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THE MAPPING BY HEIGHTS FOR QUADRATIC DIFFERENTIALS IN THE DISK

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Abstract. The heights of simple closed loops with respect to a holomorphic quadratic differential play an important role on compact Riemann surfaces. Here, the analogue is developed for quadratic differentials of finite norm in the disk. The height of a loop is replaced by the height of a cross cut, which is the same as the vertical distance, with respect to the q.d., of its end points.

1. Introduction

1.1. Let $\varphi \neq 0$ be a holomorphic quadratic differential in the unit disk $D := \{z; |z| < 1\}$. It defines invariant length elements $|\varphi(z)|^{1/2}|dz|$ and area elements $|\varphi(z)|\,dx\,dy$, $z = x + iy$.

The φ-length of an arc γ is

$$|\gamma|_\varphi := \int_\gamma |\varphi(z)|^{1/2}|dz|,$$

and the φ-distance of a pair of points z_1, z_2 is equal to

$$d_\varphi[z_1, z_2] := \inf_{\{\gamma\}} |\gamma|_\varphi,$$

where γ varies over all arcs connecting the two points. The φ-area of a point set $E \subset D$ is the integral

$$\iint_E |\varphi(z)|\,dx\,dy,$$

and the φ-area of D is the L^1-norm of φ,

$$\|\varphi\| = \iint_D |\varphi(z)|\,dx\,dy.$$

Throughout this paper we will speak about quadratic differentials of finite norm.

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Besides that φ also defines an element of height

$$\left|\operatorname{Im}\left\{\varphi(z)^{1/2}dz\right\}\right|$$

and one of horizontal length

$$\left|\operatorname{Re}\left\{\varphi(z)^{1/2}dz\right\}\right|.$$

Since the last expression is equal to the first one for the differential $-\varphi$, it is enough to look at the elements of height. Similarly as for lengths we can define the φ-height of an arc γ by

$$h_\varphi(\gamma) := \int_\gamma \left|\operatorname{Im}\left\{\varphi(z)^{1/2}dz\right\}\right|$$

and the vertical distance or the φ-height of a pair of points z_1, z_2 by

$$h_\varphi[z_1, z_2] := \inf_{\{\gamma\}} h_\varphi(\gamma),$$

with the same meaning of $\{\gamma\}$ as before (see Definition 2.1 below).

For a better visualization of the different quantities we introduce, locally and away from the zeroes, the integral of the square root of φ,

$$w = u + iv = \Phi(z) = \int^z \sqrt{\varphi(z)}\,dz.$$

The elements of the multivalued function Φ are well defined up to the transformation

$$\Phi_2(z) = \pm\Phi_1(z) + \text{const}\,.$$

The elements of length

$$|dw| = \left|\varphi(z)\right|^{1/2}|dz|$$

and of height

$$|dv| = \left|\operatorname{Im}\left\{\sqrt{\varphi(z)}\,dz\right\}\right|$$

are well defined. The height of an arc γ is nothing but the total variation of the multivalued harmonic function v along γ. If we introduce w as local parameter instead of z, the elements of length and of height become Euclidean, as the expressions show, however with branchings in the large.

1.2. Let $\overline{D}$ be the closed unit disk $|z| \leq 1$ and assign a finite number of points $\zeta_1, \ldots, \zeta_N$ on ∂D. We call $\overline{D}$ together with the distinguished points ζ_i a polygon P, an N-gon in this particular case. The ζ_i are its vertices and the intervals on ∂D between the vertices its sides or edges. A quadratic differential φ is said to belong to P, if it is meromorphic in $\overline{D}$, with at most simple poles at the vertices, and real along the sides of P (i.e. $\varphi(z)\,dz^2$ real for tangential dz).

It is easy to see (e.g. by means of a conformal mapping of D onto the upper half plane) that φ can be continued to $\hat{C}$ by reflection on ∂D. The continuation is a quadratic differential with closed trajectories which sweep out a finite number of disjoint annuli (for details see [1]). Each annulus is split into a symmetric pair of quadrilaterals by two subintervals of different sides of P, separated by at least two vertices. The quadrilaterals in D are called horizontal strips S_i. They are mapped onto Euclidean rectangles

$$0 < u < a_i, \qquad 0 < v < b_i$$

by a branch of $w = u + iv = \Phi(z)$. If we choose a trajectory α_i out of each open strip S_i, we get a system of disjoint cross cuts of D. Each cross cut connects two different sides of P, separated by at least two vertices, and different ones connect different pairs of sides.

Conversely, it was shown by H. Renelt and, simultaneously, by J. Hubbard and H. Masur (1976), that one can prescribe a system of Jordan arcs γ_i which are cross cuts in the above sense, and the numbers $b_i > 0$. Then, there exists a unique quadratic differential φ, associated with the N-gon P and such that its trajectories are homotopic to the given cross cuts γ_i and the heights of its strips (or cylinders) are the given numbers b_i. (For a proof and references to the original literature see [1]).

This theorem can be used to set up a bijection of the differentials of two different polygons. Let P and P^* be two N-gons. Given an order preserving correspondence of the vertices (and hence of the sides) of P and P^*, we assign to each polygon differential φ of P a polygon differential φ^* of P^* by the requirement that the strips of φ and φ^* connect corresponding sides of P and P^* respectively and that corresponding strips have the same heights (measured in the φ- and φ^*-metric respectively). This determines the "mapping by heights" for two given polygons with a given correspondence of the vertices.

As an example, let us consider two pentagons P and P^*. The quadratic differentials φ and φ^* associated with them are the squares of the derivatives of conformal maps $w = \Phi(z)$, $w^* = \Phi^*(z^*)$ respectively, mapping the pentagons onto bus like figures with the same heights of the upper and of the lower parts (Figure 1). The vertices go over into the corners pointing outwards. In terms of the parameters w and w^* we have, by the transformation rule for quadratic differentials, $\varphi \equiv \varphi^* \equiv 1$. The basic theorem says that the numbering of the

Figure 1.

corners and the heights of the two pieces can be given, but then the shape of a "bus" is uniquely determined.

1.3. The purpose of this note is to generalize this mapping for arbitrary quasisymmetric homeomorphisms of the boundary of the unit disk. Rather than heights of strips, which do not exist in the general case, we consider vertical distances (or heights) of pairs of boundary points. It is easy to see (using intersection numbers) that the two polygon differentials φ and φ^* induce the same vertical distances of corresponding pairs of sides. (There is no correspondence of boundary points, except for the vertices.)

Our main result will be, that every quasisymmetric selfmapping of the boundary of the disk induces a selfmapping of the space of holomorphic quadratic differentials of finite norm. Corresponding differentials φ and φ^* determine the same vertical distances of all corresponding pairs of boundary points, which is the characteristic property of the mapping. The proof goes by approximation of differentials by polygon differentials. The main ingredient is the notion of a totally regular trajectory (see [2]). A trajectory α of φ is called regular, if it does not tend, in any of its two directions, to a zero of φ. Otherwise it is called critical. It is known that a regular trajectory of quadratic differential of finite norm has two different end points on ∂D (for a proof see [1, Section 19]). A regular trajectory α is called totally regular, if for any sequence of points $\{z_n\}$ tending to a point $z \in \alpha$ and such that the trajectories $\alpha_n \ni z_n$ are regular, $\alpha_n \to \alpha$ in the Euclidean metric of the disk D. It is shown in [2] that there can be at most denumerably many regular trajectories which are not totally regular. Moreover, if $\{\varphi_n\}$ is a sequence of holomorphic quadratic differentials with uniformly bounded norm which tends locally uniformly to a differential φ not identically equal to zero, then the above statement is true with α_n a trajectory of φ_n rather than of φ. It will be shown that the vertical distance of two points is equal to the supremum of the vertical distances of pairs of totally regular trajectories separating the two points. This, together with the fact that the totally regular trajectories and their vertical distances are invariant under the constructed mapping gives the result.

For later use we state

Lemma 1.3. *Let φ be an arbitrary holomorphic quadratic differential of norm $\|\varphi\| \le M < \infty$ in the unit disk $D : |z| < 1$. Let ζ be a boundary point of D. Then, for any $\varepsilon > 0$ and $\varrho_2 > 0$ there exists a number ϱ_1, $0 < \varrho_1 < \varrho_2$, such that for some $\varrho \in [\varrho_1, \varrho_2]$*

$$L(\varrho) = \int_{\sigma_\varrho} |\varphi(z)|^{1/2} |dz| < \varepsilon,$$

with $\sigma_\varrho = \{z; |z-\zeta| = \varrho, z \in D\}$. Whereas ϱ depends on φ, ϱ_1 does not. (*For a proof see* [2, *Lemma* 1.1]).

2. Heights (vertical distances)

2.1. Let $\varphi \neq 0$ be a holomorphic quadratic differential of finite norm in the disk $D : |z| < 1$.

Definition 2.1. Let ζ_1 and ζ_2 be boundary points of D. The vertical distance or height of the pair of points ζ_1, ζ_2 with respect to φ is

$$h_\varphi[\zeta_1, \zeta_2] := \inf_{\{\gamma\}} \int_\gamma |dv|,$$

where γ runs over all locally rectifiable open Jordan arcs in D with limit points ζ_1 and ζ_2 respectively and v is the imaginary part of $w = u + iv = \Phi(z) = \int^z \sqrt{\varphi(z)}\, dz$.

Similarly one defines the vertical distance of two interior points (where the arcs γ are simply rectifiable Jordan arcs in D) or the vertical distance of an interior point and a boundary point.

The vertical distance of two point sets E_1 and E_2 is defined as usual:

$$h_\varphi[E_1, E_2] := \inf \{h_\varphi[z_1, z_2]; z_1 \in E_1, z_2 \in E_2\}.$$

A special case is the vertical distance of a pair of horizontal geodesics α_1 and α_2. The height of a pair of points $z_1 \in \alpha_1$ and $z_2 \in \alpha_2$ does not depend on their position, since one can add, to a curve γ connecting z_1 and z_2, arbitrary subintervals of α_1 and α_2 ending at z_1 and z_2 respectively (because $dv = 0$ along any horizontal interval). We therefore have, for the vertical distance of α_1 and α_2,

$$h_\varphi[\alpha_1, \alpha_2] = h_\varphi[z_1, z_2], \qquad z_1 \in \alpha_1, z_2 \in \alpha_2.$$

It follows from the fact that for any boundary point ζ of D there are circular cross cuts of D, centered at ζ, with arbitrarily short φ-length (Lemma 1.3), that the vertical distance of z_1 and z_2 is also equal to the vertical distance of two end points of the horizontal arcs.

It is immediate by computation of the norm using polar coordinates that the distance of a boundary point ζ from the center (and hence from any interior point z) is finite for a.a. $\zeta \in \partial D$. Since the vertical distance is smaller or equal to the distance, this is also true for heights.

2.2. Lower semicontinuity of the vertical distance

Lemma 2.2. *For any fixed*
$z_0 \in D$, $\zeta \in \partial D$

$$\liminf_{z \to \zeta} h_\varphi[z_0, z] \geq h_\varphi[z_0, \zeta].$$

Proof. It follows from the length area principle (Lemma 1.3), that there exists a sequence of radii $\varrho_n \to 0$ such that the φ-length $|\tau_n|_\varphi$ of the circular cross cuts

$$\tau_n := \{z \in D, |z - \zeta| = \varrho_n\}$$

tends to zero. Therefore, for every positive ε, there exists a subsequence $\{\tau_{n_i}\}$ such that

$$\sum_i |\tau_{n_i}|_\varphi < \varepsilon.$$

To simplify the notation, we call this subsequence $\{\tau_n\}$ again (Figure 2).

Figure 2.

Let us first consider the case where

$$h_\varphi[z_0, \tau_0] + h_\varphi[\tau_0, \tau_1] + h_\varphi[\tau_1, \tau_2] + \cdots = A < \infty.$$

For a given $\varepsilon > 0$ we can find an arc γ_0 connecting z_0 with τ_0 such that

$$h_\varphi(\gamma_0) < h_\varphi[z_0, \tau_0] + \frac{\varepsilon}{2},$$

and arcs γ_n connecting τ_{n-1} with τ_n such that

$$h_\varphi(\gamma_n) < h_\varphi[\tau_{n-1}, \tau_n] + \frac{\varepsilon}{2^{(n+1)}},$$

$n = 1, 2, \ldots$. The end points of γ_{n-1} and γ_n on τ_{n-1} are connected by a subinterval $\Delta\tau_{n-1}$ of τ_{n-1}. Since

$$\sum_0^\infty |\tau_n|_\varphi < \varepsilon,$$

we also have

$$\sum_0^\infty |\Delta\tau_n| < \varepsilon.$$

We thus get a curve γ, which we write

$$\gamma := \gamma_0 + \Delta\tau_0 + \gamma_1 + \Delta\tau_1 + \gamma_2 + \Delta\tau_2 + \cdots$$

connecting z_0 with ζ which has height

$$h_\varphi[z_0, \zeta] \leq h_\varphi(\gamma) < h_\varphi[z_0, \tau_0] + h_\varphi[\tau_0, \tau_1] + \cdots + 2\varepsilon = A + 2\varepsilon.$$

On the other hand, there exists an index n such that

$$h_\varphi[z_0, \tau_n] \geq h_\varphi[z_0, \tau_0] + h_\varphi[\tau_0, \tau_1] + \cdots + h_\varphi[\tau_{n-1}, \tau_n] > A - \varepsilon.$$

Therefore

$$h_\varphi[z_0, \tau_n] > h_\varphi[z_0, \zeta] - 3\varepsilon.$$

If z is separated from z_0 by τ_n,

$$h_\varphi[z_0, z] \geq h_\varphi[z_0, \tau_n],$$

which proves that

$$\liminf_{z \to \zeta} h_\varphi[z_0, z] \geq h_\varphi[z_0, \zeta].$$

Let now

$$h_\varphi[z_0, \tau_0] + \sum_{n=1}^\infty h_\varphi[\tau_{n-1}, \tau_n] = \infty.$$

Then, for any $M < \infty$ there exists an index n such that

$$h_\varphi[z_0, \tau_n] \geq h_\varphi[z_0, \tau_0] + \sum_{i=1}^n h_\varphi[\tau_{i-1}, \tau_i] > M,$$

which shows that

$$\liminf_{z \to \zeta} h_\varphi[z_0, z] = \infty.$$

2.3. Connections of smallest height; step curves

Lemma 2.3. *Let φ be a holomorphic quadratic differential in D. Then, any shortest connection of two interior points of D has minimal height.*

Proof. Let $z_0, z_1 \in D$ and let γ_0 be the shortest curve connecting the two points. Then, γ_0 is a geodesic and hence consists of φ-straight pieces satisfying the angle condition at the zeroes of φ (for details see [1, Theorem 8.1]). If γ_0 is horizontal, i.e. $\varphi(z)\,dz^2 \geq 0$ along γ_0, then $h_\varphi(\gamma_0) = 0$, and thus γ_0 has minimal height. Otherwise, γ_0 consists of non horizontal and possibly horizontal straight segments. Let γ be an arbitrary connection of z_0 and z_1. Choose a radius $r < 1$ such that the disk $D_r = \{z; |z| < r\}$ contains both γ_0 and γ. Mark the zeroes of φ on γ_0 and those points z on the non horizontal edges of γ_0 which lie on a relatively critical trajectory (i.e. one which meets a zero of φ before hitting the circle $|z| = r$). There can be only finitely many markings on γ_0. The trajectories going through non marked points on γ_0 are cross cuts of D_r which separate z_0 and z_1. They sweep out finitely many horizontal strips S_i of height b_i, say, mapped conformally onto Euclidean horizontal strips by any branch of Φ. Each strip is passed once (of course in general not vertically) by γ_0. Since the horizontal pieces of γ_0 have height zero, $h_\varphi(\gamma_0) = \sum b_i$.

On the other hand, γ must cross every strip S_i. Therefore, by Euclidean geometry in the w-plane, $h_\varphi(\gamma) \geq \sum b_i$.

Of course, there is no uniqueness of connections of minimal height, as there is for curves with minimal length. Since, locally, there always exists a shortest connection (see [1, Theorem 8.1]) we have the following

Corollary 2.3. *Every point z has a neighborhood $U(z)$ with the property that any two points $z_0, z_1 \in U(z)$ can be joined by an arc of minimal height in $U(z)$.*

We will later work with connections of a special type.

Definition 2.3. A step curve (with respect to φ) is a curve which consists of horizontal ($\varphi(z)\,dz^2 \geq 0$) and vertical ($\varphi(z)\,dz^2 \leq 0$) pieces.

Figure 3.

It is easy to see, that, locally, there always exists a connecting step curve of minimal height. It may have to pass through a zero of φ. On the other hand, there always exists a step curve connection avoiding the zero, and of height arbitrarily close to the minimum. The preceding picture shows connections of minimal height in a neighborhood of a third order zero (Figure 3). The arcs consist of two horizontal and one vertical (dotted) or two vertical and one horizontal pieces. If the two points are not in adjacent sectors, the connecting arc of minimal height goes through the zero in the center of the disk.

Of course, a union of step curves is again a step curve. The following is therefore clear.

Theorem 2.3. *Let $\varphi \neq 0$ be holomorphic in D, $z_0, z_1 \in D$. Then, every curve γ connecting z_0 and z_1 can be replaced by a step curve $\overline{\gamma}$ which is contained in an arbitrarily small neighborhood of γ and has height*

$$h_\varphi(\overline{\gamma}) \leq h_\varphi(\gamma).$$

An approximation by a step curve $\overline{\gamma}$ avoiding the zeroes of φ is possible with height

$$h_\varphi(\overline{\gamma}) < h_\varphi(\gamma) + \varepsilon,$$

for every $\varepsilon > 0$.

3. Heights and separating trajectories

3.1. The vertical distance of two points can be expressed in terms of the totally regular trajectories separating them. We begin with the case where the two points are in D.

Theorem 3.1. *The vertical distance of two points $z_0, z_1 \in D$ is zero if and only if there is no totally regular trajectory which separates them.*

Figure 4.

Remark. As the preceding picture (Figure 4) shows, the statement is wrong for regular but not totally regular trajectories. The two points z_0, z_1 have vertical distance zero, although they are separated by the regular trajectory α. The two points lie on regular trajectories α_0 and α_1 respectively, which have an end point in common with α. Note that a totally regular trajectory cannot have an end point in common with another trajectory.

Proof. Assume first that α is totally regular and separates the two points. Then, there are totally regular trajectories $\tilde{\alpha}$ in every neighborhood (in the Euclidean metric of D) of α. If we choose $\tilde{\alpha}$ in a sufficiently small neighborhood of α, it also separates the two points. Therefore all curves γ connecting z_0 and z_1 have a subinterval that connects α and $\tilde{\alpha}$. Clearly, α and $\tilde{\alpha}$ have positive vertical distance, and thus

$$h_\varphi[z_0, z_1] \geq h_\varphi[\alpha, \tilde{\alpha}] > 0.$$

Addendum. A similar argument works if α passes through z_0, but not through z_1. Then, $\tilde{\alpha}$ is chosen such that it separates α from z_1. We therefore have:

If $h_\varphi[z_0, z_1] = 0$, no totally regular trajectory α can separate the two points nor pass through one of the points without passing through the other.

It is evident, that the above proof also works if one of the points or both are boundary points of D. We will therefore not repeat it further down.

The converse is a consequence of the following

Lemma 3.1 (Trimming lemma). *Assume that no totally regular trajectory separates z_0 from z_1. Then, for every step curve $\overline{\gamma}$ connecting the two points and every $\varepsilon > 0$ there exists a variation $\overline{\gamma}_0$ of $\overline{\gamma}$ with height $h_\varphi(\overline{\gamma}_0) < \varepsilon$.*

Proof. If one of the two points lies on a totally regular trajectory σ, the other one necessarily lies on the same (because otherwise we could find separating ones). But then $\overline{\gamma}_0$ is the connecting subinterval of σ. We can therefore exclude this case in the sequel.

We connect the two points by a step curve $\overline{\gamma}$, consisting of horizontal and vertical intervals α_i and β_i respectively and avoiding the zeroes of φ (except for possible zeroes at the end points, of course). To fix the ideas, we write symbolically

$$\overline{\gamma} = z_0 + \beta_0 + \alpha_1 + \beta_1 + \cdots + \alpha_N + \beta_N + z_N$$

calling the last point z_N rather than z_1. We can always start and terminate with a vertical interval, since $\overline{\gamma}$ is not assumed to have minimal height. Moreover, by arbitrarily small shifts, if necessary, we can achieve that the horizontal intervals α_i lie on totally regular trajectories (which we also call α_i) none of which passes through z_0 or z_N.

Figure 5.

Choose a totally regular trajectory σ intersecting β_0 in a point ζ near z_0 (Figure 5). The curve $\overline{\gamma}$ has a last intersection ζ^* with σ. The subinterval of $\overline{\gamma}$ between ζ and ζ^*, which we denote by $(\overline{\gamma};\zeta,\zeta^*)$, clearly increases monotonically, if $\zeta \to z_0$, and hence the points ζ^* have a limit $z_0^* \in \overline{\gamma}$. This point is necessarily an interior point of some vertical side β_i of $\overline{\gamma}$. Otherwise it would lie on a totally regular trajectory, α_{i+1}, say, and hence z_0 would lie on the same, which we have excluded. The two pairs of points ζ, z_0 and ζ^*, z_0^* have the same distance (which is equal to their vertical distance). We choose σ such that it is smaller than $\varepsilon/2N$ and we replace the arc $(\overline{\gamma}; z_0, z_0^*)$ by $(\beta_0; z_0, \zeta) + (\sigma; \zeta, \zeta^*) + (\beta_i; \zeta^*, z_0^*)$, which means that we go, except for two short vertical intervals, along σ.

Now we look at the remaining arc $(\overline{\gamma}; z_0^*, z_N)$ of $\overline{\gamma}$. Its two end points have the property that there is no separating totally regular trajectory (because such a trajectory would also separate z_0 and z_N) and z_0^* cannot lie on a totally regular trajectory, unless z_N lies on the same. We therefore have the initial situation and we can continue the trimming process. The two intervals on β_i add to one of length (height) less than ε/N. Because there are at most N steps necessary, we have a step curve $\overline{\gamma}_0$ of height less than $N \cdot \varepsilon/N = \varepsilon$ which connects the two points z_0 and z_N.

The theorem follows immediately: because $\varepsilon > 0$ is arbitrary, $h_\varphi[z_0, z_N] = 0$.

3.2. We pass to the general case.

Theorem 3.2. *Let $z_0, z_1 \in D$ be points with the property that there exist totally regular trajectories σ separating them. Then,*

$$h_\varphi[z_0, z_1] = \sup_{\sigma', \sigma''} h_\varphi[\sigma', \sigma''],$$

where σ' and σ'' run over all totally regular trajectories which separate z_0 and z_1.

Proof. The inequality sign in one direction is evident. Take two arbitrary totally regular trajectories σ', σ'' separating z_0 and z_1. Then, every curve γ

connecting z_0 with z_1 has a subarc γ' connecting σ' and σ''. Therefore

$$\int_\gamma |\varphi(z)|^{1/2} |dz| \geq \int_{\gamma'} |\varphi(z)|^{1/2} |dz| \geq h_\varphi[\sigma', \sigma''],$$

and thus

$$h_\varphi[z_0, z_1] \geq h_\varphi[\sigma', \sigma''].$$

We conclude that

$$h_\varphi[z_0, z_1] \geq \sup_{\sigma', \sigma''} h_\varphi[\sigma', \sigma''].$$

To prove the converse inequality, connect z_0 and z_1 by a step curve $\overline{\gamma}$ as in the proof of Lemma 3.1 ($z_1 = z_N$).

Orient $\overline{\gamma}$ from z_0 to z_N. Let ζ' be the infimum and ζ'' the supremum of the intersections of separating totally regular trajectories σ with $\overline{\gamma}$. We claim that both points are interior points of certain vertical intervals β_i, except possibly $\zeta' = z_0$, $\zeta'' = z_N$. For, assume that ζ' is the end point of a β_i (Figure 6). Then α_{i+1} is itself a separating totally regular trajectory. All totally regular trajectories σ in a neighborhood of $\alpha_i + 1$ are also separating, and ζ' cannot be the infimum. If, on the other hand, ζ' is the initial point of β_i, α_i being totally regular cannot separate the two points but must pass through z_0, which we have excluded. A similar argument works for ζ''.

Figure 6.

Let $\zeta' \in \beta_i$. For a given $\varepsilon > 0$ we choose a separating totally regular trajectory σ' intersecting β_i in an ε-neighborhood of ζ'. Let $\zeta'' \in \beta_k$ and choose a totally regular trajectory σ'' cutting β_k in an ε-neighborhood of ζ''. Let γ_0 be a step curve connecting σ' and σ'' with a height

$$h_\varphi(\gamma_0) < h_\varphi[\sigma', \sigma''] + \varepsilon.$$

The subarc of $\overline{\gamma}$ connecting z_0 with ζ' is not cut by any totally regular trajectory σ separating z_0 from ζ', because σ would also separate z_0 from z_N. By Theorem 3.1

one can therefore connect z_0 and ζ' by a step curve γ' of height $< \varepsilon$. Similarly, we can connect ζ'' and z_n by a step curve γ'' of height $< \varepsilon$. Let τ' and τ'' be the two vertical intervals connecting ζ' with σ' and ζ'' with σ'' respectively. They have length and thus height smaller than ε. The step curve

$$\tilde{\gamma} = \gamma' + \tau' + \gamma_0 + \tau'' + \gamma'',$$

possibly with two subintervals of σ' and σ'' respectively, connects z_0 with z_N and has height

$$h_\varphi(\tilde{\gamma}) < h_\varphi(\gamma_0) + 4\varepsilon < h_\varphi[\sigma', \sigma''] + 5\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves that

$$h_\varphi[z_0, z_N] \leq \sup_{\sigma', \sigma''} h_\varphi[\sigma', \sigma''],$$

and the theorem is proved.

3.3. In this section and the next one the results of Sections 3.1 and 3.2 are generalized to boundary points of D.

Theorem 3.3. *The vertical distance of an interior point z from a boundary point r or of two boundary points r and s is positive if and only if there exists a totally regular trajectory which separates them.*

Proof. If there exists a totally regular trajectory σ separating the two points or ending in one of them without tending, in the opposite direction, to the other one, then $h_\varphi > 0$. This is proved as in 3.1.

The converse is first shown for an interior point and a boundary point. Let $h_\varphi[z, r] > 0$. Choose a sequence of circular cross cuts $\tau_n := \{z \in D; |z - r| = \varrho_n\}$, with $\varrho_n \to 0$ and $|\tau_n|_\varphi \to 0$. Because of the lower semicontinuity of the heights, we can find, for a fixed $0 < A < h_\varphi[z, r]$, an index n such that $|\tau|_\varphi < \frac{1}{2}A$ and $h_\varphi[z, \tau_n] > A$. For any $z_n \in \tau_n$ we clearly also have $h_\varphi[z, z_n] > A$. According to Theorem 3.2 there are two totally regular trajectories σ' and σ'' separating z and z_n such that $h_\varphi[\sigma', \sigma''] > A$. If one of these separates z and r, we are done.

Figure 7.

Figure 8.

If none of them does, by topological reasons (Figure 7), both pass through τ_n. Therefore $h_\varphi[\sigma', \sigma''] \leq |\tau_n|_\varphi < \frac{1}{2}A$, a contradiction.

In the case of two boundary points r and s, we choose two circular cross cuts

$$\tau' = \{z \in D; |z - r| = \varrho'\}, \qquad \tau'' = \{z \in D; |z - s| = \varrho''\}$$

such that $h_\varphi[\tau', \tau''] > A$ and $|\tau'|_\varphi < \frac{1}{2}A$, $|\tau''|_\varphi < \frac{1}{2}A$, for some fixed $0 < A < h_\varphi[r, s]$. The vertical distance of two arbitrary points $z' \in \tau'$, $z'' \in \tau''$ is $h_\varphi[z', z''] > A$. Then, there exist totally regular trajectories σ', σ'' separating the points z', z'' and such that $h_\varphi[z', z''] > A$ (Figure 8). If one at least of the two separates r and s, we are done. If none does, they must both cut one of the arcs τ', τ''. If both cut the same, we use the earlier argument. Otherwise we choose a totally regular trajectory σ separating σ' and σ'' and such that both $h_\varphi[\sigma', \sigma] > \frac{1}{2}A$ and $h_\varphi[\sigma'', \sigma] > \frac{1}{2}A$ (see Corollary 4.1). If σ does not separate r and s, it must intersect at least one of the cross cuts τ' or τ''. We then find a contradiction as before. Thus, if $h_\varphi[r, s] > 0$ there exists a totally regular trajectory separating the two points, as claimed.

3.4. The next theorem is the generalization of 3.2 to boundary points.

Theorem 3.4. *Let r and s be boundary points of D. If there are totally regular trajectories σ separating r and s, then*

$$h_\varphi[r, s] = \sup_{\sigma', \sigma''} h_\varphi[\sigma', \sigma''],$$

the supremum being taken over all separating pairs of totally regular trajectories. The same is true for an interior point z_0 and a boundary point .

Proof. We start with the second case. Let $z_0 \in D$, $s \in \partial D$. Let σ be a totally regular trajectory separating z_0 and s. Then, for all sufficiently small $\varrho > 0$ it also separates z_0 from the circular cross cut

$$\tau(\varrho) := \{z \in D; |z - s| = \varrho\}.$$

From Theorem 3.2 we conclude that for any $z \in \tau(\varrho)$

$$h_\varphi[z_0, z] = \sup_{\sigma', \sigma''} h_\varphi[\sigma', \sigma''],$$

σ' and σ'' running over the totally regular trajectories separating z_0 and z.

Figure 9.

Let now $0 < A < h_\varphi[z_0, s] \leq \infty$ and $\varepsilon > 0$. There exists $\varrho > 0$ such that $\left|\tau(\varrho)\right|_\varphi < \varepsilon$ and $h_\varphi[z_0, z] \geq h_\varphi[z_0, \tau(\varrho)] > A$. Let σ', σ'' be totally regular trajectories separating z_0 and z and such that $h_\varphi[\sigma', \sigma''] > A$. If they do not both separate z_0 and s (Figure 9), choose σ separating σ' from σ'' with $\varepsilon < h_\varphi[\sigma, \sigma''] < 2\varepsilon$. Then, σ cannot cut $\tau(\varrho)$. For, if it does, so does σ'', and hence $h_\varphi[\sigma, \sigma''] \leq \left|\tau(\varrho)\right|_\varphi < \varepsilon$, a contradiction. Therefore σ separates z_0 and s, and

$$h_\varphi[\sigma', \sigma] = h_\varphi[\sigma', \sigma''] - h_\varphi[\sigma, \sigma''] > A - 2\varepsilon.$$

To show the same for two boundary points r and s, choose a separating totally regular trajectory σ. Then, it is easy to see that

$$h_\varphi[r, s] = h_\varphi[r, \sigma] + h_\varphi[\sigma, s].$$

For, clearly the sign $\geq$ holds. On the other hand, any pair of arcs γ', γ'' connecting r with σ and σ with s respectively can be completed by a subinterval $\Delta\sigma$ of σ to a curve $\gamma = \gamma' + \Delta\sigma + \gamma''$ connecting r with s. Therefore

$$h_\varphi[r, s] \leq h_\varphi(\gamma) = h_\varphi(\gamma') + h_\varphi(\gamma''),$$

which gives the inequality

$$h_\varphi[r, s] \leq h_\varphi[r, \sigma] + h_\varphi[\sigma, s].$$

Let now, for any $A' < h_\varphi[r,\sigma]$, $A'' < h_\varphi[\sigma,s]$, σ' and σ'' be totally regular trajectories separating r from σ and σ from s respectively and such that $h_\varphi[\sigma',\sigma] > A'$ and $h_\varphi[\sigma,\sigma''] > A''$. Then,

$$h_\varphi[\sigma',\sigma''] = h_\varphi[\sigma',\sigma] + h_\varphi[\sigma,\sigma''] > A' + A''.$$

We end up with

$$\sup_{\sigma',\sigma''} h_\varphi[\sigma',\sigma''] \geq h_\varphi[r,\sigma] + h_\varphi[\sigma,s] = h_\varphi[r,s].$$

Since the converse inequality is evident, the theorem is proved.

4. Convergence of heights

4.1. In order to prove the convergence of heights we have to give Theorem 3.2 a more constructive form.

Definition 4.1. Given a closed, regular (i.e. without zeroes of φ) vertical interval $\tilde\beta$ with the property that the two trajectories σ and σ' through its end points are totally regular. Then, the domain bounded by σ and σ' is called an elementary horizontal strip S spanned by $\tilde\beta$. It is denoted by $(\tilde\beta;\sigma,\sigma')$.

S is said to separate the two points $z, z' \in D$, if both σ and σ' separate the two points.

Let S_j, spanned by $\tilde\beta_j$, be disjoint elementary strips separating the two points z and z'. Then, clearly

$$h_\varphi[z,z'] \geq \sum_j |\tilde\beta_j|.$$

Theorem 4.1. *For every $\varepsilon > 0$ there exists a finite system of non overlapping elementary strips S_j, spanned by $\tilde\beta_j$, such that*

$$h_\varphi[z,z'] < \sum_j |\tilde\beta_j| + \varepsilon.$$

Proof. Let

$$\overline{\gamma} = z_0 + \beta_0 + \alpha_1 + \beta_1 + \cdots + \alpha_N + \beta_N + z_N$$

be a step curve connecting the two points z_0 and $z' = z_N$. The β_i are the vertical, the α_i the horizontal intervals. We may assume that $\overline{\gamma}$ does not go through a zero of φ, with the possible exception of z_0 and z_N, and that the α_i are lying on different totally regular trajectories not going through an end point of $\overline{\gamma}$.

(1) Suppose that there are totally regular trajectories σ with intersection $\zeta = \sigma \cap \beta_0$ arbitrarily close to z_0 which do not separate the two points. Let ζ^* be the last intersection of $\overline{\gamma}$, oriented from z_0 to z_N, with σ. The point $z_0^* = \lim_{\zeta \to z_0} \zeta^*$ is called the conjugate point of z_0. It follows from the above assumptions about $\overline{\gamma}$ that z_0^* is, with exception of the trivial case $z_0^* = z_N$, an interior point of some β_i. We denote the subarc of $\overline{\gamma}$ bounded by z_0 and by z_0^* by $(\overline{\gamma}; z_0, z_0^*)$. As in the proof of Lemma 3.1, this subarc can be replaced by an arc of height $< \varepsilon/2N$. We continue with the arc $(\overline{\gamma}; z_0^*, z_N)$, which has at least one vertical side less than $\overline{\gamma}$, namely β_0. The totally regular trajectories separating z_0^* and z_N also separate z_0 and z_N.

Figure 10.

(2) Assume now that there is a half neighborhood of z_0 on β_0 which is cut only by totally regular trajectories σ separating the two points z_0 and z_N. Let $\eta_0 \in \beta_0$ be the supremum of all intersections $\zeta = \sigma \cap \beta_0$ interior to β_0. Then, every totally regular trajectory cutting β_0 between z_0 and η_0 separates z_0 and z_N. We get an elementary strip bounded by σ, with $\zeta = \sigma \cap \beta_0$ near z_0 and τ, $\eta = \tau \cap \beta_0$ near η_0 (Figure 10). Its spanning vertical interval is, in our notation, $\tilde{\beta}_0 = (\beta_0; \zeta, \eta)$. We choose the two points in such a way that the sum of the lengths of the two vertical intervals $[z_0, \zeta]$ and $[\eta, \eta_0]$ is less than $\varepsilon/2(N+1)$.

To find the remaining arc of $\overline{\gamma}$, let η^* be the last intersection of $\overline{\gamma}$ with τ, and let $\eta_0^* = \lim \eta^*$ if η approaches η_0. Eventually, all η^* lie on the same β_i, and the two vertical intervals $[\eta, \eta_0]$ and $[\eta^*, \eta_0^*]$ have the same length.

If $\eta^* \equiv \eta$, i.e. the only intersection of τ with $\overline{\gamma}$ is the point η on β_0, we have $\eta_0^* = \eta_0$. If η_0^* is an interior point of β_0, the curve $\tilde{\gamma}$ is the same as $\overline{\gamma}$, subdivided in the following way:

$$\tilde{\gamma} = (\beta_0; z_0, \zeta) + \tilde{\beta}_0 + (\beta_0; \eta, \eta_0) + (\overline{\gamma}; \eta_0, z_N).$$

It easily follows from the requirements on $\overline{\gamma}$ that the arc $(\overline{\gamma}; \eta_0, z_N)$ satisfies the same conditions.

If $\eta_0 = (\beta_0 \cap \alpha_1)$, i.e. η_0 is the end point of β_0, we set

$$\tilde{\gamma} = (\beta_0; z_0, \zeta) + \tilde{\beta}_0 + (\beta_0; \eta, \eta_0) + \alpha_1 + (\overline{\gamma}; \alpha_1 \cap \beta_1, z_N).$$

Again it is immediately clear that $(\overline{\gamma}; \alpha_1 \cap \beta_1, z_N)$, i.e. the subarc of $\overline{\gamma}$ starting with the initial point of β_1, satisfies the same conditions as $\overline{\gamma}$ itself.

Assume now that $\eta^* \neq \eta$, i.e. that the totally regular trajectory τ intersects $\overline{\gamma}$ again. Thus $\eta_0^* \neq \eta_0$. It is easy to see that η_0^* must be an interior point of β_i. For, if it is the end point, $\eta_0^* = \beta_i \cap \alpha_{i+1}$, η_0^* lies on a totally regular trajectory, namely α_{i+1}, and η_0 lies on the same. This trajectory must separate the two points z_0 and z_N, because it cannot go through z_N. If η_0 is an interior point of β_0, it therefore can be slightly pushed up, and if it is the end point, α_1 and α_{i+1} are lying on the same totally regular trajectory, which was excluded from the beginning.

Likewise one shows that η_0^* cannot lie on a totally regular trajectory α_j, $j > i+1$.

The new step curve is

$$\gamma = (\beta_0; z_0, \zeta) + \tilde{\beta}_0 + (\tau; \eta, \eta^*) + (\beta_i; \eta^*, \eta_0^*) + (\overline{\gamma}; \eta_0^*, z_N),$$

in words: starting at z_0 we follow β_0 to the lower side σ of the elementary strip S, cross it along β_0, follow the upper side τ of S from η to η^*, go on β_i to η_0^*, which is the initial point of the remaining arc $(\overline{\gamma}; \eta_0^*, z_N)$. We continue the procedure with this subarc of $\overline{\gamma}$.

Every β_i of the original step curve $\overline{\gamma}$ can give rise to at most one separating elementary strip, and there can be at most N trimmings. Therefore the total height of the final step curve $\tilde{\gamma}$ is

$$h_\varphi(\tilde{\gamma}) < \sum |\tilde{\beta}_j| + N\frac{\varepsilon}{2N} + (N+1)\frac{\varepsilon}{2(N+1)} = \sum |\tilde{\beta}_j| + \varepsilon,$$

which proves the theorem.

The constructed step curve $\tilde{\gamma}$ has almost minimal height. It is clear that among all step curves connecting z_0 and z_N there exist such elements. The special feature of $\tilde{\gamma}$ is that almost its entire height is attained by the crossings $\tilde{\beta}_j$ of disjoint elementary strips.

The following is now evident:

Corollary 4.1. *Let $h_\varphi[z_0, z_1] > 0$. Then, for every number x, $0 \leq x \leq h_\varphi[z_0, z_1]$, and every $\varepsilon > 0$ there exists a separating totally regular trajectory σ with the property*

$$x - \varepsilon < h_\varphi[z_0, \sigma] < x + \varepsilon.$$

Remember that $h_\varphi[z_0, z_1] = h_\varphi[z_0, \sigma] + h_\varphi[\sigma, z_1]$.

4.2. We now prove the convergence of vertical distances of pairs of interior points of D. An inequality in one direction is easy, even without bounded norm.

Lemma 4.2. *Let (φ_n) be a sequence of holomorphic quadratic differentials in D which tends locally uniformly to $\varphi \neq 0$. Let $z, z' \in D$. Then,*

$$\limsup h_{\varphi_n}[z, z'] \leq h_\varphi[z, z'].$$

Proof. Choose a rectifiable curve γ connecting the two points. Then,

$$h_{\varphi_n}[z, z'] \leq \int_\gamma |dv_n| \to \int_\gamma |dv|,$$

hence

$$\limsup_{n\to\infty} h_{\varphi_n}[z, z'] \leq \int_\gamma |dv|.$$

Since this is true for all γ, the lemma is proved. Evidently, if $h_\varphi[z, z'] = 0$, the heights converge.

It is easy to see, with practically the same proof, that the result is also true for point sets. Let $E, E' \subset D$. Then

$$\limsup_{n\to\infty} h_{\varphi_n}[E, E'] \leq h_\varphi[E, E'].$$

On the other hand, the lemma does not hold for boundary points r, s, even if $(\varphi_n) \to \varphi$ in norm. Counterexamples can readily been given using conformal mappings.

4.3. An inequality in the other direction can be shown for sequences (φ_n) which are bounded in norm.

Lemma 4.3. *Let (φ_n) be a sequence of holomorphic quadratic differentials in D, with uniformly bounded norm $\|\varphi_n\| \leq M < \infty$, for all n. Let $(\varphi_n) \to \varphi \neq 0$ locally uniformly in D. Then, for each elementary strip $S : (\beta; \sigma, \sigma')$ of φ*

$$\lim_{n\to\infty} h_{\varphi_n}[\sigma, \sigma'] = h_\varphi[\sigma, \sigma'] = |\beta|_\varphi.$$

Proof. After the end of Section 4.2 we only have to show that

$$\liminf_{n\to\infty} h_{\varphi_n}[\sigma, \sigma'] \geq h_\varphi[\sigma, \sigma'].$$

Let z and z' be the end points of β on σ and σ' respectively. Let β_n be the vertical φ_n-interval, starting at z and ending at a point $z_n' \in \sigma'$. Then, by

Figure 11.

Lemma 2.3 and because β_n is vertical, $h_{\varphi_n}[z, z'_n] = |\beta_n|_{\varphi_n}$. On the other hand, the right hand term clearly tends to $|\beta|_\varphi = h_\varphi[z, z']$. Since $z'_n \to z'$, we have

$$\lim_{n\to\infty} h_{\varphi_n}[z, z'] = \lim_{n\to\infty} h_{\varphi_n}[z, z'_n] = h_\varphi[z, z'].$$

Choose two points ζ, ζ' on β, near z and z' respectively (Figure 11) and such that all the trajectories through ζ and ζ' of φ and φ_n are totally regular. Denote these trajectories by α, α', α_n, α'_n respectively. From $\alpha_n \to \alpha$, $\alpha'_n \to \alpha'$ we get for all sufficiently large n,

$$h_{\varphi_n}[\sigma, \sigma'] \geq h_{\varphi_n}[\alpha_n, \alpha'_n] = h_{\varphi_n}[\zeta, \zeta'],$$

and hence

$$\liminf h_{\varphi_n}[\sigma, \sigma'] \geq h_\varphi[\zeta, \zeta'].$$

Since this is true for all ζ, ζ' the result follows.

We now proceed to the general case.

Theorem 4.3. *Let φ, φ_n be as in Lemma 4.3. Let $z, z' \in D$ and $z_n \to z$, $z'_n \to z'$. Then*

$$\lim_{n\to\infty} h_{\varphi_n}[z_n, z'_n] = h_\varphi[z, z']. \tag{1}$$

Similarly, for two totally regular trajectories σ, σ'

$$\lim_{n\to\infty} h_{\varphi_n}[\sigma, \sigma'] = h_\varphi[\sigma, \sigma']. \tag{2}$$

Proof. We first prove equation (1). Evidently Lemma 4.2 also holds for $z_n \to z$, $z'_n \to z'$ instead of $z_n \equiv z$ and $z'_n \equiv z'$. We therefore have again

$$\limsup_{n\to\infty} h_{\varphi_n}[z_n, z'_n] \leq h_\varphi[z, z'].$$

To show the reversed inequality, let, for any given $\varepsilon > 0$, $\tilde{\gamma}$ be a step curve connecting z and z' as in Lemma 4.1. Let $S_j : (\beta_j, \sigma_j, \sigma_j')$ be a system of disjoint elementary strips separating z and z' and such that

$$\sum |\beta_j| > h_\varphi[z, z'] - \varepsilon.$$

Then,

$$h_{\varphi_n}[z, z'] \geq \sum_j h_{\varphi_n}[\sigma_j, \sigma_j'] \to \sum_j |\beta_j| > h_\varphi[z, z'] - \varepsilon$$

and hence

$$\liminf_{n\to\infty} h_{\varphi_n}[z, z'] \geq h_\varphi[z, z'].$$

Because of the locally uniform convergence $\varphi_n \to \varphi$ and the triangle inequality for heights we can replace z and z' by z_n and z_n' respectively, which proves part one of the theorem.

To prove (2), we choose two arbitrary points $z \in \sigma$ and $z' \in \sigma'$. We have $h_{\varphi_n}[\sigma, \sigma'] \leq h_{\varphi_n}[z, z']$ and $h_\varphi[\sigma, \sigma'] = h_\varphi[z, z']$. Therefore

$$\limsup_{n\to\infty} h_{\varphi_n}[\sigma, \sigma'] \leq \lim_{n\to\infty} h_{\varphi_n}[z, z'] = h_\varphi[z, z'] = h_\varphi[\sigma, \sigma'].$$

On the other hand, with the same setting as above,

$$h_{\varphi_n}[\sigma, \sigma'] \geq \sum_j h_{\varphi_n}[\sigma_j, \sigma_j']$$

and hence

$$\begin{aligned}\liminf_{n\to\infty} h_{\varphi_n}[\sigma, \sigma'] &\geq \lim_{n\to\infty} \sum_j h_{\varphi_n}[\sigma_j, \sigma_j'] = \sum_j h_\varphi[\sigma_j, \sigma_j'] \\ &= \sum_j |\beta_j| > h_\varphi[z, z'] - \varepsilon = h_\varphi[\sigma, \sigma'] - \varepsilon.\end{aligned}$$

This is true for every positive ε and thus proves the assertion.

Using Theorem 4.3, we can derive an inequality for pairs of boundary points.

Corollary 4.3. *Let r and s be boundary points of D. Then, under the assumptions of Lemma* 4.3, *we have*

$$\liminf_{n\to\infty} h_{\varphi_n}[r, s] \geq h_\varphi[r, s].$$

Proof. If $h_\varphi[r, s] = 0$, there is nothing to prove. So let $h_\varphi[r, s] > 0$ and let σ and σ' be two separating totally regular trajectories. Then, by Theorem 4.3

$$h_{\varphi_n}[r, s] \geq h_{\varphi_n}[\sigma, \sigma'] \to h_\varphi[\sigma, \sigma'],$$

and by Theorem 3.4 the last expression is arbitrarily close to $h_\varphi[r, s]$.

5. Existence of the mapping

5.1. We use approximation of φ by polygon differentials to prove the existence of a differential ψ that has the same heights as φ.

Thoerem 5.1. *Let $\varphi \neq 0$ be a holomorphic quadratic differential of finite norm in the unit disk $D : |z| < 1$. Then, φ can be approximated in norm by a sequence of polygon differentials φ_n in D for which the maximal length of the sides of the polygons tends to zero.*

Proof. Choose a sequence of radii $r_n \to 1$. The differential $\varphi_n : \varphi_n(z) = \varphi(r_n z)$ is holomorphic in the closed disk $\overline{D}$. Choose a number $K > 1$ and a sequence of polygons P_j on $\overline{D}$ with side lenghts tending uniformly to zero. By the well known frame mapping criterion, applied to φ_n, the dilatation K and the polygons P_j, there exists a sequence of polygon differentials φ_{nj} such that

$$\|\varphi_n - \varphi_{nj}\| \to 0, \qquad j \to \infty.$$

On the other hand, it is easy to see that

$$\|\varphi_n - \varphi\| \to 0, \qquad n \to \infty.$$

It follows that there exists a subsequence $(j_n) \to \infty$ with the property that

$$\|\varphi - \varphi_{nj_n}\| \to 0, \qquad n \to \infty.$$

This is the desired approximating sequence; we denote it by (φ_n) again.

5.2. Let $w = f(z)$ be a K-qc selfmapping of the unit disk D, and let $\varphi \neq 0$ be a holomorphic quadratic differential of finite norm in D. Let (φ_n) be a sequence of polygon differentials with uniformly bounded norm which converges locally uniformly to φ. The vertices of the polygons P_n are denoted by ζ_{in} and it is assumed that the maximum of their side lengths tends to zero.

Let P'_n be the polygon with the vertices $\zeta'_{in} = f(\zeta_{in})$ and denote by ψ_n the image by heights of φ_n in the polygon P'_n. The totally regular trajectories of the polygon differentials are the interior trajectories of the horizontal strips, which is the same as the subintervals, in the disk, of the closed trajectories in $\hat{C}$. In corresponding strips of the two differentials φ_n and ψ_n we define a mapping of the interior trajectories by equality of heights: if α of φ_n subdivides a strip S in a certain ratio, α' of ψ_n subdivides S' in the same ratio. This establishes a mapping by heights for the trajectories: corresponding pairs of trajectories have the same vertical distance.

It follows from a minimum property of quadratic differentials with closed trajectories (and hence also for polygon differentials) that

$$\frac{1}{K}\,\|\varphi_n\| \leq \|\psi_n\| \leq K\,\|\varphi_n\|$$

(for a proof see e.g. [3]). We can now show

Theorem 5.2. *The sequence* (ψ_n) *converges locally uniformly to a differential* $\psi \neq 0$. *Two boundary points* p *and* q *of* D *are connected by a totally regular trajectory* α *of* φ *if and only if the image points* $p' = f(p)$ *and* $q' = f(q)$ *are connected by a totally regular trajectory* α' *of* ψ. *Corresponding pairs of totally regular trajectories* α, $\tilde{\alpha}$ *and* α', $\tilde{\alpha}'$ *have the same vertical distance (measured in terms of* φ *and* ψ *respectively).*

Notice that in the proof we only use the fact that the sequence (φ_n) tends to φ locally uniformly and has uniformly bounded norm.

Proof. (1) Let α be a totally regular trajectory of φ in D, with end points p and q. Let $p' = f(p)$, $q' = f(q)$. Choose a totally regular trajectory α_n of φ_n, for each n, such that $\alpha_n \to \alpha$ for $n \to \infty$. (It is enough to have a sequence of points $z_n \in \alpha_n$ tending to a point $z \in \alpha$.) The end points p_n, q_n of α_n tend to the end points p and q of α. Denote by α_n' the totally regular trajectory of ψ_n which has been assigned to α_n. By assumption, the distances of neighboring vertices of the polygons P_n tend to zero, and by the continuity of f on ∂D the same is true for the polygons P_n'. Therefore the end points p_n', q_n' of the trajectories α_n' tend to p' and q' respectively. (Note that in general $p_n' \neq f(p_n)$, $q_n' \neq f(q_n)$; for polygon differentials this is only true at the vertices.)

We claim that every boundary point ζ' of D which is different from p' and q' has an ε-neighborhood $U_\varepsilon(\zeta')$ which is free from α_n for all n.

Figure 12.

Assume the contrary (Figure 12). Then, there is a point ζ' and a subsequence of curves α_n' (which we call (α_n') again, to avoid double indices) with some $z_n' \in \alpha_n'$, $z_n' \to \zeta'$. We choose a second totally regular trajectory $\tilde{\alpha}$, separating $\zeta = f^{-1}(\zeta')$ from α, with end points $\tilde{p}$ and $\tilde{q}$. Let $\tilde{\alpha}_n$ with end points $\tilde{p}_n$ and $\tilde{q}_n$ be totally regular trajectories of φ_n, with $\tilde{\alpha}_n \to \tilde{\alpha}$ for $n \to \infty$. By Theorem 4.3 (1) the vertical distance $h_{\varphi_n}[\alpha_n, \tilde{\alpha}_n]$ tends to $h_\varphi[\alpha, \tilde{\alpha}] > 0$ for $n \to \infty$. Since by definition

$$h_{\varphi_n}[\alpha_n, \tilde{\alpha}_n] = h_{\psi_n}[\alpha_n', \tilde{\alpha}_n'],$$

the vertical distances of the pairs α'_n, $\tilde{\alpha}'_n$ are bounded away from zero. On the other hand, an application of Lemma 1.3 to the point ζ' shows that the ψ_n-distance of the trajectories becomes arbitrarily small (see Figure 12, right side). This is a contradiction and thus proves the assertion.

Figure 13.

(2) The differentials ψ_n have uniformly bounded norm. Therefore there exists a subsequence (which we denote by (ψ_n) again) which converges locally uniformly to a holomorphic quadratic differential ψ. We claim that $\psi \neq 0$, in other words the sequence (ψ_n) does not degenerate. To this end we draw a cross cut τ' of the disk D which separates the two pairs of points p', $\tilde{p}'$ and q', $\tilde{q}'$ (Figure 13). We conclude from (1) that the trajectories α'_n, $\tilde{\alpha}'_n$ cut τ' in a compact subinterval. If $\psi_n \to 0$ locally uniformly in D, the ψ_n-distance

$$d_{\psi_n}[\alpha'_n, \tilde{\alpha}'_n] \geq h_{\psi_n}[\alpha'_n, \tilde{\alpha}'_n]$$

tends to zero, a contradiction.

5.3. The next step is to show, that the points p' and q' are connected by a horizontal geodesic of ψ.

Lemma 5.3. *Let (ψ_n) be a sequence of holomorphic quadratic differentials in D which tends locally uniformly to a differential $\psi \neq 0$. Assume that the points z'_n, z''_n are connected by a ψ_n-geodesic γ_n which is contained in a disk $D_r : |z| \leq r < 1$ for all n. If $z'_n \to z'$, $z''_n \to z''$, then, z' and z'' are connected by a ψ-geodesic γ in D_r and $\gamma_n \to \gamma$ uniformly in the Euclidean metric (Figure 14).*

Proof. The lengths of the ψ_n-geodesics γ_n are bounded, $|\gamma_n|_{\psi_n} \leq M$, say. This is so, because the ψ_n are bounded in D_r and therefore the ψ_n-distance of any two points in D_r is bounded. Choose $d > 0$ such that any two points z_1,

Figure 14.

z_2 in D_r with ψ-distance $\leq d$ can be joined by a ψ-geodesic, not necessarily in D_r (see [1, Theorem 8.1]). Fix N and subdivide each γ_n into N pieces of equal ψ_n-length less than $\frac{1}{2}d$. Let $z_{n0} = z'_n$, $z_{n1}, \ldots, z_{nN} = z''_n$ be the subdividing points. By passing to a subsequence we can assume that $z_{nk} \to z_k \in D_r$ for every $k = 0, 1, \ldots, N$, $z_0 = z'$, $z_N = z''$. Clearly, the ψ_n-geodesic connection of z_{nk} and $z_{n,k+1}$, which is the subinterval of γ_n connecting the two points, tends to the ψ-geodesic between z_k and z_{k+1} which is therefore in D_r. Moreover, the arc z_{k-1}, z_k, z_{k+1} is the shortest ψ-connection between z_{k-1} and z_{k+1}. Therefore, altogether, the points z' and z'' are connected by a ψ-geodesic γ in D_r and $\gamma_n \to \gamma$ uniformly. Because of the uniqueness of γ, the original sequence (γ_n) converges to γ.

Figure 15.

We are now able to show that p' and q' are connected by a horizontal geodesic of ψ. Fix a double sequence of circles σ_k, $-\infty < k < \infty$, centered at p' and q' and tending to these points for $k \to \pm\infty$ respectively (Figure 15). The trajectory α'_n of ψ_n, connecting $p'_n \to p'$ with $q'_n \to q'$ has (for large enough n) a last intersection with σ_k ($k < 0$) and a first one with σ_l ($l > 0$). The subinterval of α'_n between these two points is denoted by $\overline{\alpha}_n$. Because of (1) there is a subsequence of the sequence $(\overline{\alpha}_n)$ which tends uniformly to a ψ-geodesic between two points of σ_k and σ_l in D. Passing on to σ_{k-1}, σ_{l+1} etc. we end up with a diagonal sequence which tends uniformly in D, to a geodesic α' of ψ. It contains sequences of points (on the σ_k, σ_l) tending to p' and q' respectively. Therefore it connects the two points. Since all the α'_n are horizontal with respect to ψ_n, and the sequence (ψ_n) tends locally uniformly to ψ, α' itself must be a horizontal geodesic of ψ. Let α and $\tilde{\alpha}$ be totally regular trajectories of φ. It follows from Theorem 4.3 (1) that the corresponding horizontal geodesics α' and $\tilde{\alpha}'$ have the same vertical distance.

Figure 16.

5.4. It is now easy to see that α' is in fact a totally regular trajectory of ψ. Assume, first, that α' is not regular. Then, it passes through a zero w of ψ, where at least one other trajectory γ' starts, which can be continued as a horizontal geodesic to a boundary point r' of D (Figure 16, right side). Of course, r' is different from p' and q', and hence $r = f^{-1}(r')$ is different from p and q. Since α is totally regular, it is approximated by totally regular trajectories of φ. Choose such a trajectory $\tilde{\alpha}$, with end points $\tilde{p}$ and $\tilde{q}$ separating r from p and q (Figure 16 left side). The points $\tilde{p}'$, $\tilde{q}'$ are connected by a horizontal geodesic $\tilde{\alpha}'$. By the invariance of heights we have

$$h_\psi[\alpha', \tilde{\alpha}'] = h_\varphi[\alpha, \tilde{\alpha}] > 0.$$

This is impossible, because α' and $\tilde{\alpha}'$ necessarily belong to the same component of the horizontal graph of ψ and hence have vertical distance zero. This proves that α' is regular.

Now α is assumed to be totally regular. It can therefore be approximated, from either side, by a sequence of totally regular trajectories α_n of φ. Their end points p_n and q_n tend to p and q respectively. Therefore, the trajectories α'_n of

ψ have end points $p'_n \to p'$, $q'_n \to q'$. But then, it is easy to see that the sequence (α'_n) tends itself to α'. Otherwise, there would exist a vertical interval β' with initital point on α', pointing to α'_n but disjoint from all α'_n. We would then have a regular trajectory $\alpha'' \neq \alpha'$ with end points p' and q', contradicting the uniqueness of geodesic connections of boundary points. Since the approximation can be performed from either side, α' is totally regular.

Conversely: Assume that p' and q' are connected by a totally regular trajectory α' of ψ. Then, $p = f^{-1}(p')$ and $q = f^{-1}(q')$ are connected by a totally regular trajectory α of φ.

To see that, we use the same approximating sequences of polygon differentials $\varphi_n \to \varphi$, $\varphi_n \leftrightarrow \psi_n$ $\psi_n \to \psi$. We now just reverse the argument. Let (α'_n) be a sequence of totally regular trajectories of the differentials ψ_n which tends to α'. (It suffices to choose a sequence of points $w_n \to w \in \alpha'$ such that w_n lies on a totally regular trajectory α'_n of ψ_n.) Let α_n be the trajectory of φ_n which corresponds to α'_n. Then, the end points p_n and q_n of α_n tend to p and q respectively, because the end points p'_n and q'_n tend to p' and q' respectively. The argument is a repetition of the last part of the earlier one, showing that p and q are connected by a totally regular trajectory α of φ. Because the two points p' and q' can only be connected by one trajectory, α' is the one corresponding to α. We therefore have a 1-1- correspondence of the totally regular trajectories of φ and those of ψ.

5.5. It follows readily that φ and ψ generate the same vertical distance for *all* corresponding pairs of boundary points. For, let $r' = f(r)$, $s' = f(s)$. If $h_\varphi[r,s] = 0$, there are no totally regular trajectories of φ separating r and s. Since the totally regular trajectories of φ and ψ correspond to each other, there are no totally regular trajectories of ψ separating r' and s'. Thus, $h_\varphi[r',s'] = 0$. The same argument goes in the reversed direction.

Let $h_\varphi[r,s] > 0$. Let σ be a totally regular trajectory separating r and s. Then, the totally regular trajectory σ' separates r' and s', and conversely. Moreover, the vertical distances of corresponding pairs of totally regular trajectories are the same. Therefore, by Theorem 3.4,

$$h_\varphi[r,s] = \sup h_\varphi[\sigma,\tau] = \sup h_\psi[\sigma',\tau'] = h_\psi[r',s'],$$

where σ and τ are running over all totally regular trajectories of φ which separate r and s.

Theorem 5.5. *Let f be a quasisymmetric mapping of ∂D onto itself. Then, to every holomorphic quadratic differential φ of finite norm corresponds a differential ψ which satisfies*

$$h_\varphi[r,s] = h_\psi[r',s'] \qquad \text{for all } r,s \in \partial D,$$

with $r' = f(r)$, $s' = f(s)$.

5.6. The following uniqueness theorem is based on the vertical distance of pairs of boundary points.

Theorem 5.6 (Uniqueness)**.** *Let φ and $\tilde{\varphi}$ be holomorphic quadratic differentials of finite norm in the disk D. Assume that the vertical distance of any pair of boundary points p, q is the same with respect to φ as with respect to $\tilde{\varphi}$. Then, $\varphi = \tilde{\varphi}$.*

Clearly, $\varphi = 0$ if and only if all its heights are zero. For, if $\varphi \neq 0$, it has a regular vertical trajectory β, connecting two points r and s. The vertical distance of r and s is equal to the length of β, which is

$$|\beta|_\varphi = \int_\beta |\varphi(z)|^{1/2} |dz| > 0.$$

Let $\varphi \neq 0$. Choose a denumerable dense set of regular horizontal trajectories α_ν of φ. By the vertical strip S_ν based on α_ν we mean the domain swept out by the set of vertical trajectories which intersect α_ν. Progressive cancelling of intersections leads to a system $\{S_\nu\}$ of non overlapping strips which cover D up to the critical points of φ (for details see [1, Theorem 19.2]).

Let β be a regular vertical trajectory of φ connecting the boundary points r and s. Then, with

$$\tilde{w} = \tilde{u} + i\tilde{v} = \tilde{\Phi}(z) = \int^z \sqrt{\tilde{\varphi}(z)}\, dz$$

we find

$$\int_\beta |d\tilde{v}| \geq h_{\tilde{\varphi}}[r,s] = h_\varphi[r,s] = \int_\beta dv,$$

where $w = u + iv = \Phi(z) = \int^z \sqrt{\varphi(z)}\, dz$. The strips S_ν are oriented in the increasing direction of v, which is well determined on each individual S_ν.

We now introduce the parameter w in the individual strips S_ν. We then get, by first integrating the above inequality over u in each S_ν and then summing up,

$$\iint_{\Sigma S_\nu} \left|\frac{\partial \tilde{v}}{\partial v}\right| du\, dv \geq \iint_{\Sigma S_\nu} du\, dv = \|\varphi\|.$$

The Schwarz inequality leads to

$$\begin{aligned} \|\varphi\|^2 &\leq \|\varphi\| \iint_{\Sigma S_\nu} \left(\frac{\partial \tilde{v}}{\partial v}\right)^2 du\, dv \leq \|\varphi\| \iint_{\Sigma S_\nu} \left\{\left(\frac{\partial \tilde{v}}{\partial u}\right)^2 + \left(\frac{\partial \tilde{v}}{\partial v}\right)^2\right\} du\, dv \\ &= \|\varphi\| \cdot \left\|\tilde{\Phi}'\right\|^2 = \|\varphi\| \cdot \|\tilde{\varphi}\|. \end{aligned}$$

This gives $\|\varphi\| \leq \|\tilde{\varphi}\|$, and by reversing the argument, $\|\varphi\| = \|\tilde{\varphi}\|$. But then, $\partial\tilde{v}/du \equiv 0$, and hence $\tilde{v} = \tilde{v}(v)$. From the Schwarz inequality, applied to the union of the strips S_ν in terms of the parameter w we find that $\tilde{v} = a \cdot v + b$. Now the equality of the norms gives $a = \pm 1$, hence $\tilde{\Phi} = \pm\Phi +$ const, and finally $\tilde{\varphi} = \varphi$, as claimed.

Corollary 5.6. *Let f be a K-quasiconformal selfmapping of the disk D and let ψ be the induced image by heights of φ. Then,*

$$\frac{1}{K}\|\varphi\| \leq \|\psi\| \leq K\|\varphi\|.$$

Proof. Let (φ_n) be a sequence of polygon differentials approximating φ in norm, $\|\varphi_n - \varphi\| \to 0$. The image by heights ψ_n of φ_n satisfies

$$\frac{1}{K}\|\varphi_n\| \leq \|\psi_n\| \leq K\|\varphi_n\|.$$

Since $\psi_n \to \psi$ locally uniformly, we have

$$\|\psi\| \leq \liminf_{n\to\infty} \|\psi_n\| \leq K\|\varphi\|.$$

Let the sequence of polygon differentials $(\tilde{\psi}_n)$ approximate ψ in norm, and let $\tilde{\varphi}_n$ be the image by heights of $\tilde{\psi}_n$. Let $\tilde{\varphi}$ be the locally uniform limit of the sequence $(\tilde{\varphi}_n)$. It produces the same heights as ψ, and thus because of the uniqueness theorem $\varphi = \tilde{\varphi}$. We find, as above,

$$\|\varphi\| = \|\tilde{\varphi}\| \leq K\|\psi\|,$$

which completes the double inequality.

5.7. The definition and the uniqueness of the mapping by heights is based on the vertical distance of pairs of boundary points, whereas for the existence we use totally regular trajectories and their vertical distance. The two properties are in fact equivalent.

Theorem 5.7. *Let f be a qc selfmapping of the unit disk D. Let φ and ψ be holomorphic quadratic differentials of finite norm in D. Then, the following two properties are equivalent:*

(1) $$h_\psi[r', s'] = h_\varphi[r, s], \qquad r' = f(r), s' = f(s),$$

for all $r, s \in D$.

(2) a) *p and $q \in \partial D$ are connected by a totally regular trajectory α of φ if and only if $p' = f(p)$ and $q' = f(q)$ are connected by a totally regular trajectory α' of ψ.*

b) *If α, $\tilde{\alpha}$ and α', $\tilde{\alpha}'$ are corresponding pairs of totally regular trajectories, then*

$$h_\varphi[\alpha, \alpha'] = h_\psi[\alpha', \tilde{\alpha}'].$$

We call ψ the image by heights of φ and set $\psi = H_f(\varphi)$. H_f is the mapping by heights induced by the quasisymmetric mapping $f \mid \partial D$.

Proof. It has been shown in Section 5.5 that (2) implies (1).

Let (1) hold. By Theorem 5.2 we construct a quadratic differential $\tilde{\psi}$ which has the properties (2). Again by Section 5.5 $\tilde{\psi}$ satisfies (1). The Uniqueness Theorem 5.6 shows that $\tilde{\psi} = \psi$. Therefore ψ has the properties (2).

5.8. Every quasisymmetric mapping of ∂D onto itself induces a mapping by heights of the space of holomorphic quadratic differentials of finite norm. We now show that conversely, if a homeomorphism f of ∂D onto itself induces a mapping by heights with a Lipschitz condition for the norm, then it is quasisymmetric.

Theorem 5.8. *Let $f\colon \partial D \to D'$ be a homeomorphism. Assume that there is a bijection $H_f\colon \varphi \to \psi = H_f(\varphi)$ of the space of holomorphic quadratic differentials of finite norm onto itself satisfying*

$$\frac{1}{K}\,\|\varphi\| \le \|\psi\| \le K\,\|\varphi\|$$

for some constant $K \ge 1$. Then, f is quasisymmetric.

Figure 17.

Proof. Choose four points ζ_1, ζ_2, ζ_3, ζ_4 in this order on ∂D and let $\Phi\colon z \to \zeta^* = \Phi(z)$ be the conformal mapping of the quadrilateral $Q = (D; \zeta_1, \ldots, \zeta_4)$ onto a rectangle R with side lengths a and b (Figure 17).

The square of the derivative of Φ, i.e., $\varphi = (d\Phi/dz)^2$ is a quadratic differential associated with the given quadrilateral Q. All its trajectories in D are totally regular, and the vertical distance of any two boundary points is the Euclidean vertical distance of the corresponding boundary points of the rectangle R. Let $\psi = H_f(\varphi)$. All its trajectories in D' are again totally regular. Therefore it has no zeroes and the function $\Psi = \int \sqrt{\psi}$ is a conformal mapping onto a domain

S shaped in Figure 17. The trajectories are the horizontal crosscuts, and the Euclidean vertical distances in R and in S are the same. They connect boundary points corresponding by f. This serves as an illustration for the mapping by heights.

However, for our present purposes, we map the quadrilateral $Q' = (D'; \zeta_1', \dots, \zeta_4')$, $\zeta_i' = f(\zeta_i)$, conformally onto a rectangle R'. We double R' by reflection on one of its vertical sides and identify the two free vertical sides of the new rectangle R'' to form a cylinder. The quadratic differential ψ has a representation in terms of the parameter ζ of the R'-plane. By the Dirichlet principle, applied to the cylinder, we find that

$$a'b' \le \|\psi\| \le K\,\|\varphi\| = Kab.$$

We end up, because of $b' = b$, with

$$\frac{a'}{b'} \le K\frac{a}{b},$$

which is the module inequality for inscribed quadrilaterals. It is well known that the inequality proves the quasisymmetry of f.

5.9. The same considerations which served to show the existence of an image by heights can be used to prove the weak convergence of these images.

Theorem 5.9. *Let f be a quasiconformal selfmapping of D and let $\psi = H_f(\varphi)$ be the image by heights of φ. Assume that the sequence (φ_n) tends to $\varphi \neq 0$ locally uniformly and has uniformly bounded norm. Then, the images by heights $\psi_n = H_f(\varphi_n)$ converge to ψ in the same sense.*

Proof. (This is a repetition of the proof of Theorem 5.2) (1) From the norm inequality $\|\psi_n\| \le K\,\|\varphi_n\|$ and the boundedness of the sequence $(\|\varphi_n\|)$ we conclude that the sequence $(\|\psi_n\|)$ is bounded. Therefore there exists a subsequence (ψ_{n_i}) which converges locally uniformly to a differential $\tilde{\psi}$ of finite norm.

(2) Let p and q be the end points of a totally regular trajectory α of φ. Choose, for each n, a totally regular trajectory α_n of φ_n, such that $\alpha_n \to \alpha$ in the Euclidean metric (it suffices to have $z_n \in \alpha_n$, $z_n \to z \in \alpha$). Denote the end points of α_n by p_n and q_n respectively. We have $p_n \to p$, $q_n \to q$, and hence $p_n' = f(p_n) \to p' = f(p)$, $q_n' = f(q_n) \to q' = f(q)$. The points p_n', q_n' are connected by a totally regular trajectory α_n' of ψ_n. Using these α_n' it follows as before that $\tilde{\psi} \neq 0$.

(3) In the next step we show, as in the earlier proof, that the sequence (α_n') actually converges pointwise to a horizontal geodesic γ of $\tilde{\psi}$ with end points p' and q'.

(4) Let α and $\tilde{\alpha}$ be two totally regular trajectories of φ, with end points p, q and $\tilde{p}$, $\tilde{q}$ respectively. The points $\tilde{p}' = f(\tilde{p})$, $\tilde{q}' = f(\tilde{q})$ are connected by a horizontal geodesic $\tilde{\gamma}$ of $\tilde{\psi}$. It follows from Theorem 4.3 that

$$h_\varphi[\alpha, \tilde{\alpha}] = h_{\tilde{\psi}}[\gamma, \tilde{\gamma}].$$

From this we easily conclude that γ is actually a regular trajectory $\tilde{\alpha}'$ of $\tilde{\psi}$.

(5) We then show, again as before, that $\tilde{\alpha}'$ is totally regular. Theorem 3.4 then shows that the heights (vertical distances of pairs of boundary points) with respect to ψ and $\tilde{\psi}$ are the same, and by Theorem 5.6 $\tilde{\psi} = \psi$. A standard argument then gives $\psi_n \to \psi$ locally uniformly for the original sequence. It seems reasonable to expect that if the convergence $\varphi_n \to \varphi$ is in norm, so is the convergence $\psi_n \to \psi$.

6. Extremal Teichmüller mappings

6.1. A Teichmüller mapping $f\colon D \to D'$ is a quasiconformal mapping with a complex dilatation of the form $\kappa = k\overline{\varphi}/|\varphi|$, where φ is a holomorphic quadratic differential and k a real constant $0 < k < 1$ (we do not admit conformal mappings, where $k = 0$). It is well known that the mapping generates a holomorphic quadratic differential ψ in the image domain and that in the Φ- and Ψ-planes the mapping is represented by a horizontal stretching by the factor $K = (1+k)/(1-k)$ (Figure 18).

Figure 18.

Setting $z = x + iy$, $w = u + iv$ for the variables in the Φ- and Ψ-plane respectively the mapping f has locally and away from the zeroes the respresentation

$$f = \Psi^{-1} \circ F \circ \Phi, \qquad w = F(z) = Kx + iy.$$

The two differentials φ and ψ are called the Teichmüller differentials associated with f. One can read off several properties of f from the figure, in particular:

(1) The trajectories of φ (which are the horizontals in the Φ-plane) are taken into the trajectories of ψ. Vertical distances of pairs of trajectories stay the same.

(2) The height of a curve γ with respect to φ, which is the Euclidean height in the Φ-plane, is the same as the height of its image with respect to ψ. Therefore the vertical distances of the pairs of boundary points in the z-plane and their images in the w-plane are the same (in terms of φ and ψ respectively). This means that ψ is the image by heights of φ.

(3) The vertical trajectories of φ (which are the verticals in the Φ-plane) are taken into the vertical trajectories of ψ. The horizontal distance of corresponding pairs of vertical trajectories is multiplied by K.

(4) The norm of φ, which is the Euclidean area in the Φ-plane, is multiplied by K: $\|\psi\| = K\,\|\varphi\|$.

It is also known that a Teichmüller mapping associated with a quadratic differential of finite norm is uniquely extremal for its boundary values.

6.2. The mapping by heights permits to formulate the existence problem for Teichmüller mappings associated with quadratic differentials of finite norm as an extremum problem for norms.

Theorem 6.2. *Let $f\colon \partial D \to \partial D'$ be a quasisymmetric mapping, and let H_f be the associated mapping by heights. Then, f allows for an extension to a Teichmüller mapping associated with the holomorphic quadratic differentials φ and ψ of finite norm if and only if the quotient $\|\psi\| \,/\, \|\varphi\|$, $\psi = H_f(\varphi)$, assumes a maximum greater than one. The differentials φ_0, $\psi_0 = H_f(\varphi_0)$ of the maximum are the Teichmüller differentials of the mapping and the maximum is its dilatation K_0.*

Proof. Let

$$L = \max\left\{\frac{\|\psi\|}{\|\varphi\|}, \frac{\|\varphi\|}{\|\psi\|}\right\} > 1$$

and suppose that it is assumed by φ_0, $\|\psi_0\| = L\cdot\|\varphi_0\|$. Let β be an open, regular vertical interval with end points on totally regular trajectories α_1, α_2 of φ. They connect p_1, q_1 and p_2, q_2 respectively on ∂D. The domain swept out by the

Figure 19.

trajectories through β is called the horizontal strip S. Let α be a variable totally regular trajectory through β.

The images α_1', α_2' of α_1 and α_2, connecting $p_1' = f(p_1)$, $q_1' = f(q_1)$ etc., form the upper and lower boundaries of a horizontal strip S' in D'. We now apply the conformal mappings Φ, Ψ to the strips S, S' respectively. Using the same notations downstairs, we arrive at Figure 19. Note that the widths of the strips S and S' as well as the heights of α and its image α' are the same in the two planes.

We now apply the mapping by heights to $-\psi$ in the reversed direction, i.e. we look at $\tilde{\varphi} = (H_f)^{-1}(-\psi) = H_{f^{-1}}(-\psi)$. We partition D into non overlapping horizontal strips S_i and represent $\tilde{\varphi}$ in each strip in terms of the z-parameter (for details see [1, Theorem 19.2]). We have $\tilde{w} = \tilde{u} + i\tilde{v} = \tilde{\Phi} = \int \sqrt{\tilde{\varphi}}$, and the computation of $\|\tilde{\varphi}\|$ looks as follows. We first integrate $|d\tilde{v}|$ along α in the z-plane and remember that its heights are the same as the heights of $-\psi$, which are the horizontal lenghts of ψ. We get

$$\int_\alpha |d\tilde{v}| = \int_\alpha \left|\frac{\partial \tilde{v}}{dx}\right| dx \geq h_{\tilde{\varphi}}[p,q] = h_{-\psi}[p',q'] = \int_{\alpha'} du.$$

Since φ and ψ have the same vertical distances, we get, in S and S', $dy = dv$.

Therefore integration over y in S and over v in S' gives

$$\iint_S \left|\frac{\partial \tilde{v}}{dx}\right| dx\, dy \geq \iint_{S'} du\, dv.$$

A summation over all S_i and subsequent application of the Schwarz inequality yields

$$\|\psi\| = \iint_{\Sigma S_i'} du\, dv \leq \iint_{\Sigma S_i} \left|\frac{\partial \tilde{v}}{\partial x}\right| dx\, dy,$$

$$\|\psi\|^2 \leq \left\{\iint_{\Sigma S_i} \left|\frac{\partial \tilde{v}}{\partial x}\right| dx\, dy\right\}^2 \leq \iint_{\Sigma S_i} dx\, dy \cdot \iint_{\Sigma S_i} \left(\frac{\partial \tilde{v}}{\partial x}\right)^2 dx\, dy$$
$$\leq \|\varphi\| \iint_{\Sigma S_i} \left\{\left(\frac{\partial \tilde{v}}{\partial x}\right)^2 + \left(\frac{\partial \tilde{v}}{\partial y}\right)^2\right\} dx\, dy = \|\varphi\| \cdot \|\tilde{\varphi}\|.$$

Remember that $\|\psi\| = L\,\|\varphi\|$ with maximal L. Therefore

$$L\,\|\psi\| \leq \|\tilde{\varphi}\| \leq L\,\|-\psi\| = L\,\|\psi\|\,,$$

which gives $\|\tilde{\varphi}\| = L\,\|\psi\|$. We thus must have equality all over. This implies $\partial\tilde{v}/\partial y \equiv 0$. Therefore the orthogonal trajectories of φ (locally the curves $x =$const) are the trajectories of $\tilde{\varphi}$ (locally the curves $\tilde{v} =$const). Therefore $\tilde{\varphi} = c(-\varphi)$, with a positive constant c. From $\|\tilde{\varphi}\| = L\,\|\psi\| = L^2\,\|\varphi\|$ we finally get $\tilde{\varphi} = L^2(-\varphi)$.

Since $\tilde{\varphi} = H_{f^{-1}}(-\psi)$, $H_f(\tilde{\varphi}) = H_f\big(L^2(-\varphi)\big) = -\psi$. The interpretation of this functional equation is as follows.

Let β be a totally regular vertical trajectory of φ, connecting the boundary points r, s. Then, ψ has a totally regular vertical trajectory β' connecting $r' = f(r)$ with $s' = f(s)$. The horizontal distances of β_1 and β_2, connecting r_1, s_1 and r_2, s_2 respectively, measured in terms of the differential $L^2\varphi$ is the same as the horizontal distance of β_1', β_2' connecting r_1', s_1' and r_2', s_2' respectively, measured in terms of ψ. In other words, measuring in terms of φ and ψ, the horizontal distance of corresponding pairs of totally regular vertical trajectories is multiplied by L.

So far we only have a mapping of trajectories and orthogonal trajectories. We now define a mapping of points. Let $z \in D$ be the intersection of a totally regular trajectory α of φ connecting p and q with a totally regular vertical trajectory β of φ connecting r and s. Then, their images α' and β' cut at a point w, because their pairs of end points separate each other. This is a bijection of a dense set of points of D onto a dense set of points on D'. We are actually right back at Figure 18: the completion of the mapping can be carried out in the Φ- and Ψ-planes, as horizontal stretching by the factor L of small rectangles with sides parallel to the axes. We end up with an L-quasiconformal mapping of the disk D punctured at the zeroes of φ onto D' punctured at the zeroes of ψ. It can therefore be extended to these zeroes and to the boundary. Because the end points of the totally regular horizontal and vertical trajectories are everywhere dense on ∂D, the boundary values are the given ones, and we have found an L-qc extension of the quasisymmetric boundary mapping which is a Teichmüller mapping associated with the differentials φ and $\psi = H_f(\varphi)$. Such a mapping is known to be uniquely extremal, and so is φ, if we normalize it by $\|\varphi\| = 1$. In correspondence with the usual notation we set $L = K_0$.

We have the horizontal stretching version of the Teichmüller mapping from left to right. If the maximum were taken by the quotient $\|\varphi\| / \|\psi\| = L$, we would have a horizontal stretching from the right to the left. Replacing φ by $\tilde{\varphi} = (-\varphi)/L^2$, ψ by $\tilde{\psi} = -\psi$, we have again $\|\tilde{\psi}\|/\|\tilde{\varphi}\| = L$, so we are back at the former case.

The converse is immediate. Let f be a Teichmüller mapping associated with the differentials φ in D and ψ in D', of finite norm, in the horizontal stretching version. Then, $\psi = H_f(\varphi)$, $\|\psi\| = K_0 \|\varphi\|$, with K_0 the extremal dilatation. Since the mapping by heights satisfies the double inequality

$$\frac{1}{K_0} \|\varphi\| \leq \|\psi\| \leq K_0 \|\varphi\|,$$

φ gives the maximal value of $\|\psi\| / \|\varphi\|$.

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