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MEROMORPHIC FUNCTIONS WHOSE JULIA SETS CONTAIN A FREE JORDAN ARC

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Abstract. In the main result of this paper we show that if the Julia set of a meromorphic function f contains a free analytic Jordan arc then it must in fact be a straight line, circle, segment of a straight line or an arc of a circle. If f is transcendental then the Julia set is unbounded and so consists of one or two straight line segments. We construct examples of functions whose Julia sets are of this form.

1. Introduction

Let a *standard* meromorphic function f be a meromorphic function which is not rational of degree less than two and denote by f^n, $n \in \mathbf{N}$, the nth iterate of f. We define the set of normality, $N(f)$, to be the set of points z in $\widehat{\mathbf{C}}$ such that $(f^n)_{n\in\mathbf{N}}$ is well-defined, meromorphic and forms a normal family in some neighbourhood of z. The complement $J(f)$ of $N(f)$ is called the Julia set of f. It is clear that $N(f)$ is open and completely invariant under f.

We define $E(f)$ to be the set of exceptional values of f, that is, the points whose inverse orbit

$$O^-(z) = \bigl\{w : f^n(w) = z \text{ for some } n \in \mathbf{N}\bigr\}$$

is finite. The set $E(f)$ contains at most two points. Standard meromorphic functions can be divided into the following four classes:

I rational functions of degree at least two,
II transcendental entire functions,
III transcendental meromorphic functions with $\infty \in E(f)$ and with one pole,
IV transcendental meromorphic functions with $\infty \notin E(f)$ and with at least one pole.

The iteration of functions in classes I and II was studied in detail by Fatou [11, 12] and by Julia [14]. If f is a function in class III then we may assume without loss of generality that it has a pole at the point 0 and it then follows that f must be an analytic map of the punctured plane $\widehat{\mathbf{C}} \setminus \{0\}$ onto itself. The iteration of such maps was first studied by Rådström [20]. Various authors have produced

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work following on from this [3], [8], [15], [16] and [18]. In a recent series of papers [4, 5, 6] Baker, Kotus and Lü Yinian have studied the iteration of functions in class IV.

A Jordan arc γ in $\widehat{\mathbf{C}}$ is defined to be the image of the real interval $[0,1]$ under a homeomorphism φ. In the following work γ is said to be an *analytic* arc if the homeomorphism φ has a meromorphic extension in a neighbourhood of $[0,1]$. If, in addition, the extended φ is univalent at each point of $[0,1]$ then γ is said to be a *regular* arc. If the interval $[0,1]$ is replaced by the unit circle in the above definitions then γ is said to be a Jordan *curve*. Finally, if f is a standard meromorphic function, α is said to be a *free* Jordan arc in $J(f)$ if there exists a homeomorphism ψ of the open unit disc onto a domain D in $\widehat{\mathbf{C}}$ such that $J(f)\cap D$ is the image of $(-1,1)$ under ψ and α is the image of some real interval $[a,b]$ where $-1<a<b<1$.

The first theorem which we prove is as follows.

Theorem A. *If f is a standard meromorphic function and $J(f)$ contains a free Jordan arc α, then $J(f)$ is a Jordan arc or a Jordan curve. Further, if α is analytic, then $J(f)$ is also analytic.*

Using Theorem A we are able to prove the main result of this paper.

Theorem B. *If f is a standard meromorphic function and $J(f)$ contains a free analytic Jordan arc, then $J(f)$ is a straight line, circle, segment of a straight line or an arc of a circle.*

Töpfer [21, p. 69] showed that if f is an entire transcendental function then it is impossible for $J(f)$ to contain a free Jordan arc. We are able to extend his result to cover functions in class III and so Theorems A and B follow trivially for functions in class II or III.

Theorem B was shown to be true for rational functions by Fatou [11, p. 225]. It is well known that if $f_1(z)=z^2$ then $J(f_1)=\{z:|z|=1\}$. It is also known that if $f_2(z)=z^2-2$ then $J(f_2)=[-2,2]$ (see, for example, [9, Theorem 12.1]). By conjugating f_1 and f_2 with Möbius transformations it is possible to obtain examples of rational functions which show that each possible type of behaviour given in Theorem B can in fact occur.

If f is in class IV then $J(f)$ must be unbounded and so cannot be a circle or an arc of a circle. We prove the following result which shows that the other possible types of behaviour given in Theorem B do in fact occur for functions in class IV.

Theorem C. *There exist transcendental meromorphic functions g_0, g_1 and g_2 such that*

(i) $J(g_0)=\mathbf{R}\cup\{\infty\}$,
(ii) $J(g_1)=[0,\infty]$,
(iii) $J(g_2)=[-\infty,-1]\cup[1,\infty]$.

Finally, we prove the following.

Theorem D. *Suppose that f is a standard meromorphic function and that $J(f)$ is a Jordan arc or a Jordan curve. Then, if $J(f)$ is not a straight line, circle, segment of a straight line or an arc of a circle, $J(f)$ contains no differentiable arc.*

Fatou [11, p. 229] showed that this is true for rational functions.

2. Background material

In this section we give some results that will be needed to prove Theorems A, B, C and D. These are results which have either been proved elsewhere or which are easy extensions of existing results. The first result concerns normal families.

Lemma 2.1 (see, for example, [13, Theorem 6.3]). *A family $\{f_\alpha\}$ of meromorphic functions in the domain D is normal if and only if the 'spherical derivatives'*

$$\left\{\frac{\left|(f_\alpha)'(z)\right|}{1+\left|f_\alpha(z)\right|^2}\right\}$$

form a locally bounded family in D.

The second result concerns the set of normality. It was proved for rational functions by Fatou (see, for example, [9, Theorem 5.2]) and the same method of proof can be used to show that the result holds for all meromorphic functions.

Lemma 2.2 [4, Lemma 5]. *Suppose that f is a function in class IV and that in some component N_0 of $N(f)$ some subsequence of f^n has a non-constant limit function. Then there exists $p \geq 0$ such that $f^p(N_0)$ belongs to a component N_1 of $N(f)$, and $q > 0$ such that f^q maps N_1 univalently onto N_1.*

The next three results concern the Julia set. The first of these was proved for functions in class I by Fatou (see, for example, [9, Lemma 2.2]) and this proof can also be used for functions in class II. The proof for functions in class IV [4, Lemma 1] can also be applied to functions in class III.

Lemma 2.3. *If f is a standard meromorphic function and $\alpha \in \widehat{\mathbf{C}} \setminus E(f)$ then $J(f) \subset O^-(\alpha)'$ and so, if $\alpha \in J(f)$, $J(f) = O^-(\alpha)'$. In particular, if f is in class IV, then $J(f) = O^-(\infty)'$.*

The remaining results about Julia sets are concerned with the expanding properties of the iterates f^n in a neighbourhood of a point in the Julia set of f.

Lemma 2.4. *If f is a standard meromorphic function then the repelling periodic points of f are dense in $J(f)$.*

For details of the proof of the above result see [11, p. 45] for rational functions, [2, Theorem 1] for functions in class II, [8, Theorem 5.2] for functions in class III and [4, Theorem 1] for functions in class IV.

Lemma 2.5. *Let f be a standard meromorphic function. If L is compact, $L \cap E(f) = \emptyset$, $z \in J(f)$ and V is an open neighbourhood of z then there exists $M \in \mathbf{N}$ such that for all $n \geq M$ we have $f^n(V) \supset L$.*

Proof. This result was proved by Fatou [11] and [12] for functions in class I or class II.

If f is a function in class IV, $z \in J(f)$ and V is an open neighbourhood of z, then it follows from Lemma 2.3 that $O^-(\infty) \cap V \neq \emptyset$. We take the smallest value of $r \in \mathbf{N}$ such that f^r has a pole in V. It is clear that there are only finitely many poles, say $p_1, \dots, p_m$, of f^r in V and that f^{r+1} is a meromorphic function in $V \setminus \{p_1, \dots, p_m\}$ with an essential singularity at each of the points $p_1, \dots, p_m$. It follows from Picard's theorem that

$$f^{r+1}(V) \supset \mathbf{C} \setminus E(f^{r+1}) = \mathbf{C} \setminus E(f)$$

and that $f\bigl(\mathbf{C} \setminus E(f)\bigr) \supset \mathbf{C} \setminus E(f)$. Thus, if $L \cap E(f) = \emptyset$, we have $f^n(V) \supset \bigl(\mathbf{C} \setminus E(f)\bigr) \supset L$ for all $n \geq r+1$.

If f is a function in class III, then $E(f) = \{\alpha, \infty\}$, where α is the pole of f. If $z = \alpha$, then α is an essential singularity of f^2 and the proof follows as for functions in class IV. If $z \in J(f) \setminus \{\alpha\}$ then we can use the same method of proof as used for functions in class I or II.

In the study of iteration we are frequently interested in showing that the distortion $L(f, A)$ of a function f holomorphic in A, defined by

$$L(f, A) = \sup_{z_1, z_2 \in A} \bigl|f'(z_1)/f'(z_2)\bigr|,$$

is bounded in A. The following result follows easily from [10, Theorem 2.5].

Lemma 2.6. *Let $D(w, r)$ denote a disc of radius r, centre w. Then, for $0 < s < r$, there is a constant $K(s/r) = \bigl\{(r+s)/(r-s)\bigr\}^4$ such that, for every univalent map $g\colon D(w, r) \to \mathbf{C}$, the distortion of g in $D(w, s)$ is bounded by $K(s/r)$.*

The next few results are concerned with Ahlfors' five-islands theorem. The form of this theorem given below can easily be deduced from that given by Tsuji in [22, Theorem VI.13 using p. 252 for the definition of an island].

Lemma 2.7. *Suppose that* i) *f is meromorphic in $D(z, R)$ and* ii) *E_i, $1 \leq i \leq 5$, are simply connected domains in $\mathbf{C}$ bounded by sectionally-analytic Jordan curves such that the $\overline{E}_i$ are disjoint. Then there exists a constant $C = C(E_1, \dots, E_5)$ independent of f such that if*

$$\frac{R\bigl|f'(z)\bigr|}{1 + \bigl|f(z)\bigr|^2} > C$$

then some subdomain of $D(z, R)$ is mapped by f univalently onto one of the E_i.

An open Riemann surface R is said to be *regularly exhaustible* if R can be exhausted by a sequence of compact Riemann surfaces $R_0 \subset R_1 \subset \cdots \to R$ where $\overline{R}_n \subset R_{n+1}$ and the boundary Γ_n of R_n consists of a finite number of analytic Jordan curves such that

$$\lim_{n\to\infty} L_n/|R_n| = 0,$$

where L_n is the spherical length of Γ_n and $|R_n|$ is the spherical area of R_n.

An alternative form of Ahlfors' five-islands theorem to Lemma 2.7 is that given below (see, for example, [22, Theorem VI.8]).

Lemma 2.8. *Let R be a simply connected open Riemann surface spread over the complex sphere K and let E_i, $1 \le i \le 5$, be simply connected domains in K bounded by sectionally-analytic Jordan curves such that the $\overline{E}_i$ are disjoint. Then if the spherical area of R, $|R|$, is equal to ∞ and R is regularly exhaustible there are infinitely many domains in R that are mapped univalently by f onto one of the E_i.*

We prove the following result.

Lemma 2.9. *If, for some $r_0 > 0$, f is a meromorphic function in $r_0 \le |z| < \infty$ with an essential singularity at ∞ then the Riemann surface R generated by f is regularly exhaustible.*

Proof. Put

$$L(r) = \int_{|z|=r} \frac{|f'(z)|}{1+|f(z)|^2}\,|dz| = \int_0^{2\pi} \frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2} r\,d\theta$$

and

$$A(r) = \int_{r_0\le|z|\le r} \frac{|f'(z)|^2}{\{1+|f(z)|^2\}^2}\,|dz|^2 = \int_{r_0}^{r}\int_0^{2\pi} \frac{|f'(\varrho e^{i\theta})|^2}{\{1+|f(\varrho e^{i\theta})|^2\}^2}\varrho\,d\theta\,d\varrho.$$

Thus $A(r)$ is the spherical area of $R(r) = f\big(\{z : r_0 \le |z| \le r\}\big)$ and $L(r)+L(r_0)$ is the spherical length of the boundary of $R(r)$. We note that $A(r) \to \infty$ as $r \to \infty$ since by Picard's theorem $f(z)$ takes all but at most two values infinitely often in $r_0 \le |z| < \infty$.

It follows from the Cauchy–Schwarz inequality that

$$L^2(r) \le \int_0^{2\pi} r\,d\theta \int_0^{2\pi} \frac{|f'(re^{i\theta})|^2}{\{1+|f(re^{i\theta})|^2\}^2} r\,d\theta = 2\pi r\frac{dA(r)}{dr}.$$

If, for some constant h and some $r' > r_0$, we have $A(r) \le hL(r)$ for $r \ge r'$ then it follows from the above inequality that

$$A^2(r) \le h^2L^2(r) \le h^2 2\pi r\frac{dA(r)}{dr}$$

for $r \geq r'$, and so

$$\int_{r'}^{\infty} \frac{dr}{r} \leq \int_{r'}^{\infty} \frac{2\pi h^2}{A^2(r)}\, dA(r) = \frac{2\pi h^2}{A(r')}.$$

As the left hand side of the above expression is unbounded it follows that $A(r') = 0$ and hence $f' \equiv 0$. This, however, is a contradiction and so there must be an infinite sequence $r_n \to \infty$ such that $r_0 < r_1 < \cdots < r_n < \cdots$ and $L(r_n)/A(r_n) \to 0$ as $n \to \infty$. We know that $A(r) \to \infty$ as $r \to \infty$ and so

$$\big[L(r_n) + L(r_0)\big]/A(r_n) \to 0$$

as $n \to \infty$. Thus R is indeed regularly exhaustible.

Using the two preceding results one obtains the following form of the five-islands theorem which we will use in the proof of Theorem B.

Lemma 2.10. *If, for some $r_0 > 0$, f is a meromorphic function in $r_0 \leq |z| < \infty$ with an essential singularity at ∞, and E_i, $1 \leq i \leq 5$, are simply connected domains bounded by sectionally-analytic Jordan curves such that the $\overline{E}_i$ are disjoint then there are infinitely many domains in $r_0 \leq |z| < \infty$ that are mapped univalently by f onto one of the E_i.*

The final result of this section is a generalisation of Gross's star theorem (see, for example, [19, p. 287]) which we need for the proof of Theorem B.

Lemma 2.11. *If R is a branch, analytic at z_0, of the inverse of a function g that is meromorphic in $\mathbf{C}$ or in $\mathbf{C} \setminus \{0\}$ then R can be continued analytically along almost every ray from z_0 to ∞.*

Proof. Gross's star theorem covers the case when g is meromorphic in $\mathbf{C}$ and so we may assume that g is meromorphic only in $\mathbf{C} \setminus \{0\}$. We consider the function $G = gf$ where $f(z) = e^z$ and note that G is meromorphic in $\mathbf{C}$. If we take a branch g_0 of g^{-1} that is analytic at z_0 then, as $g^{-1}(z_0) \neq 0$, there exists a branch f_0 of f^{-1} that is analytic at $g^{-1}(z_0)$ and hence a branch G_0 of G^{-1} that is analytic at z_0. We know that G_0 can be continued analytically along almost every ray from z_0 to ∞. If G_1 is an analytic continuation of G_0 along one such ray then it is clear that fG_1 is an analytic continuation of g_0 along the same ray. This completes the proof.

3. Proof of Theorem A

Töpfer [21, p. 69] showed that the Julia set of a transcendental entire function cannot contain a free Jordan arc. By using his method of proof we obtain the more general result given below.

Lemma 3.1. *If f is a function in class II or III then $J(f)$ cannot contain a free Jordan arc.*

Proof. Suppose that f is such a function and that $J(f)$ contains a free Jordan arc α. As $E(f)$ contains at most two points we may assume without loss of generality that $\alpha \cap E(f) = \emptyset$. It follows from Lemma 2.4 that we can choose a point $z_0 \in \alpha$ such that $f^m(z_0) = z_0$ for some $m \in \mathbf{N}$. We now take three distinct points $z_i \in \alpha$, $1 \le i \le 3$, with neighbourhoods N_i such that the N_i are mutually disjoint and such that $J(f) \cap N_i \subset \alpha$, $1 \le i \le 3$. It then follows from Lemma 2.5 that for some $n \in \mathbf{N}$ there exist three distinct points $w_i \in N_i$ such that $f^n(w_i) = z_0$ and hence $w_i \in \alpha$.

We now take the subarc γ of α which passes through all three points w_i and has two of these points as its endpoints. As $\gamma \cap E(f) = \emptyset$ and $\infty \in E(f)$, $f^n(\gamma) = \Gamma$ is clearly compact and we claim that it is in fact a component of $J(f)$. For, if $z \in \Gamma$, there exists $w \in \gamma$ such that $f^n(w) = z$ and a neighbourhood N of w such that $J(f) \cap N \subset \gamma$. It then follows that $f^n(N)$ is a neighbourhood of z and $J(f) \cap f^n(N) \subset \Gamma$. It is also clear that Γ is connected and contained in $J(f)$. Thus Γ is indeed a bounded component of $J(f)$.

As $z_0 \in \Gamma$ and $f^m(z_0) = z_0$, it now follows that $f^m(\Gamma) \subset \Gamma$ and hence $f^{pm}(\Gamma) \subset \Gamma$ for all $p \in \mathbf{N}$. As f is transcendental, $J(f)$ must be unbounded and so there exists a point $z_4 \in J(f)$ and a compact set L such that $z_4 \in L$ and $L \cap \big(E(f) \cup \Gamma\big) = \emptyset$. We now take a neighbourhood N_0 of z_0 such that $N_0 \cap J(f) \subset \Gamma$. It follows from Lemma 2.5 that there exists $M \in \mathbf{N}$ such that $f^N(N_0) \supset L$ for all $N \ge M$. This, however, is a contradiction as for each $M \in \mathbf{N}$ there exists $p \in \mathbf{N}$ such that $pm > M$ and $J(f) \cap f^{pm}(N_0) \subset \Gamma$. This completes the proof.

It follows that Theorem A is true in a trivial sense for functions in class II or III and so we need only consider functions which belong to class I or class IV. We begin by proving the following two results. We define

$$\alpha_n(f) = \{z \in \alpha : f^n \text{ is meromorphic in a neighbourhood of } z\}.$$

Lemma 3.2. *If f is a standard meromorphic function and $J(f)$ contains a free Jordan arc α then each point $z \in f^n\big(\alpha_n(f)\big) \subset J(f)$ has an open neighbourhood N such that either*

(i) *$J(f) \cap \overline{N}$ is a Jordan arc with no end points in N, or*
(ii) *$J(f) \cap \overline{N}$ is a Jordan arc with an endpoint at z.*

Proof. If α is a free Jordan arc of $J(f)$ then it follows from the definitions given in the introduction that there exists a homeomorphism φ of the open disc $D(0,1)$ with $J(f) \cap \varphi\big(D(0,1)\big) = \varphi\big((-1,1)\big)$ such that $\alpha = \varphi\big([a,b]\big)$ for some $-1 < a < b < 1$. We take a value $n \in \mathbf{N}$ and put $\psi = f^n\varphi$. If $z \in f^n\big(\alpha_n(f)\big)$ then it is clear that there exists a point $w \in \alpha$ and $t_0 \in [a,b]$ such that $z = f^n(w) = \psi(t_0)$. If f^n is univalent at w and hence in a neighbourhood of w it is clear that there exists $\delta > 0$ such that ψ is a homeomorphism of $D(t_0,\delta) \subset D(0,1)$ onto an open neighbourhood N of z of form (i).

We now consider the case where f^n has valency $k > 1$ at w. As such points are isolated it is clear that, for some $\varepsilon_1 > 0$, ψ is univalent at each point of $[t_0 - \varepsilon_1, t_0) \subset (-1, 1)$ and $\{\psi(t_0 - \varepsilon_1)\} \cap \psi((t_0 - \varepsilon_1, t_0]) = \emptyset$. We claim that in fact $\psi([t_0 - \varepsilon_1, t_0])$ is a Jordan arc.

For let

$$\tau = \sup\{t : t_0 - \varepsilon_1 \le t < t_0, \psi([t_0 - \varepsilon_1, t]) \text{ is a Jordan arc }\}.$$

Suppose that $\psi([t_0 - \varepsilon_1, \tau])$ is not a Jordan arc. Then there exists $\tau' \in [t_0 - \varepsilon_1, \tau)$ such that $\psi(\tau) = \psi(\tau')$. If we take $\tau' < \tau'' < \tau$ then we see that $\psi([t_0 - \varepsilon_1, \tau'])$, $\psi([\tau', \tau''])$, $\psi([\tau'', \tau])$ are three distinct arcs in $J(f)$ which meet at the point $\psi(\tau)$. It follows from the complete invariance of $J(f)$ under f^n that at least three arcs of $J(f)$ meet at $\varphi(\tau) \in \alpha$ which is clearly a contradiction. Thus $\psi([t_0 - \varepsilon_1, \tau])$ is a Jordan arc.

We now suppose that $\tau < t_0$ so that ψ is a homeomorphism on $[\tau - \mu, \tau + \mu]$ for some $\mu > 0$. As $\psi([t_0 - \varepsilon_1, \tau])$ is a Jordan arc we know that

$$\psi([t_0 - \varepsilon_1, \tau - \mu]) \cap \psi((\tau - \mu, \tau]) = \emptyset \tag{3.1}$$

and also that

$$\psi([\tau - \mu', \tau + \mu']) \cap \psi([t_0 - \varepsilon_1, \tau - \mu]) = \emptyset, \tag{3.2}$$

for some $0 < \mu' < \mu$. (3.1) and (3.2) together imply that $\psi([t_0 - \varepsilon_1, \tau + \mu'])$ is a Jordan arc which contradicts the definition of τ. Thus $\tau = t_0$ and so $\gamma_1 = \psi([t_0 - \varepsilon_1, t_0])$ is indeed a Jordan arc. A similar argument can be used to show that, for some $\varepsilon_2 > 0$, $[t_0, t_0 + \varepsilon_2] \subset (-1, 1)$ and $\gamma_2 = \psi([t_0, t_0 + \varepsilon_2])$ is a Jordan arc.

We now show that there exists $\delta \in (0, \varepsilon_2]$ such that

$$\psi([t_0, t_0 + \delta]) \subset \gamma_1. \tag{3.3}$$

We begin by supposing that there exists $\varepsilon' \in (0, \varepsilon_2]$ such that $\psi([t_0, t_0 + \varepsilon']) \cap \gamma_1 = \emptyset$. If this is the case then there are at least two distinct Jordan arcs in $J(f)$ which end at $\psi(t_0) = z$. As f^n has valency $k > 1$ at $w = \varphi(t_0)$, the complete invariance of $J(f)$ under f^n implies that there are at least $2k$ Jordan arcs of $J(f)$ which meet at $w \in \alpha$. This is clearly a contradiction and so for every $\varepsilon' \in (0, \varepsilon_2]$ we have $\psi([t_0, t_0 + \varepsilon']) \cap \gamma_1 \ne \emptyset$.

We now choose $\delta \in (0, \varepsilon_2]$ so small that for $t \in [t_0, t_0 + \delta]$ we have

$$|\psi(t) - z| < \tfrac{1}{2}|\psi(t_0 - \varepsilon_1) - z| \tag{3.4}$$

and claim that this δ satisfies (3.3). If this is not the case then there exists $\delta' \in (0,\delta]$ such that $\psi(t_0+\delta') \notin \gamma_1$. We know from the previous paragraph that

$$A = \{t : t \in (t_0, t_0+\delta'), \psi(t) \in \gamma_1\} \neq \emptyset$$

and so $\tau_1 = \sup A$ is well-defined. Clearly $\tau_1 \in (t_0, t_0+\delta')$, $\psi(\tau_1) \in \gamma_1$ and hence, from (3.4), there exists $T \in (t_0-\varepsilon_1, t_0)$ such that $\psi(T) = \psi(\tau_1)$. If we take $T' \in (t_0-\varepsilon_1, T)$ the arcs $\psi([T',T])$, $\psi([T,t_0])$ and $\psi([\tau_1, t_0+\delta'])$ belong to $J(f)$ and are disjoint except for their common endpoint $\psi(T) = \psi(\tau_1)$. The complete invariance of $J(f)$ under f^n implies that there must be at least three disjoint arcs of $J(f)$ which meet at $\varphi(T) \in \alpha$. This is clearly a contradiction and so δ does indeed satisfy (3.3).

If we take a domain $U \subset D(0,1)$ such that $U \cap (-1,1) = (t_0-\varepsilon_1, t_0+\delta)$ then it is clear that $\psi(U)$ is an open neighbourhood N of z of form (ii).

Lemma 3.3. *If f is a standard meromorphic function and $J(f)$ contains a free Jordan arc α then if, for some $n \in \mathbf{N}$, $\alpha_n(f) = \alpha$ it follows that $f^n(\alpha)$ must be either a Jordan arc or a Jordan curve.*

Proof. It follows from Lemma 3.2 that either

(i) $f^n(\alpha)$ has two endpoints b and b' and $f^n(\alpha) \setminus \{b,b'\}$ is a non-compact, connected one-dimensional manifold or

(ii) $f^n(\alpha)$ has no endpoints and is a compact, connected one-dimensional manifold.

In case (i) $f^n(\alpha) \setminus \{b,b'\}$ must be homeomorphic to an open interval and hence $f^n(\alpha)$ is a Jordan arc. In case (ii) $f^n(\alpha)$ must be homeomorphic to a circle and is hence a Jordan curve. (See, for example, [7, Theorem 3.4.1]).

Proof of Theorem A. If f is a rational function then $E(f) \subset N(f)$ (see, for example, [9, Lemma 2.2]) and so if $J(f)$ contains a free Jordan arc α then it follows from Lemma 2.5 that $J(f) = f^N(\alpha)$ for some $N \in \mathbf{N}$. As $\alpha_N(f) = \alpha$ it then follows from Lemma 3.3 that $J(f)$ must be either a Jordan arc or a Jordan curve.

Now suppose that f is a function in class IV and that $J(f)$ contains a free Jordan arc α. If we denote by A the set whose members are the endpoints of α then it follows from Lemma 2.3 that there exist arbitrarily large values of $n \in \mathbf{N}$ such that f^n has a pole in $\alpha \setminus A$. We denote by k the minimum such value of n and take an arc γ contained in α such that γ contains precisely one pole p of f^k and such that the endpoints of γ are not poles of f^k. There exists a simply connected domain D in $\mathbf{C}$ such that $J(f) \cap \overline{D} = \gamma$ and hence a closed neighbourhood $U = f^k(\overline{D})$ of ∞ such that $J(f) \cap U = f^k(\gamma)$. It then follows from Picard's theorem that there are at most two points in $J(f)$ not having pre-images in U, that is in $f^k(\gamma) \setminus \{\infty\}$. It easily follows from Lemma 2.3 that $J(f)$ is perfect and so $J(f) = \overline{f(f^k(\gamma) \setminus \{\infty\})}$.

We now show that $J(f)$ is in fact equal to $f\big(f^k(\gamma)\setminus\{\infty\}\big)$. It follows from Lemma 3.3 that $f^k(\gamma)$ is a Jordan arc or curve containing ∞. If ∞ is an endpoint of $f^k(\gamma)$ or if $f^k(\gamma)$ is a Jordan curve then $f\big(f^k(\gamma)\setminus\{\infty\}\big)=C$ is clearly connected and so $J(f)$ consists of one component $\overline{C}$. Otherwise, $f^k(\gamma)\setminus\{\infty\}$ is the union of two connected sets L and R, neither of which can reduce to a single point, and $J(f)=\overline{f(L)}\cup\overline{f(R)}$. It is now clear that $J(f)$ has at most two components, neither of which can reduce to a single point.

If J_1 is a component of $J(f)$ then, as p is an essential singularity of f^{k+1}, it follows from Picard's theorem that for some $z_1\in J_1$ there exist three points w_i, $1\le i\le 3$, such that as we go along γ we meet w_1, then w_2, followed by w_3 and finally p, and such that, in the subarc γ' of γ with endpoints w_1 and w_3, the solutions of $f^{k+1}(w)=z_1$ are w_1, w_2 and w_3.

We claim that $f^{k+1}(\gamma')$ is equal to J_1. For if $z\in f^{k+1}(\gamma')$ then there exists $w\in\gamma'$ such that $f^{k+1}(w)=z$ and a neighbourhood N of w such that $J(f)\cap N\subset\gamma'$. It follows that $f^{k+1}(N)$ is a neighbourhood of z and $J(f)\cap f^{k+1}(N)\subset f^{k+1}(\gamma')$. It is also clear that $f^{k+1}(\gamma')$ is closed, connected and contained in $J(f)$. It is thus a component of $J(f)$ and hence equal to J_1.

It also follows from Lemma 3.3 that $J_1=f^{k+1}(\gamma')$ must be either a Jordan arc or a Jordan curve. It is clear that f^{k+1} is meromorphic in a neighbourhood of γ' and so, if α and hence γ' is an analytic arc, then J_1 must be either an analytic Jordan arc or an analytic Jordan curve.

Finally we show that $J(f)$ is in fact connected. If not, then we have shown that $J(f)$ must have precisely two components J_1 and J_2 one of which, say J_1, must be bounded. It is clear that $\infty\in J_2$ and we may assume without loss of generality that $0\in J_1$.

It follows from Lemma 2.3 that there exist arbitrarily large values of $n\in\mathbf{N}$ such that f^n has a pole in J_1. We denote by m the minimum such value of n. We now take a simple closed curve Γ in $N(f)$ which separates J_1 and J_2 and has winding number 1 round 0. As f^m is meromorphic in $\widehat{\mathbf{C}}\setminus J_2$ we are able to apply the argument principle to f^m in the domain V bounded by Γ. Since $f^m(J_1)$ is a connected, unbounded subset of $J(f)$ we must have $f^m(J_1)\subset J_2$ and so f^m has no zeros in V. Thus the number of zeros of f^m in V minus the number of poles of f^m in V is strictly negative and so the winding number of $f^m(\Gamma)$ round 0 is also negative. $f^m(\Gamma)$ is clearly a closed curve in $N(f)$ and so it separates J_1 and J_2.

We now take a simple closed curve $\Gamma_1\subset f^m(\Gamma)$ which separates J_1 and J_2 and repeat the argument. We see that, for any $k\in\mathbf{N}$, $f^{mk}(\Gamma)$ contains a simple closed curve $\Gamma_k\subset N(f)$ which separates J_1 and J_2. Thus, denoting the spherical diameter of a set Ω by $d(\Omega)$, we see that

$$d\big(f^{mk}(\Gamma)\big)>\min\big(d(J_1),d(J_2)\big)$$

and so (f^{mk}) has some non-constant limit function in the component N_1 of $N(f)$ between J_1 and J_2. It is clear that N_1 is mapped into itself by f^m and so it follows from Lemma 2.2 that f^m maps N_1 univalently onto itself.

It is easy to see that $N(f)$ can have at most three components N_j, $1 \le j \le 3$, since each of J_1, J_2 is a Jordan arc or curve. We know that f^m takes all but at most two values in N_1 infinitely often and so, as f^m is univalent in N_1 it follows that $f^m(N_k) \subset N_1$ for some $k \neq 1$. It is then clear that there exists $j \neq 1$ such that, for each $w \in N_j$, $f^m(z) = w$ has not solutions. This is clearly a contradiction and so $J(f)$ must be connected.

We have thus shown that $J(f)$ is either a Jordan arc or a Jordan curve and, in the case where α is analytic, $J(f)$ is also analytic.

Note. Suppose that $J(f)$ is an analytic Jordan curve or arc and that $w \in J(f)$ is not an endpoint of $J(f)$. Then it follows from the proof of Lemma 3.2 that there exists an open neighbourhood N of w such that $J(f) \cap \overline{N}$ is a regular Jordan arc.

4. Proof of Theorem B

This theorem has already been shown to be true for functions in class I and so it follows from Lemma 3.1 that we need only consider functions in class IV. Having proved Theorem A, it is sufficient to show that if f is a function in class IV and $J(f)$ is an analytic Jordan curve or arc then $J(f)$ is in fact a straight line. We begin with the following result.

Lemma 4.1. *Suppose that f is a function in class IV and that $J(f)$ is a Jordan arc with precisely one finite endpoint, a. Put $P(z) = z^2 + a$. For some z_1 such that $fP(z_1) = \alpha \neq a, \infty$ take a fixed branch of $P^{-1}(w) = (w-a)^{1/2}$ at $w = \alpha$. Then $F = P^{-1} f P$ continues analytically to a function in class IV and $J(F)$ is a Jordan curve. If $J(f)$ is analytic then $J(F)$ is also analytic.*

Proof. In order to show that F is single-valued it is sufficient to show that fP is 2-valent at each point in the set

$$B = \{z : fP(z) = a \text{ or } \infty\}.$$

We first consider the point 0 and note that it is in B. It is clear that P is 2-valent at 0 and, as only one component of $J(f) \setminus \{a\}$ ends at $P(0) = a$, the complete invariance of $J(f)$ under f implies that $f(a) = a$ or ∞ and that f is univalent at $P(0)$.

If $z \in B \setminus \{0\}$ then P is clearly univalent at z. As $P(z) \notin \{a, \infty\}$, exactly two segments of $J(f)$ end at $P(z)$ and, as only one segment of $J(f)$ ends at $fP(z)$, we again use the complete invariance of $J(f)$ under f to deduce that f is 2-valent at $P(z)$. Thus fP is 2-valent at each point in B.

It follows that F is a transcendental meromorphic function with at least one pole and so belongs to class III or class IV. We know that $\infty \notin E(f)$ and this implies that $\infty \notin E(F)$ thus showing that F belongs to class IV.

We note from Lemma 2.3 that if g is a function in class IV then $J(g)$ is the closure of the set of poles of all g^n. Since $F^n = P^{-1} f^n P$ we see that $P(\beta)$ is a pole of f^n if and only if β is a pole of F^n. Hence $J(F) = P^{-1}(J(f))$, the complete inverse image of $J(f)$ under P, and the result follows.

Proof of Theorem B. Let f denote a function in class IV. In proving the theorem we have to consider three cases:

I) $J(f) = \Gamma$ is an analytic Jordan curve. Since Γ must pass through ∞, $N(f)$ has precisely two components, D^+ and D^-, both of which are simply connected. We have either

IA) $f(D^+) \subset D^+$ and $f(D^-) \subset D^-$, or

IB) $f(D^+) \subset D^-$ and $f(D^-) \subset D^+$.

II) Γ is an analytic Jordan arc and both the endpoints of Γ are finite. In this case $N(f)$ has one component D and this is also simply connected.

III) Γ is an analytic Jordan arc and has one end at ∞ and one finite endpoint.

Proof in cases IA and II. (1) We begin by introducing a conformal map h of the upper half plane H^+ which has a continuous extension to $\mathbf{R}$. In case IA, h is a map of H^+ onto D^+ with $h(\infty) = \infty$. We set $A = \emptyset$ and $B = \{\infty\}$. In case II, h is a map of H^+ onto D with $h(\infty) = h(0) = \infty$. We denote the unique pre-images of b and b', the endpoints of Γ, by a and a' respectively. We set $A = \{a, a'\}$ and $B = \{0, \infty\}$.

Figure 4.1.

If $z \in \mathbf{R} \setminus A$ then it follows from the note at the end of Section 3 that there exists a neighbourhood V of z, symmetric about $\mathbf{R}$, such that h can be continued to give a meromorphic function in V by use of the reflection principle.

We now take g to be the map defined by

$$g = h^{-1} f h$$

in H^+. We apply the reflection principle to g and deduce that g can be continued to a meromorphic map in the plane, except for essential singularities at the points in B, such that $g(H^+) \subset H^+$, $g(H^-) \subset H^-$ and $g(\mathbf{R} \setminus B) \subset \mathbf{R} \cup \{\infty\}$.

We note that g is univalent at each point $z \in \mathbf{R} \setminus B$. For if g has valency k at z then, as exactly two segments of $\mathbf{R}$ meet at $g(z)$, there must be $2k$ segments of $g^{-1}(\mathbf{R}) \subset \mathbf{R}$ which meet at z and hence k must be equal to 1. We also note that

$$h = f h g^{-1} \tag{4.1}$$

for all branches of g^{-1} on $H^+ \cup \mathbf{R}$.

(2) The main part of the proof is to obtain a meromorphic continuation of h to the whole of $\widehat{\mathbf{C}}$. The nature of the extended h will enable us to use the fact that $h(\mathbf{R}) = \Gamma$ to deduce that Γ must be a straight line. In this section we use (4.1) to obtain a meromorphic continuation of h to a neighbourhood of ∞.

We know that f has a pole $p' \in \Gamma$ and so g has a pole $p'' \in \mathbf{R}$ which satisfies $h(p'') = p'$. It is clear that p' cannot be an endpoint of Γ as this would force $f(p') = \infty$ to be an endpoint of Γ contrary to our initial assumption. We are therefore able to take a neighbourhood V' of $p'' \in \mathbf{R} \setminus (A \cup B)$ in which h is defined and such that g is a univalent map of V' onto a neighbourhood U of ∞. We are now able to use (4.1) to obtain a meromorphic continuation of h to U by taking the branch G_0 of g^{-1} that maps U univalently onto V' and putting $h = f h G_0$ in U.

We note that $h G_0(U \setminus \mathbf{R}) \subset N(f)$ and hence

$$h(U \setminus \mathbf{R}) = f h G_0(U \setminus \mathbf{R}) \subset N(f). \tag{4.2}$$

We denote by K the compact set $\mathbf{C} \setminus U$.

(3) We shall now show that all possible meromorphic continuations of h in the plane lead to a single-valued meromorphic function. First note that g^{-1} has no algebraic singularities on $\mathbf{R}$ as $g^{-1}(\mathbf{R}) \subset \mathbf{R} \setminus B$ and g is univalent at each point of $\mathbf{R} \setminus B$. Together with Picard's theorem this shows that there are at most two points on $\mathbf{R}$ at which there are only finitely many analytic branches of g^{-1}.

If all possible meromorphic continuations of h do not lead to a single-valued function then we can take a point $r \in \mathbf{R} \setminus A$ at which g^{-1} has infinitely many analytic branches and two polygonal paths γ_1, γ_2 from r to a point $q \in K$ together with a neighbourhood D_r of r and a neighbourhood D_q of q such that h is meromorphic in D_r and has a meromorphic continuation along γ_i to a function H_i that is meromorphic in D_q, $i = 1, 2$, where $H_1 \neq H_2$.

We know that g is meromorphic in $\mathbf{C} \setminus B$, i.e. in $\mathbf{C}$ or in $\mathbf{C} \setminus \{0\}$ and so it follows from Lemma 2.11 that we can modify the closed polygonal path $\gamma_1 \setminus \gamma_2$ by changing γ_1 to a polygonal path λ_1 from r to a point $q' \in D_q$ and $-\gamma_2$ to a polygonal path $-\lambda_2$ from q' to some $r' \in \mathbf{R}$ such that $[r, r'] \subset D_r$ and such that

(i) for $i = 1, 2$, h has a meromorphic continuation along λ_i to the function H_i at q' with $H_1(q') \neq H_2(q')$,

(ii) every branch of g^{-1} that is analytic at r continues analytically round $\Lambda = \lambda_1 \setminus \lambda_2$ to $r' \in \mathbf{R}$.

We now show that there is a branch G of g^{-1} that is analytic at r and which satisfies $G(\Lambda) \subset \mathbf{C} \setminus K$. We first note that at any point s of Λ at most finitely many segments $\sigma_1, \ldots, \sigma_k$ of Λ intersect and there exists a neighbourhood of s which meets Λ only in $\sigma_1, \ldots, \sigma_k$.

We consider the branches $G_1, G_2 \ldots$ of g^{-1} that are analytic at r and hence along Λ. Suppose that, for infinitely many i, $G_i(\Lambda) \cap K \neq \emptyset$. Then there exist $z_i \in \Lambda$ with $G_i(z_i) = w_i \to w_0 \in K$ and hence $z_i \to z_0 = g(w_0) \in \Lambda$. Now, as g has valency v at w_0 and there are only finitely many segments $\sigma_1, \ldots, \sigma_n$ of Λ which meet at z_0, it is clear that there are exactly vn segments $\omega_1, \ldots, \omega_{vn}$ of $g^{-1}(\Lambda)$ which meet at w_0. By taking a sufficiently small neighbourhood N_w of w_0 in which g has valency v, we can ensure that Λ meets $N_z = g(N_w)$ only in $\sigma_1, \ldots, \sigma_n$ and that $g^{-1}(\Lambda)$ meets N_w only in $\omega_1, \ldots, \omega_{vn}$ with w_0 being the only point in N_w at which two segments ω_i, ω_j meet for $1 \le i,\ j \le vn$. We then take a point z_k on one of the segments σ_m, $1 \le m \le n$, such that $G_k(z_k) = w_k$ is a point on one of the segments ω_s, $1 \le s \le vn$. Continuing G_k along σ_m to z_0 we have that $G_k(z_0) = w_0$ and so g must in fact be univalent in N_w as G_k is analytic on Λ and hence at z_0. Thus $v = 1$.

We also note that G_k must map σ_m onto ω_s. Similarly, if for some $j \neq k$ there exists a point $z_j \in \sigma_m$ with $G_j(z_j) = w_j \in \omega_s$, then we must have $G_j(\sigma_m) = \omega_s$. But g is univalent in N_w and hence on ω_s so $G_j \equiv G_k$ which is a contradiction. It follows that there are at most n different values of i with $z_i \in \sigma_m$ and $G_i(z_i) \in N_w$ and hence at most n^2 different values of i with $w_i \in N_w$. This is clearly a contradiction and so there is a branch G of g^{-1} that is analytic at r and maps Λ into $\mathbf{C} \setminus K = U$.

We now take this branch G of g^{-1} and put $H = fhG$ which is well-defined and meromorphic on the path Λ which joins r to r' as $G(\Lambda) \subset U$. It follows from (4.1) that H is also meromorphic on $[r, r']$ and that $H \equiv h$ on $[r, r']$. Thus H gives a meromorphic continuation of h along λ_1 to q' and also along λ_2 to q'. This contradicts (i) and so all possible meromorphic continuations of h lead to a single-valued function.

(4) We now show that h can be continued to give a meromorphic function in the whole of $\widehat{\mathbf{C}}$. Suppose that this is not the case and that there are five points $a_k \in K$, $1 \le k \le 5$, to which h cannot be continued to give a meromorphic function. We take five simply connected domains E_k bounded by sectionally-analytic Jordan curves with $a_k \in E_k$ such that the closures of the E_k are disjoint and such that $E_k \cap H^+ \ne \emptyset$. It is clear that there exists $r_0 > 0$ such that g is a meromorphic function in $\{z : r_0 \le |z| < \infty\} \subset U$ with an essential singularity at ∞ and so it follows from Lemma 2.10 that there is a subdomain U' of U that is mapped univalently by g onto one of the E_k, say E_5. We then take the branch $\hat{G}$ of g^{-1} that maps E_5 univalently onto U' and put $\hat{H} = fh\hat{G}$. $\hat{H}$ is well-defined and meromorphic in E_5 and, from (4.1), agrees with h in $E_5 \cap H^+$. It therefore gives a meromorphic continuation of h to E_5 and hence to a_5.

Thus h can be continued to the whole of $\widehat{\mathbf{C}}$ apart from at most four points $a_k \in K$, $1 \le k \le 4$, to give a single-valued meromorphic function. We denote by C the set whose members are the essential singularities of h and show that in fact $C = \emptyset$.

We first show that if $s \in H^- \setminus C$ then $h(s) \in N(f)$. For suppose that there exists a point $s \in H^- \setminus C$ with $h(s) \in J(f)$. If we take a neighbourhood N of s such that $N \subset H^- \setminus C$ then $h(N)$ will contain an arc of $J(f)$ and so we can take five distinct points $b_k \in N$, $1 \le k \le 5$, such that $h(b_k) \in J(f)$. We then take five simply connected domains B_k bounded by sectionally-analytic Jordan curves with $b_k \in B_k$ such that the closures of the B_k are disjoint and such that $B_k \cap H^+ \ne \emptyset$. By applying Lemma 2.10 as above we are able to show that for one of the domains B_k, say B_5, there exists a branch G^* of g^{-1} that maps B_5 univalently onto a domain $U'' \subset U$ and $h = fhG^*$ in B_5. It is clear that $G^*(b_5) \in H^- \cap U$ and so, from (4.2), $hG^*(b_5) \in N(f)$ and hence $h(b_5) = fhG^*(b_5) \in N(f)$ which is a contradiction. Thus $h(H^- \setminus C) \subset N(f)$ as claimed and so, by Picard's theorem, $C \cap H^- = \emptyset$.

We now know that C must be contained in A. If $w \in A$, we take a neighbourhood W of w such that h is analytic in $W \setminus \{w\}$. We know that $h(W \cap H^-) \subset N(f)$ and so, since $h(z)$ tends to an endpoint of Γ as $z \to w$ on any path in $H^+ \cup \mathbf{R}$, it follows from Picard's theorem that h is analytic at w and hence $C = \emptyset$.

(5) In case IA we have obtained an analytic continuation of h to the whole of $\widehat{\mathbf{C}}$ with $h(H^+) \subset D^+$ and $h(\mathbf{R}) \subset \Gamma$. We know that $h(H^-) \subset N(f)$ and, as h is continuous and $h(H^-) \cap D^- \ne \emptyset$, it follows that $h(H^-) \subset D^-$. As h is conformal in H^+ it follows that h is a conformal map of $\widehat{\mathbf{C}}$. As h has no poles it must in fact be linear and so $J(f) = \Gamma = h(\mathbf{R})$ is a straight line as claimed.

In case II we have obtained a meromophic continuation of h to the whole of $\widehat{\mathbf{C}}$ with $h(H^+) = h(H^-) = N(f) = D$ and $h(\mathbf{R}) = \Gamma = J(f)$. As h is conformal in H^+ it is easy to see that for each point $w \in \Gamma$ there are at most two real solutions of $h(z) = w$ and hence at most two solutions in $\mathbf{C}$ to $h(z) = w$. Thus, by Picard's theorem, h has a pole at ∞ and so must be a rational function of degree two. We know that h has precisely one finite pole and that this is at the point 0. We have

$$h(z) = [az^2 + bz + c]/z \tag{4.3}$$

for some constants a, b, c.

We know that h is a conformal map of H^+ onto D and so h is also univalent in H^-. Thus the solutions of $h'(z) = 0$ must be real. Differentiating (4.3) we see that $h'(z) = 0$ if and only if $az^2 - c = 0$ and so $\mu^2 = c/a \in [0, \infty)$. Rewriting (4.3) as

$$h(z) = a[z + \mu^2/z] + b$$

we see that $J(f)$ is the image of

$$[-\infty, -2\mu] \cup [2\mu, \infty]$$

under the map $w(t) = at + b$ and is therefore a straight line segment as claimed.

Proof in case IB. (1) We note that f^2 is meromorphic in $\mathbf{C}$ apart from essential singularities in Γ. Denoting by E the set whose members are the essential singularities of f^2 we see that $f^2(D^+) \subset D^+$, $f^2(D^-) \subset D^-$ and $f^2(\Gamma \setminus E) \subset \Gamma$.

As in case IA we take a conformal map h of the upper half plane H^+ such that $h(H^+) = D^+$ and $h(\infty) = \infty$. By use of the reflection principle we are able to obtain an analytic continuation of h to a neighbourhood N of $\mathbf{R}$ such that h is univalent in $H^+ \cup N$.

We now take G to be the map defined by

$$G = h^{-1} f^2 h$$

in H^+. By using the reflection principle we are able to obtain a continuation of G that is meromorphic in the plane apart from essential singularities at the points in $h^{-1}(E) \subset \mathbf{R}$ such that the continued function G satisfies $G(H^+) \subset H^+$, $G(H^-) \subset H^-$ and $G\big(\mathbf{R} \setminus h^{-1}(E)\big) \subset \mathbf{R} \cup \{\infty\}$. We note that

$$h = f^2 h G^{-1} \tag{4.4}$$

for all branches of G^{-1} on $H^+ \cup \mathbf{R}$ and that G is univalent on $\mathbf{R} \setminus h^{-1}(E)$.

(2) In this section we use (4.4) to obtain an analytic continuation of h to a neighbourhood of ∞. We know that f^2 has a pole $P' \in \Gamma$ and hence G has a pole $P'' \in \mathbf{R}$ which satisfies $h(P'') = P'$. we take a neighbourhood $V \subset N$ of P'' such that G is a univalent map of V onto a neighbourhood U of ∞ that is symmetric about $\mathbf{R}$. We now continue h analytically to U by taking the branch G_0 of G^{-1} that maps U univalently onto V and putting $h = f^2 h G_0$ in U.

From the invariance of $J(f)$ under f^2 it is clear that f^2 is univalent at P' and so by keeping V sufficiently small we are able to ensure that h is a univalent map of U onto $h(U) = W$. Letting $U^- = U \cap H^-$ and $U^+ = U \cap H^+$ we note that $h(U^-) \subset D^-$ and $h(U^+) \subset D^+$.

(3) We have shown that h can be continued analytically to a function that is univalent in $H^+ \cup \mathbf{R} \cup U \cup N$, where N is a neighbourhood of $\mathbf{R}$, and so h^{-1} is analytic in $D^+ \cup \Gamma \cup W \cup h(N)$. The main part of the proof is to show that h^{-1} can be continued analytically to the whole of $\mathbf{C}$. We denote by K the compact set $\mathbf{C} \setminus W$.

We take g to be the function defined by

$$g = h^{-1} f h$$

in $U^- \cup N'$, where N' is a neighbourhood of $\mathbf{R}$ satisfying $fh(N') \subset h(N)$. By use of the reflection principle we are able to obtain a meromorphic continuation of g to $U \cup N'$. We note that

$$h^{-1} = g h^{-1} f^{-1} \tag{4.5}$$

on Γ for all branches of f^{-1}. It follows from the invariance of $J(f)$ under f that f is univalent at each point of $f^{-1}(\Gamma) = \Gamma$ and so it follows from Picard's theorem that there are at most two points in Γ at which f^{-1} does not have infinitely many analytic branches.

(4) We now show that all possible analytic continuations of h^{-1} lead to a single-valued function. If not then we can take a point $r \in \Gamma$ at which f^{-1} has infinitely many analytic branches and two polygonal paths γ_1, γ_2 from r to a point $q \in K$ together with a neighbourhood D_r of r and a neighbourhood D_q of q such that h^{-1} is analytic in D_r and can be continued analytically along γ_i to a function H_i that is analytic in D_q, $i = 1, 2$, where $H_1 \neq H_2$.

It follows from Lemma 2.11 that we can modify the closed polygonal path $\gamma_1 \setminus \gamma_2$ by changing γ_1 to a polygonal path λ_1 from r to a point $q' \in D_q$ and $-\gamma_2$ to a path $-\lambda_2$ from q' to some $r' \in \Gamma$ such that there is an arc $\Lambda_r \subset \Gamma \cap D_r$ with endpoints r and r' and such that

(i) for $i = 1, 2$, h^{-1} continues analytically along λ_i to the function H_i at q' with $H_1(q') \neq H_2(q')$,

(ii) every branch of f^{-1} that is analytic at r continues analytically round $\Lambda = \lambda_1 \setminus \lambda_2$ to $r' \in \Gamma$.

By using the same type of argument as that used in section (3) of the proof in case IA it can be shown that there exists a branch F of f^{-1} which satisfies $F(\Lambda) \subset \mathbf{C} \setminus K = W$. We take this branch F of f^{-1} and put $H = g h^{-1} F$ which is well-defined and analytic on Λ which joins r to r'. It follows from (4.5) that H is also analytic on Λ_r and that $H \equiv h^{-1}$ on this arc. Thus H gives an analytic continuation of h^{-1} along λ_1 from r to q' and also along λ_2 from r' to q'. this contradicts (i) and so all possible analytic continuations of h^{-1} do in fact lead to a single-valued function.

(5) By modifying the arguments used in the proof in case IA as in the previous section, it can be shown that h^{-1} can be continued to give an entire function with $h^{-1}(D^-) \subset H^-$, $h^{-1}(D^+) \subset H^+$ and $h^{-1}(\Gamma) \subset \mathbf{R}$. We know that h^{-1} is a conformal map of D^+ and so h^{-1} must in fact be a conformal map of $\mathbf{C}$. As h^{-1} has no poles it must in fact be linear and so h is also linear. Thus $\Gamma = h(\mathbf{R})$ is indeed a straight line as claimed.

Proof in case III. If $J(f)$ is an analytic Jordan arc with one end at ∞ and one finite endpoint, a, we consider the function $F = P^{-1} f P$ where $P(z) = z^2 + a$. It follows from Lemma 4.1 that $J(F)$ is an analytic Jordan curve and, as we have proved Theorem B in case I, it follows that $J(F)$ must be a straight line. As $J(f) = P\bigl(J(F)\bigr)$ and $0 \in J(F)$ it follows that $J(f)$ is a half-line.

5. Proof of Theorem C

(i) We consider the function g_0 where $g_0(z) = \tan(z)$. In [4] several examples are given of functions whose Julia sets are contained in the real line. In particular it is shown that $J(g_0) = \mathbf{R} \cup \{\infty\}$ and is hence an analytic Jordan curve.

(ii) We now consider the function g_1 where $g_1(z) = \bigl[\tan(z^{1/2})\bigr]^2$. This is a well-defined function in class IV. Clearly $g_1 = I g_0 I^{-1}$ where $I(z) = z^2$ and so by induction we obtain $(g_1)^n = I (g_0)^n I^{-1}$. It follows that the pre-images of ∞ under the iterates of g_1 are contained in and dense in $\mathbf{R}^+$ since the pre-images of ∞ under the iterates of g_0 are contained in and dense in $\mathbf{R}$. Thus, by Lemma 2.3, $J(g_1) = \mathbf{R}^+$ which is an analytic Jordan arc with an endpoint at ∞.

(iii) We construct a transcendental meromorphic function g_2 such that

I) g_2 is real on $\mathbf{R}$,

II) if $I_1 = (-\infty, -1]$ and $I_2 = [1, \infty)$, then $g_2(z) \in I_1 \cup I_2 \cup \{\infty\}$ if and only if $z \in I_1 \cup I_2$,

III) g_2 is an odd function.

The above properties of g_2 imply that $N_1 = \mathbf{C} \setminus (I_1 \cup I_2)$ is an invariant domain of g_2 and so $N_1 \subset N(g_2)$ which has only one component. It now follows from Lemma 2.2 that no subsequence of f^n can have a non-constant limit function in $N(g_2)$. We see from (III) that $g_2(0) = 0$ and so $(g_2)^n(z) \to 0$ as $n \to \infty$ for each $z \in N(g_2)$. It then follows from (II) that $J(g_2) = I_1 \cup I_2 \cup \{\infty\}$ and is thus of the required form. The remainder of this section is devoted to the construction of the function g_2.

We begin by giving an outline of the method of construction. We start with the function $g(z) = 1/(\sin z)$. This is an odd function and has the property that $g(z) \in I_1 \cup I_2$ if and only if $z \in \mathbf{R}$. Figure 5.1 shows the values of $g(z)$ for real values of z. We then obtain a rational function h of degree three which behaves on the real axis as shown in Figure 5.2.

The main idea behind the construction is to modify g using quasiconformal surgery so that it behaves like h in a neighbourhood of the origin and retains its original behaviour elsewhere. To do this we take the disc $D = D(0,b)$ and construct a quasiconformal map φ of D such that the function

$$F = \begin{cases} \varphi h & \text{for } z \in \overline{D} \\ g & \text{for } z \in \mathbf{C} \setminus D \end{cases}$$

is continuous. The behaviour of F on the real axis is shown in Figure 5.3.

The next step is to obtain a quasiconformal map ν such that $f = F\nu^{-1}$ is meromorphic. Finally, we show that, for some $\delta > 0$, the function f defined by $f_\delta(z) = f(\delta z)$ is the required function g_2.

Before starting the construction we give some basic definitions and results concerning quasiconformal maps. Given $K > 1$, a homeomorphism φ of a domain D is said to be K-quasiconformal in D if it is absolutely continuous on horizontal and vertical lines and if the complex dilatation of φ,

$$\mu_\varphi = [\partial\varphi/\partial\bar{z}]/[\partial\varphi/\partial z],$$

satisfies $|\mu_\varphi(z)| \le (K-1)/(K+1)$ almost everywhere (a.e.) in D. φ is conformal in D if and only if $\mu_\varphi(z) = 0$ a.e. in D. If $H\colon D_1 \to D_2$ is quasiconformal in D_1 and $G\colon D_2 \to D_3$ is quasiconformal in D_2 then GH is quasiconformal in D_1 and

$$\text{(5.1)} \qquad \mu_{GH}(z) = \frac{\mu_H(z) + \mu_G\big(H(z)\big)\exp\big(-2i\arg\big((\partial H/\partial z)(z)\big)\big)}{1 + \mu_H(z)\mu_G\big(H(z)\big)\exp\big(-2i\arg\big((\partial H/\partial \bar{z})(z)\big)\big)}$$

a.e. in D_1.

Lemma 5.1 (The measurable Riemann mapping theorem). *Given a measurable function μ on the plane such that $\|\mu\|_\infty < 1$, there exists a unique sense-preserving quasiconformal homeomorphism φ of $\widehat{\mathbf{C}}$ to $\widehat{\mathbf{C}}$ such that $\mu_\varphi = \mu$ a.e. and φ fixes ± 1 and ∞.*

For more details concerning these and other results about quasiconformal maps see, for example, [17].

We define S to be the class of functions, f, each defined in a domain Ω_f which contains 0 and is symmetric with respect to both the real and imaginary axes and such that f has the following properties in Ω_f:

(i) f is an odd function,
(ii) $f(\bar{z}) = \overline{f(z)}$,
(iii) $f(z) \in \mathbf{R} \cup \{\infty\}$ if and only if $z \in \mathbf{R} \cap \Omega_f$.

It is clear that if two functions $f_1\colon \Omega_1 \to \Omega_2$ and $f_2\colon \Omega_2 \to \Omega_3$ are in S then their composition $f_2 f_1$ also belongs to S. We are now ready to begin the construction of g_2.

Figures 5.1, 5.2 and 5.3.

(1) In this section we construct the rational function h. We begin by considering the function R defined by

$$R(z) = \frac{(1-\alpha^2 z^2)z}{(z^2-\alpha^2)}$$

where $0 < \alpha < 1$. We claim that R belongs to the class S with $\Omega_R = \mathbf{C}$. It is not difficult to see that R has properties (i) and (ii) and that it is real on $\mathbf{R}$. It can be seen that $R(-\alpha) = R(\alpha) = R(\infty) = \infty$. By considering the behaviour of R near each of these points it follows that

$$R\big((-\infty,-\alpha)\big) = R\big((-\alpha,\alpha)\big) = R\big((\alpha,\infty)\big) = \mathbf{R}.$$

Thus each point in $\mathbf{R}$ has three pre-images in $\mathbf{R}$. As R is of degree three it follows that R is real only on $\mathbf{R}$ and so it has property (iii) as claimed.

If we take Δ to be the unit disc then clearly R maps $\partial\Delta$ to itself. By choosing α sufficiently small we can ensure that R is a conformal homeomorphism of a neighbourhood of $\partial\Delta$. For the zeros of R' satisfy

$$\alpha^2 z^4 + z^2(1-3\alpha^4) + \alpha^2 = 0$$

and so if $R'(z_0) = 0$ for some $z_0 \in \partial\Delta$ then

$$|1-3\alpha^4| = \big|(z_0)^2(1-3\alpha^4)\big| = |-\alpha^2-\alpha^2 z^4| \leq 2\alpha^2$$

which is clearly impossible for sufficiently small values of α, say $\alpha < \alpha_0$. In the remaining work we will assume that α has been chosen to be less than α_0. It follows from the argument principle that the winding number of $R(\partial\Delta)$ about 0 is -1 and so R is a conformal homeomorphism of $\partial\Delta$ onto itself. It is now clear that there must exist a neighbourhood of $\partial\Delta$ in which R is also a conformal homeomorphism.

We take a value b such that $1 < b < \pi/2$, put $h(z) = bR(z/b)$ and $\gamma = \partial D$ where $D = D(0,b)$.

(2) We recall that $g(z) = 1/(\sin z)$. In this section we show that there exists a quasiconformal map $\varphi\colon \mathbf{C} \to \mathbf{C}$ in S such that

$$\varphi|_\gamma = gh^{-1}|_\gamma = g|_\gamma h^{-1}|_\gamma.$$

Putting $W = \big\{z : |\operatorname{Re} z| < \pi/2\big\}$, we note that g is univalent in W and that $g \in S$ with $\Omega_g = W$. We let k be the unique conformal map of the open disc D onto the bounded component of $\mathbf{C} \setminus g(\gamma)$ which maps 0 to 0 and real positive values to real positive values. We note that k has a continuous extension to γ and belongs to the class S.

We recall that h is a conformal homeomorphism of a neighbourhood of γ onto itself and so $k^{-1}gh^{-1}$ is a well-defined map of γ onto itself, preserving orientation. From the symmetry of the situation we have, for $z = be^{i\theta}$, $k^{-1}gh^{-1}(z) = be^{i\psi(\theta)}$, where $\psi(0) = 0$, $\psi(\pi) = \pi$, $\psi(-\theta) = -\psi(\theta)$, $\psi(\theta + \pi) = \psi(\theta) + \pi$ and $\psi'(\theta) > 0$. In fact there must exist constants $0 < m \le M < \infty$ such that

$$m \le \psi'(\theta) \le M \tag{5.2}$$

for $0 \le \theta \le 2\pi$. We now put

$$\tau(re^{i\theta}) = re^{i\psi(\theta)}$$

for $0 \le r \le b$. It is clear that τ belongs to the class S, is a homeomorphism of $\overline{D}$ onto itself and $\tau|_\gamma = k^{-1}gh^{-1}|_\gamma$. The complex dilatation of τ is

$$\mu_\tau = [\partial\tau/\partial\bar{z}]/[\partial\tau/\partial z] = e^{2i\theta}\{\tau_r + (i/r)\tau_\theta\}/\{\tau_r - (i/r)\tau_\theta\}$$

where the subscripts denote partial derivatives and so, from (5.2),

$$|\mu_\tau| = \left|\frac{1-\psi'(\theta)}{1+\psi'(\theta)}\right| \le \max\left(\frac{1-m}{1+m}, \frac{M-1}{M+1}\right) < 1.$$

Hence τ is a quasiconformal map of D onto itself.

It now follows that $k\tau$ is a quasiconformal function in the class S that maps D onto the bounded component of $\mathbf{C} \setminus g(\gamma)$ with

$$k\tau|_\gamma = kk^{-1}gh^{-1}|_\gamma = gh^{-1}|_\gamma.$$

By using a similar argument we are able to extend gh^{-1} to a quasiconformal function in S that maps $\mathbf{C} \setminus \overline{D}$ to the unbounded component of $\mathbf{C} \setminus g(\gamma)$. This completes the argument to show the existence of the function φ.

(3) We now define

$$F = \begin{cases} \varphi h & \text{for } z \in \overline{D} \\ g & \text{for } z \in \mathbf{C} \setminus D \end{cases}$$

which is continuous in $\mathbf{C}$. We note that F is an odd function, $F(\bar{z}) = \overline{F(z)}$ and F is real on $\mathbf{R}$. Clearly

$$\mu_F(z) = (\partial F/\partial\bar{z})/(\partial F/\partial z) = 0$$

a.e. in $\mathbf{C} \setminus D$ and $\mu_F(z) = \mu_{\varphi h}(z)$ in $\overline{D}$. As $\mu_h(z) = 0$ a.e. in $\overline{D}$ it follows from (5.1) that

$$\left|\mu_F(z)\right| = \left|\mu_\varphi\big(h(z)\big)\right|$$

a.e. in $\overline{D}$. φ is quasiconformal in $\widehat{\mathbf{C}}$ and so it follows that there exists $K > 1$ such that

$$\left|\mu_F(z)\right| \le (K-1)/(K+1)$$

a.e. in $\mathbf{C}$.

(4) In this section we obtain a quasiconformal map ν such that $f = F\nu^{-1}$ is meromorphic in $\mathbf{C}$. It follows from Lemma 5.1 that there exists a unique quasiconformal map, ν, of $\widehat{\mathbf{C}}$ which fixes ± 1 and ∞ and whose quasiconformal dilatation is given by $\mu_\nu = \mu_F$ a.e. in $\mathbf{C}$. Since F is an odd function and $F(\bar z) = \overline{F(z)}$, it follows that μ_ν is an even function with $\mu_\nu(\bar z) = \overline{\mu_\nu(z)}$. If we put $\lambda(z) = -z$ it then follows from (5.1) that ν and $\lambda\nu\lambda$ have the same complex dilatation and both fix the points ± 1 and ∞. Thus ν is identically equal to $\lambda\nu\lambda$ or, in other words, ν is an odd function. In a similar way it can be shown that $\nu(\bar z) = \overline{\nu(z)}$ and, as ν is a homeomorphism of $\widehat{\mathbf{C}}$, we can then deduce that ν is real precisely on the real line and so belongs to S.

We now put $f = F\nu^{-1}$. Note that since $F = (F\nu^{-1})\nu$ and ν have the same complex dilatation μ_ν, it follows from (5.1) that $\mu_f = 0$ a.e. Thus f is conformal except perhaps at the isolated points z such that F is not univalent at $\nu^{-1}(z)$. Since f is continuous at these points it follows that f has no finite singularities other than poles, i.e. f is meromorphic. Since g is transcendental and ν^{-1} is a homeomorphism, f must in fact be a transcendental meromorphic function. As ν is in the class S and F is an odd function with $F(\bar z) = \overline{F(z)}$ that is real on $\mathbf{R}$, it follows that f is also odd, $f(\bar z) = \overline{f(z)}$ and f is real on $\mathbf{R}$.

(5) Finally, putting $f_\delta(z) = f(\delta z)$, we claim that, for some $\delta > 0$, f_δ has the properties I, II, and III and is thus the required function g_2. It is clear that f_δ has properties I and III for any $\delta > 0$. It remains to show that there exists $\delta > 0$ such that $f(z) \in I_1 \cup I_2 \cup \{\infty\}$ if and only if $z \in (-\infty, -\delta] \cup [\delta, \infty)$.

We begin by showing that if $f(z) \in I_1 \cup I_2 \cup \{\infty\}$ then z must be real. This is clearly true in the case where $f(z) = \varphi h \nu^{-1}(z)$ as each of φ, h and ν^{-1} is real precisely on $\mathbf{R}$. We now consider the case where $f(z) = g\nu^{-1}(z)$. It is clear that if $z \in \mathbf{C} \setminus \mathbf{R}$ and $g(z) \in \mathbf{R}$ then $\bigl|g(z)\bigr| < 1$. Thus $g(z) \in I_1 \cup I_2 \cup \{\infty\}$ implies that $z \in \mathbf{R}$. As ν^{-1} is real only on $\mathbf{R}$ it follows that, in both cases, $f(z) \in I_1 \cup I_2 \cup \{\infty\}$ only if $z \in \mathbf{R}$.

It remains to show that, for real z, there exists $\delta > 0$ such that $\bigl|f(z)\bigr| \ge 1$ if and only if $|z| \ge \delta$. We note that ν^{-1} is a homeomorphism that fixes 0, ± 1 and ∞ and so increases with z on $\mathbf{R}$. If $z \in \mathbf{R}$ and $|z| \ge \nu(b)$ then it follows that $f(z) = g\nu^{-1}(z)$ and, as $\bigl|g(z)\bigr| \ge 1$ for all real values of z, it follows that $\bigl|f(z)\bigr| \ge 1$ for all real z satisfying $|z| \ge \nu(b)$.

We now consider the real points satisfying $|z| < \nu(b)$ and hence $f(z) = \varphi h \nu^{-1}(z)$. As φ and ν^{-1} are both odd homeomorphisms such that $|\varphi|$ and $|\nu^{-1}|$ increase with $|z|$ on $\mathbf{R}$ it follows from Figure 5.3 that $\bigl|f(z)\bigr| > \bigl|\varphi h(b)\bigr| = \bigl|g(b)\bigr| > 1$ for all real values satisfying $\nu(b\alpha) \le |z| < \nu(b)$. It is also clear that there exists a value of δ such that, on the interval $\bigl[-\nu(b\alpha), \nu(b\alpha)\bigr]$, $f(z)$ takes values in $I_1 \cup I_2 \cup \{\infty\}$ precisely for those values of z satisfying $|z| \ge \delta$. It now follows that, for this value of δ, the function f_δ is of the required form.

6. Proof of Theorem D

We take a meromorphic function f whose Julia set, $J(f)$, is a Jordan curve or a Jordan arc. Let $w \in J(f)$ be a repelling periodic point of f, that is $f^p(w) = w$ for some $p \in \mathbf{N}$ and $|(f^p)'(w)| = \lambda' > 1$. We assume that $J(f)$ is differentiable at w. Thus there exists a constant $C \in [0, 2\pi)$ such that if (w_n) is a sequence of points in $J(f)$ with $w_n \to w$ as $n \to \infty$ then, for some subsequence $(w_{n(k)})$,

$$\lim_{n\to\infty} \arg(w - w_{n(k)}) = C \qquad \text{or} \qquad C + \pi. \tag{6.1}$$

We now take a value $\lambda \in (1, \lambda')$ and note that, for some $r > 0$, the branch h of f^{-p} which maps w to itself is univalent in $D(w, r)$ and, further, $|h'(z)| < 1/\lambda$ for $z \in D(w, r)$. Thus

$$h\bigl(D(w, r)\bigr) \subset D(w, r/\lambda) \subset D(w, r)$$

and indeed, for each $k \in \mathbf{N}$,

$$h^k\bigl(D(w, r)\bigr) \subset D(w, r/\lambda^k).$$

We now choose $z_0 \in \bigl(D(w, r/2) \setminus \{w\}\bigr) \cap J(f)$ and define the functions G_k in $D(w, r)$ by

$$G_k(z) = \bigl[h^k(z) - w\bigr] / \bigl[(h^k)'(z_0)\bigr]. \tag{6.2}$$

It is clear that each function G_k is univalent in $D(w, r)$ and so it follows from Lemma 2.6 that the distortion of G_k in $D(w, r/2)$ is bounded above by $K(1/2) = 81$. Thus for each $z \in D(w, r/2)$ and each $k \in \mathbf{N}$

$$\frac{|(G_k)'(z)|}{1 + |G_k(z)|^2} \le |(G_k)'(z)| \le 81 |(G_k)'(z_0)| = 81 \tag{6.3}$$

and so by Lemma 2.1 the functions G_k form a normal family in $D(w, r/2)$. We note that $G_k(w) = 0$ and so, from (6.3),

$$G_k\bigl(D(w, r/2)\bigr) \subset D(0, 81r/2). \tag{6.4}$$

We now take a subsequence $G_{k(r)}$ of the functions G_k which converges uniformly to a function φ in $D(z_0, R) \subset \bigl(D(w, r/2) \setminus \{w\}\bigr)$. As each function $G_{k(r)}$ is univalent and $G_{k(r)}(z) \ne 0$ in $D(z_0, R)$ it follows from Hurwitz' theorem (see, for example, [1, p. 178]) that φ must be univalent or a constant and that $\varphi(z) \ne 0$ in $D(z_0, R)$. It follows from (6.4) that $\varphi(z) \ne \infty$ for $z \in D(z_0, R)$ and, as $(G_{k(r)})'(z_0) = 1$ for each $k(r)$, we see that φ must be univalent in $D(z_0, R)$.

We now have, from (6.2), that

$$h^{k(r)}(z) - w = (h^{k(r)})'(z_0)\big[\varphi(z) + \varepsilon_{k(r)}(z)\big] \tag{6.5}$$

where $\varepsilon_{k(r)} \to 0$ uniformly in $D(z_0, R)$. We let γ be an arc in $J(f) \cap D(z_0, R)$ and take any point $z \in \gamma$. Now

$$h^{k(r)}(z) \in J(f) \cap \big[D(w, r/\lambda^{k(r)}) \setminus \{w\}\big]$$

and so it follows from (6.1) that, for some subsequence $\big(h^{k(r(s))}(z)\big)$,

$$\lim_{s\to\infty} \arg\big(h^{k(r(s))}(z) - w\big) = C \qquad \text{or} \qquad C + \pi$$

where C is a constant that is independent of $z \in \gamma$.

If we let

$$\limsup_{s\to\infty} \arg\big((h^{k(r(s))})'(z_0)\big) = C_1$$

then, as $\varphi(z) \neq 0$ in $D(z_0, R)$, it follows from (6.5) that

$$\arg\varphi(z) = C - C_1 = C_2 \qquad \text{or} \qquad C + \pi - C_1 = C_2 + \pi$$

where C_2 is a constant that is independent of $z \in \gamma$ and hence $\varphi(\gamma)$ must be a straight line segment. As φ is univalent, we see that γ must be an analytic arc and so it follows from Theorem B that $J(f)$ must be a straight line, circle, segment of a straight line or an arc of a circle. As we know from Lemma 2.4 that the repelling periodic points of f are dense in $J(f)$, Theorem D follows.

Note. It is known that it is possible to have a meromorphic function in class IV whose Julia set is a Jordan curve and not a straight line. In [4] several examples are given of functions in class IV whose Julia sets are quasicircles and not straight lines.

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