0 satisfies

$$(N_0-1)\eta, (N_0-1)\varepsilon - 1 > 1, \tag{10.20}$$

then we can choose L so large that

$$E \leq \tfrac{1}{2}\big|f_{J,K}(x) - f_{J,K}(y)\big|,$$

because of (10.17). Combining this with (10.17) and (10.14) we get (10.11) in this case (where $j(x,y) > j_0$, $y \in 2Q_{J,K}$).

It remains to take care of (10.11) when $j(x,y) > j_0$ and $y \in \mathbf{R}^n \setminus 2Q_{J,K}$. In this case we proceed using g_i instead of f_i, but the arguments are otherwise essentially identical.

We have

$$\begin{aligned} \big|f(x) - f(y)\big| &\geq \big|g_i(x) - g_i(y)\big| \\ &\geq \big|g_{J,K}(x) - g_{J,K}(y)\big| - \sum_{(j,k)\in Z}\big|g_{j,k}(x) - g_{j,k}(y)\big|, \end{aligned} \tag{10.21}$$

where Z is the same as before. Because $x \in \overline{Q_{J,K}}$ but $y \notin 2Q_{J,K}$ we have $g_{J,K}(x) \geq C^{-1}\delta_J(x)$ and $g_{J,K}(y) = 0$, by (9.10) and (9.6). We also have $\delta(x,y) \leq C\delta_J(x)$ because (10.9) holds with $j = J$. Thus

$$\big|g_{J,K}(x) - g_{J,K}(y)\big| \geq C^{-1}\delta(x,y). \tag{10.22}$$

Let E' denote the sum on the right side of (10.21). Essentially the same arguments as before imply that E' is dominated by the right side of (10.19). (The proof of the analogue of (10.18) is slightly different this time, because we do not have $y \in 2Q_{J,K}$ now. However, the fact that (10.9) holds with $j = J$ provides an adequate substitute.) Hence

$$E' \leq \tfrac{1}{2}\big|g_{J,K}(x) - g_{J,K}(y)\big|$$

if N_0 satisfies (10.20) and L is large enough. Combining this with (10.22) and (10.21) gives (10.11) in this case.

This completes the proof of Lemma 10.10, and also of Theorem 5.2.

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