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DEGENERATION OF QUASICIRCLES: INNER AND OUTER RADII OF TEICHMÜLLER SPACES

John A. Velling

Brooklyn College, Department of Mathematics
Brooklyn, NY 11210, U.S.A.; jvelling@bklyn.bitnet

Abstract. A univalent function $f\colon \mathbf{D} \to \hat{\mathbf{C}}$ with Schwarzian derivative having sup norm 2 can always be normalized to be arbitrarily close to $\log(1+z)/(1-z)$ on a given compact subset of $\mathbf{D}$. Using this, necessary and sufficient conditions for the Bers embedding of a Teichmüller space (centered at a given surface) to have minimum possible inner radius are established in terms of hyperbolic geometry of the given surface. These conditions are the existence of points with arbitrarily large injectivity radius or simple closed geodesics with arbitrarily wide geodesic annular neighborhoods.

1. Statements of the main results

Let Ω denote a domain in the complex plane. If $f\colon \Omega \to \hat{\mathbf{C}}$ is a nonconstant meromorphic function we let

$$\mathscr{S}_f(z) = \frac{f'''}{f'} - \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2 \tag{1.1}$$

denote the Schwarzian derivative of f. This is analytic on Ω if and only if f is locally univalent. It satisfies the transformation law

$$\mathscr{S}_{\phi\circ f\circ\psi}(z) = \mathscr{S}_f\big(\psi(z)\big)\cdot\psi'(z)^2 + \mathscr{S}_\psi(z) \tag{1.2}$$

for $\phi \in$ Möb, the group of orientation preserving Möbius transformations of $\hat{\mathbf{C}}$. Furthermore, $\mathscr{S}_f \equiv 0$ if and only if $f \in$ Möb.

Now let $\mathbf{D}$ denote the unit disk in $\mathbf{C}$. Nehari [19] showed that if $f\colon \mathbf{D} \to \hat{\mathbf{C}}$ satisfies

$$\|\mathscr{S}_f\|_\infty = \sup_{z\in\mathbf{D}} \frac{(1-|z|^2)^2}{4}\big|\mathscr{S}_f(z)\big| \le \frac{1}{2} \tag{1.3}$$

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then f is univalent. We let $\mathrm{M\ddot{o}b}(\mathbf{D})$ denote the subgroup of Möb preserving $\mathbf{D}$, and observe, from (1.2), that if $\psi \in \mathrm{M\ddot{o}b}(\mathbf{D})$, then $\|\mathscr{S}_{f\circ\psi}\|_\infty = \|\mathscr{S}_f\|_\infty$. Also note from (1.2) that with respect to the $\mathrm{M\ddot{o}b}(\mathbf{D})$ changes of coordinates on $\mathbf{D}$, $\mathscr{S}_f$ is a quadratic differential on $\mathbf{D}$. We denote by $\mathrm{Q}^\infty(\mathbf{D})$ the space of holomorphic quadratic differentials on $\mathbf{D}$ which are bounded in the sense of (1.3). This is an infinite dimensional complex Banach space.

Ahlfors and Weill [2] strengthened Nehari's result by showing that if $\|\mathscr{S}_f\|_\infty < \frac{1}{2}$ then $f(\mathbf{D})$ is a quasidisk, *i.e.* f extends continuously to $\mathbf{S}^1 = \partial\mathbf{D}$, and $f(\mathbf{S}^1)$ is a quasicircle. In [6], Gehring and Pommerenke showed that if $\|\mathscr{S}_f\|_\infty = \frac{1}{2}$ then f extends continuously to $\mathbf{S}^1$ and either $f(\mathbf{D})$ is a Jordan domain ($f(\mathbf{S}^1)$ is a Jordan curve) or else $f(\mathbf{D})$ is a region in $\hat{\mathbf{C}}$ bounded between two circles tangent at a point. Herein the first order of business is to identify the complete obstruction to $f(\mathbf{S}^1)$ being a quasicircle under the hypothesis that $\|\mathscr{S}_f\|_\infty = \frac{1}{2}$.

If $z \in \mathbf{C}$, let $\mathrm{Re}\,(z)$ and $\mathrm{Im}\,(z)$ denote the real and imaginary parts of z. The univalent function

$$f^*(z) = \log\frac{1+z}{1-z} = 2\tanh^{-1} z \qquad (z \in \mathbf{D}) \tag{1.4}$$

satisfies $(1-z^2)^2\mathscr{S}_{f^*}(z) \equiv 2$ and maps $\mathbf{D}$ onto the parallel strip

$$\mathscr{T} = \Big\{t : -\frac{\pi}{2} < \mathrm{Im}\,(t) < \frac{\pi}{2}\Big\}. \tag{1.5}$$

Any region between two circles tangent at a point can be mapped by a Möbius transformation to $\mathscr{T}$. If $f\colon \mathbf{D} \to \hat{\mathbf{C}}$ has $\|\mathscr{S}_f\|_\infty = \frac{1}{2}$ we will show that this map f^* embodies the complete obstruction to $f(\mathbf{S}^1)$ being a quasicircle.

Theorem 1. *If $f\colon \mathbf{D} \to \hat{\mathbf{C}}$ has $\|\mathscr{S}_f\|_\infty = \frac{1}{2}$ and $f(\mathbf{S}^1)$ is not a quasicircle, then there exist sequences of Möbius transformations $\{\psi_n \in \mathrm{M\ddot{o}b}(\mathbf{D})\}_{n=1}^\infty$ and $\{\phi_n \in \mathrm{M\ddot{o}b}\}_{n=1}^\infty$ such that the maps*

$$f_n = \phi_n \circ f \circ \psi_n\colon \mathbf{D} \to \hat{\mathbf{C}} \tag{1.6}$$

converge to f^* uniformly on compacta.

This is proved in Section 2. In a similar vein we establish in Section 3

Theorem 2. *If $\{f_n\colon \mathbf{D} \to \hat{\mathbf{C}}\}_{n=1}^\infty$ is a sequence of locally univalent functions such that no f_n is univalent, $\|\mathscr{S}_{f_n}\|_\infty < \infty$, and $\|\mathscr{S}_{f_n}\|_\infty \searrow \frac{1}{2}$ as $n \to \infty$, then there exist sequences of Möbius transformations, $\big\{\psi_n \in \mathrm{M\ddot{o}b}(\mathbf{D})\big\}_{n=1}^\infty$ and $\{\phi_n \in \mathrm{M\ddot{o}b}\}_{n=1}^\infty$, and a sequence $\big\{r_n \in (0,1)\big\}_{n=1}^\infty$ with $r_n \to 1$ as $n \to \infty$, such that the maps given by*

$$g_n(z) = \phi_n \circ f_n\big(r_n\psi_n(z)\big) \tag{1.7}$$

converge to f^* uniformly on compacta.

The author does not know whether the r_n are actually needed in this theorem.

We apply Theorem 1 to a study of Teichmüller spaces to obtain several rather immediate results. Theorems 3 and 4 below are essentially corollaries of Theorem 1 and known results. As such they are presented without formal proof.

Let $\mathscr{R}$ be a hyperbolic Riemann orbifold, *i.e.* if $\mathbf{H}^2$ denotes two dimensional hyperbolic space with curvature $\equiv -1$ then $\mathscr{R} = \mathbf{H}^2/\Gamma$ where Γ is a discrete group of isometries of $\mathbf{H}^2$. Giving $\mathbf{D}$ the metric $ds = 2\,|dz|/\bigl(1-|z|^2\bigr)$ makes $\mathbf{D}$ a conformal model of $\mathbf{H}^2$. In this case the group of orientation perserving isometries of $\mathbf{H}^2$ is precisely $\mathrm{Möb}(\mathbf{D})$. Discrete subgroups of $\mathrm{Möb}(\mathbf{D})$ are called *Fuchsian* groups. Let $\mathrm{Q}^\infty(\mathbf{D};\Gamma)$ denote the subspace of $\mathrm{Q}^\infty(\mathbf{D})$ which is Γ-invariant, *i.e.* $A \in \mathrm{Q}^\infty(\mathbf{D};\Gamma)$ if and only if $A \in \mathrm{Q}^\infty(\mathbf{D})$ and

$$A\bigl(\gamma(z)\bigr)\bigl(\gamma'(z)\bigr)^2 = A(z) \qquad \text{for all } z \in \mathbf{D},\ \gamma \in \Gamma. \tag{1.8}$$

We will need the following equivalence, established but not stated in this form by Nakanishi and Yamamoto in [18].

Theorem. [18] *Let $\{\Gamma_n\}_{n=1}^\infty$ be a sequence of Fuchsian groups. Then for some $\alpha \in \mathbf{C}$ (and hence for any $\alpha \in \mathbf{C}$) there exists a sequence $\bigl\{\mathscr{S}_n \in \mathrm{Q}^\infty(\mathbf{D};\Gamma_n)\bigr\}_{n=1}^\infty$ converging uniformly to $\alpha\,dz^2/(1-z^2)^2$ on compacta in $\mathbf{D}$ if and only if the sequence of Riemann surfaces $\{\mathscr{R}_n = \mathbf{D}/\Gamma_n\}_{n=1}^\infty$ has the property that, for given $r > 0$, eventually any $\mathscr{R}_n$ contains either an isometrically embedded geodesic ball of radius r or a collar of width r about a simple closed geodesic.*

In this case we say that the $\mathscr{R}_n$ have either larger and larger balls or longer and longer tubes.

Let $\mathrm{Q}^\infty(\mathscr{R})$ denote the complex Banach space of bounded quadratic differentials on $\mathscr{R}$. If $\mathscr{R} = \mathbf{D}/\Gamma$ then this is canonically isomorphic to $\mathrm{Q}^\infty(\mathbf{D};\Gamma)$. The Bers embedding $B_{\mathscr{R}}\colon \mathbf{T}(\mathscr{R}) \to \mathrm{Q}^\infty(\mathscr{R})$ maps the Teichmüller space $\mathbf{T}(\mathscr{R})$ of equivalence classes of marked Riemann surfaces quasiconformally equivalent to $\mathscr{R}$ into $\mathrm{Q}^\infty(\mathscr{R})$. This is an injection, realizing $\mathbf{T}(\mathscr{R})$ as a bounded domain in $\mathrm{Q}^\infty(\mathscr{R})$. For background on Bers' embeddings of Teichmüller spaces, the reader is referred to [3], a beautiful expository paper on this and related topics.

The *inner radius* $i\bigl(\mathbf{T}(\mathscr{R})\bigr)$ of $\mathbf{T}(\mathscr{R})$ is the supremum of radii of balls in $\mathrm{Q}^\infty(\mathscr{R})$ centered at the origin which are contained in $B_{\mathscr{R}}\bigl(\mathbf{T}(\mathscr{R})\bigr)$. A consequence of the Ahlfors–Weill paper, [2], is that if $\mathscr{R}$ carries a hyperbolic metric of curvature $\equiv -1$ then

$$i\bigl(\mathbf{T}(\mathscr{R})\bigr) \ge \frac{1}{2} \qquad \text{whenever} \qquad B_{\mathscr{R}}(\mathbf{T}(\mathscr{R})) \ne \{0\}. \tag{1.9}$$

Throughout the paper we will assume this curvature condition.

If Γ is an elementary group then we have equality in (1.9), [8]. On the other hand it can be shown, following Gehring and Pommerenke [6], that the inequality in (1.9) is strict for cofinite Γ ([12], [15]). A rather direct consequence of Theorem 1 and the result of [18] above is

Theorem 3. *Let $\mathscr{R}$ be a hyperbolic surface such that $i\big(\mathbf{T}(\mathscr{R})\big) = \frac{1}{2}$. Then one of the following two conditions holds:*

(O_1) *for any $r > 0$, a hyperbolic geodesic ball of radius r embeds isometrically in $\mathscr{R}$, or*

(O_2) *for any $r > 0$, a collar of width r exists about some simple closed geodesic of $\mathscr{R}$.*

If $\mathscr{R}$ satisfies either (O_1) or (O_2) of Theorem 3, we will say that $\mathscr{R}$ has either *big balls*—(O_1)—or *long tubes*—(O_2). Now if Γ is dissipative on $\mathbf{S}^1$, $\mathscr{R}$ has big balls. The following proposition is thus almost immediate, given information available in [21].

Proposition 1. *If $\mathscr{R} = \mathbf{D}/\Gamma$, where the action of Γ on $\mathbf{S}^1$ is dissipative, then $i\big(\mathbf{T}(\mathscr{R})\big) = \frac{1}{2}$ and $o\big(\mathbf{T}(\mathscr{R})\big) = \frac{3}{2}$.*

As mentioned above, for any hyperbolic $\mathscr{R}$, $B_{\mathscr{R}}\big(\mathbf{T}(\mathscr{R})\big)$ is a bounded domain in $\mathrm{Q}^\infty(\mathscr{R})$. We denote by $o\big(\mathbf{T}(\mathscr{R})\big)$ the outer radius of $\mathbf{T}(\mathscr{R})$, *i.e.* the infimum of radii of balls in $\mathrm{Q}^\infty(\mathscr{R})$ centered at the origin which contain $B_{\mathscr{R}}\big(\mathbf{T}(\mathscr{R})\big)$. A well-known theorem of Kraus [11] implies that

$$o\big(\mathbf{T}(\mathscr{R})\big) \le \frac{3}{2}. \tag{1.10}$$

In [18] it was shown that $o\big(\mathbf{T}(\mathscr{R})\big) = \frac{3}{2}$ if and only if either (O_1) or (O_2) of Theorem 3 hold. In [16] and [17] it was shown that if either (O_1) or (O_2) hold on $\mathscr{R}$, then $i\big(\mathbf{T}(\mathscr{R})\big) = \frac{1}{2}$. Thus we have

Theorem 4. *For a given hyperbolic Riemann surface $\mathscr{R}$, the following are equivalent:*

1. $i\big(\mathbf{T}(\mathscr{R})\big) = \frac{1}{2}$;
2. $o\big(\mathbf{T}(\mathscr{R})\big) = \frac{3}{2}$;
3. *$\mathscr{R}$ has either big balls or long tubes.*

Some consequences of these results will be discussed in Section 4.

It follows from [18] that for univalent $f\colon \mathbf{D} \to \hat{\mathbf{C}}$ with either $\|\mathscr{S}_f\|_\infty = \frac{1}{2}$ for which $f(\mathbf{D})$ is not a quasidisk, or $\|\mathscr{S}_f\|_\infty = \frac{3}{2}$, whence $f(\mathbf{D})$ is necessarily not a quasidisk, we have a sequence $\{\psi_n \in \mathrm{M\ddot{o}b}(\mathbf{D})\}_{n=1}^\infty$ such that for some $\alpha \in \mathbf{C}$, $\mathscr{S}_{f\circ\psi_n} \to \alpha\, dz^2/(1-z^2)^2$ uniformly on compacta. We ask if the family of quadratic differentials $\alpha\, dz^2/(1-z^2)^2$ on $\mathbf{D}$ contain the complete obstruction to any univalent map $f\colon \mathbf{D} \to \hat{\mathbf{C}}$ having a quasidisk as its image. This is answered in the negative in Section 5 by giving explicit examples. We have

Theorem 5. *There exist univalent maps* $f\colon \mathbf{D} \to \hat{\mathbf{C}}$, *with* $f(\mathbf{D})$ *not a quasidisk, such that for no sequence* $\{\psi_n \in \text{Möb}(\mathbf{D})\}_{n=1}^{\infty}$ *does* $\mathscr{S}_{f\circ\psi_n} \to \alpha\, dz^2/(1-z^2)^2$ *uniformly on compacta for any* $\alpha \in \mathbf{C}$. *Thus the special collection of quadratic differentials* $\alpha\, dz^2/(1-z^2)^2$ *does not contain the full obstruction for a simply connected domain uniformized by* $\mathbf{D}$ *to be a quasidisk.*

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2. Proof of Theorem 1

A domain $\Omega \subset \mathbf{C}$ is said to have a c-accessible boundary if each w_1, $w_2 \in \partial\Omega$ can be joined by an open arc $A_{w_1,w_2} \subset \Omega$ such that

$$\min_{j=1,2} |w - w_j| \le c \operatorname{dist}(w, \partial\Omega) \qquad \text{for} \qquad w \in A_{w_1,w_2}. \tag{2.1}$$

It follows that $c \ge 1$.

Gehring and Pommerenke [5, Theorem III.2.3], [6] characterize quasicircles via domains with c-accessible boundaries. In particular, we will need their

Lemma 1 [6]. *Let* Ω *be a Jordan domain in* $\hat{\mathbf{C}}$. *That* Ω *is a quasidisk is equivalent to the existence of a constant* c *such that, for all* $\phi \in$ Möb *with* $\phi(\Omega) \subset \mathbf{C}$, *the domains* $\phi(\Omega)$ *have* c*-accessible boundaries. If* Ω *satisfies this, then* $\partial\Omega$ *is a quasicircle with constant* $M \le 2c$.

It follows that if $\Omega \subset \hat{\mathbf{C}}$ is a Jordan domain but not a quasidisk, then for any $c \ge 1$ there is some $\phi \in$ Möb (with $\phi(\Omega) \subset \mathbf{C}$) such that for some w_1, $w_2 \in \partial\phi(\Omega)$, any curve $A_{w_1,w_2} \subset \phi(\Omega)$ joining w_1 to w_2 has some w_0 on it satisfying

$$\min_{j=1,2} |w_0 - w_j| > c \operatorname{dist}\bigl(w_0, \partial\phi(\Omega)\bigr). \tag{2.2}$$

Assume now that we are given $f\colon \mathbf{D} \to \hat{\mathbf{C}}$ with $\|\mathscr{S}_f\|_\infty = \frac{1}{2}$ and $f(\mathbf{D}) = \Omega$ not a quasidisk. Let $c = n \in \mathbf{Z}_+$, and $\phi_n \in$ Möb take Ω into $\mathbf{C}$ such that the conditions of the last paragraph hold. Giving $\phi_n(\Omega)$ hyperbolic geometry, we take $w_{1,n}$, $w_{2,n} \in \partial\phi_n(\Omega)$, let $A_{w_{1,n},w_{2,n}}$ be the hyperbolic geodesic joining $w_{1,n}$ and $w_{2,n}$ in $\phi_n(\Omega)$, and $w_{0,n} \in A_{w_{1,n},w_{2,n}}$ be such that (2.2) is satisfied. Then as (2.2) is invariant under postcomposition by a complex affine transformation we may assume ϕ_n is chosen so that $w_{0,n} = 0$. Precomposing by a Möbius transformation ψ_n, we let $f_n = \phi_n \circ f \circ \psi_n$ and may assume that $f_n(-1) = w_{1,n}$, $f_n(1) = w_{2,n}$, and $f_n(0) = 0$. As we still have freedom left in our choice of ϕ_n, we may also assume that $f_n'(0) = 1$. It follows that in this case we have $\frac{1}{4} \le \operatorname{dist}(w_0, \partial\Omega) \le 1$, [13, Section 28, Satz 6], so that

$$\min_{j=1,2} |w_{j,n}| > \frac{n}{4}. \tag{2.3}$$

We now establish the following

Lemma 2. *If* $\{f_n\colon \mathbf{D} \to \mathbf{C}\}_{n=1}^{\infty}$, *normalized by* $f_n(z) = z + a_n z^2 + O(z^3)$, *is a sequence of maps satisfying* $\|\mathscr{S}_{f_n}\|_\infty \le \frac{1}{2}$, *and* $|f_n(1)|$, $|f_n(-1)| > n$, *then*

$$f_n(z) \to \tanh^{-1}(z) \tag{2.4}$$

uniformly on compacta in $\mathbf{D}$.

Immediate consequences of this are

1. $\operatorname{Re}\big(\mathscr{S}_{f_n}(0)\big) \to 2$,
2. $a_n \to 0$, and
3. the verity of Theorem 1. (The f_n given preceding Lemma 2 satisfy the criteria of Lemma 2.)

To establish Lemma 2, we use a technique of Hawley and Schiffer [7] (see also [6]). We precompose with $h\colon T = \{t \in \mathbf{C} : |\operatorname{Im}(t)| < \frac{1}{2}\pi\} \to \mathbf{D}$ given by $h(t) = \tanh(\frac{1}{2}t)$. Let $g_n = f_n \circ h$, so that

$$g_n(t) = t + a_n t^2 + O(t^3) \tag{2.5}$$

and

$$\mathscr{S}_{g_n}(t) = -\frac{1}{2} + \frac{1}{4}\big(1 - h(t)^2\big)^2 \mathscr{S}_{f_n}\big(h(t)\big). \tag{2.6}$$

Evidently $\operatorname{Re}\big(\mathscr{S}_{g_n}(t)\big) \le 0$ with equality if and only if $\mathscr{S}_{f_n}\big(h(t)\big) = 2/\big(1-h(t)^2\big)^2$. It suffices to show

$$g_n \to \mathrm{id} \tag{2.7}$$

uniformly on compacta in T.

Defining

$$v_n(t) = |g_n'(t)|^{-1/2} \qquad \text{for} \qquad t \in \mathbf{R}, \tag{2.8}$$

we note that since $g_n\colon T \to \mathbf{C}$ is univalent in T, v_n never vanishes. Also

$$\frac{v_n'}{v_n} = -\frac{1}{2}\operatorname{Re}\Big(\frac{g_n''}{g_n'}\Big), \qquad \frac{v_n''}{v_n} - \Big(\frac{v_n'}{v_n}\Big)^2 = \frac{1}{2}\operatorname{Re}\Big(\frac{d}{dt}\frac{g_n''}{g_n'}\Big) \tag{2.9}$$

so that

$$v_n''(t) = p_n(t) \cdot v_n(t) \tag{2.10}$$

where

$$p_n(t) = -\frac{1}{2}\operatorname{Re}\big(\mathscr{S}_{g_n}(t)\big) + \frac{1}{2}\operatorname{Im}\Big(\frac{g_n''(t)}{g_n'(t)}\Big)^2 \ge 0. \tag{2.11}$$

We now verify that (2.7) does indeed hold.

Suppose that $v_n(1) \geq 1+\varepsilon$. Assume for the moment that $v_n(t) \geq 1$ for $t \in (0,1)$. The mean value theorem tells us that for some $\tau_n \in (0,1)$, $v_n'(\tau_n) = \varepsilon$, so that $v_n'(1) \geq \varepsilon$. As v_n is convex, $v_n(t) \geq 1+\varepsilon t$ for $t \geq 1$. Thus $|g_n'(t)| \leq (1+\varepsilon t)^{-2}$, so that

$$|w_{2,n}| \leq \int_0^\infty |g_n'(t)|\,dt \leq 1 + \int_1^\infty \frac{1}{(1+\varepsilon t)^2}\,dt \leq 1 + \frac{1}{\varepsilon}. \tag{2.12}$$

We see that $\frac{1}{4}n \leq \bigl(1+(1/\varepsilon)\bigr)$, or that $n \leq 4\bigl(1+(1/\varepsilon)\bigr)$. Similarly if $v_n(-1) \geq 1+\varepsilon$ and $v_n(t) \geq 1$ for $t \in (-1,0)$, then $n \leq 4\bigl(1+(1/\varepsilon)\bigr)$. The convexity of v_n implies that indeed $v_n(t) \geq 1$ on either $t \in (-1,0)$ or $(0,1)$, so that as $n \to \infty$ we see that $v_n(\pm 1) \to 1$.

As we may, for any $t \in \mathbf{R}$ rather than just $t = \pm 1$, study in the same fashion the behavior of $v_n(t)$, this means that as $n \to \infty$, $v_n(t) \to 1$ for all $t \in \mathbf{R}$. Note that if $\{f_n\}_{n=1}^\infty$ is a sequence of positive convex functions on $\mathbf{R}$ such that $f_n(0) = 1$ for all n, and, for all $t \in \mathbf{R}$, $f_n(t) \to 1$, then $f_n \to 1$ uniformly on compact sets in $\mathbf{R}$. We conclude that $v_n \to 1$ uniformly on compact sets in $\mathbf{R}$.

It follows from the identity theorem that $g_n \to \mathrm{id}$ uniformly on compacta, as desired. □

3. Proof of Theorem 2

We pick r_n so that $F_n\colon \mathbf{D} \to \hat{\mathbf{C}}$ given by $F_n(z) = f_n(r_n z)$ is univalent on $\mathbf{D}$ but not on $\mathbf{S}^1$. Such clearly exist, and that $r_n \to 1$ as $n \to \infty$ follows from an estimate of Kra and Maskit [10, Lemma 5.1]. Pick $z_{z,1}$, $z_{n,2} \in \mathbf{S}^1$ such that $F_n(z_{n,1}) = F_n(z_{n,2})$. Take $\psi_n \in \text{Möb}(\mathbf{D})$ such that $\psi_n(z_{n,1}) = -1$ and $\psi_n(z_{n,2}) = 1$. By pre- and post-composition with Möbius transformations we may assume $G_n(z) = \phi_n \circ F_n \circ \psi_n(z)$ is univalent on $\mathbf{D}$, extends holomorphically to $\mathbf{S}^1$, and $G_n(-1) = G_n(1) = \infty$. Note that we still have two degrees of freedom left in the choice of ϕ_n, one in the choice of ψ_n.

With $h\colon \mathscr{T} \to \mathbf{D}$ as in Section 2, let $g_n = G_n \circ h$. We wish to choose ϕ_n, ψ_n so that $g_n \to \mathrm{id}$ uniformly on compacta. As in Section 2, let $v_n = |g_n'|^{-1/2}$.

Lemma 3. *Let $t \in \mathbf{R}$. As $t \to \pm\infty$, $v_n(t) = O\bigl(\exp(-|t/2|)\bigr)$.*

Proof of Lemma 3. With G_n as in the preceding paragraph, we let $V_n(z) = |G_n'(z)|^{-1/2}$. As G_n has a pole at both ± 1, G_n' has a double pole at both. Thus V_n vanishes at ± 1. One readily checks that

$$v_n(t) = V_n\bigl(h(t)\bigr) \cdot \left|\frac{dh}{dt}\right|^{-1/2} \qquad \text{with} \qquad z = h(t) = \tanh\left(\frac{t}{2}\right). \tag{3.1}$$

Thus, as $V_n(z) = O\bigl(|z-1|\bigr)$ as $z \to 1$,

$$\begin{aligned} v_n(t) &= O\bigl(\tanh(\tfrac{1}{2}t) - 1\bigr) \cdot O\bigl(\cosh(\tfrac{1}{2}t)\bigr) \\ &= O(e^{-t}) \cdot O(e^{t/2}) \qquad\qquad \text{as } t \to \infty. \\ &= O(e^{-t/2}) \end{aligned} \tag{3.2}$$

Since the same argument works as $t \to -\infty$, the lemma is established. □

Lemma 4. *For any fixed ϕ_n such that our normalization holds,*

$$\max_{t \in \mathbf{R}} v_n(t) \tag{3.3}$$

is defined and $\neq \infty$, independent of the freedom of choice of ψ_n.

Proof of Lemma 4. By Lemma 3, there exists a maximum over $\mathbf{R}$ for $v_n(t)$. Our choice of ψ_n is determined up to post-composition by an element of $\mathrm{Möb}(\mathbf{D})$ fixing 1 and -1. Since if $\psi \in \mathrm{Möb}(\mathbf{D})$ fixes both 1 and -1, $\psi \circ \psi_n \circ h(t) = \psi_n \circ h(t+\tau)$ for some fixed $\tau \in \mathbf{R}$, the proof of our lemma reduces down to showing that $\max_{t \in \mathbf{R}} v_n(t) = \max_{t \in \mathbf{R}} v_n(t+\tau)$. But this is immediate, proving the lemma. □

We now choose ψ_n so that the maximum in (3.3) occurs at $t = 0$, and choose ϕ_n so that $g_n(0) = 0$, $g_n'(0) = 1$.

Since the g_n are univalent on $\mathscr{T}$ and normalized as in the last paragraph, we consider any convergent subsequence (still called $\{g_n\}_{n=1}^{\infty}$ for convenience) which converges uniformly on compacta in $\mathscr{T}$. Then v_n also converges uniformly on compacta in $\mathscr{T}$. Let g^* be the limit of the g_n, and v^* the limit of the v_n. As in (2.8), v^* satisfies an ODE

$$(v^*)''(t) = p^*(t) \cdot v^*(t), \tag{3.4}$$

where $p^* \geq 0$. Thus v^* is convex for $t \in \mathbf{R}$.

Noting that if $\left\{f_n \colon \mathbf{R} \to [0,1], f_n(0) = 1\right\}_{n=1}^{\infty}$ is a sequence of continuous functions which converge pointwise to a convex function f^*, then $f^* \equiv 1$, we deduce that $v^* \equiv 1$. As this limit is independent of the subsequence chosen in the last paragraph from the original sequence $\{g_n\}_{n=1}^{\infty}$, this completes the proof of the theorem. □

4. Some consequences

Let $\mathscr{R}_1$ and $\mathscr{R}_2$ be two quasiconformally equivalent hyperbolic Riemann surfaces. If $\mathscr{R}_1$ satisfies either condition (O_1) or (O_2) then $\mathscr{R}_2$ does also: if $\mathscr{R}_1$ has big balls then the fact that K-quasiconformal maps are bi-Hölder of exponent $1/K$ (see [1, Chapter 1]) implies that $\mathscr{R}_2$ does as well, and $\mathscr{R}_1$ having long tubes is equivalent to annuli of arbitrarily large moduli being embedded in $\mathscr{R}_1$—a property which is preserved by quasiconformal deformation (see [4, p. 159]). We may thus say that the satisfaction of (O_1) or (O_2) is determined by the underlying quasiconformal structure, and have

Corollary 1. *If $\mathscr{R}$ is a quasiconformal hyperbolic surface and $\mathscr{R}_1$, $\mathscr{R}_2$ are two conformal structures on $\mathscr{R}$ compatible with its quasiconformal structure, then the following conditions are equivalent:*

1. $i\big(\mathbf{T}(\mathscr{R}_1)\big) = \frac{1}{2}$;
2. $o\big(\mathbf{T}(\mathscr{R}_1)\big) = \frac{3}{2}$;
3. $i\big(\mathbf{T}(\mathscr{R}_2)\big) = \frac{1}{2}$;
4. $o\big(\mathbf{T}(\mathscr{R}_2)\big) = \frac{3}{2}$;
5. *$\mathscr{R}$ has big balls or long tubes (in the sense of Theorem 3), independent of the compatible complete hyperbolic structure.*

Following this line of thought, we let $K_{1/3}$ denote the middle thirds Cantor set, and $\hat{\mathbf{C}} \setminus K_{1/3}$ its complement in the number sphere. As this hyperbolic Riemann surface is quasiconformally equivalent to a normal cover of a genus two surface, with $\mathbf{Z} * \mathbf{Z}$ acting as the group of deck transformations (see [22] for the construction), we have that hyperbolically $\hat{\mathbf{C}} \setminus K_{1/3}$ satisfies neither (O_1) nor (O_2). This yields

Corollary 2. *For the Riemann surface $\hat{\mathbf{C}} \setminus K_{1/3}$ one has $i\big(\mathbf{T}(\hat{\mathbf{C}} \setminus K_{1/3})\big) > \frac{1}{2}$ and $o\big(\mathbf{T}(\hat{\mathbf{C}} \setminus K_{1/3})\big) < \frac{3}{2}$.*

Theorems 3 and 4 may be seen to hold in the more general case of sequences of Riemann surfaces. One has, in particular,

Corollary 3. *Let $\{\mathscr{R}_n\}_{n=1}^{\infty}$ be a sequence of hyperbolic Riemann surfaces. The following are equivalent:*

1. $i\big(\mathbf{T}(\mathscr{R}_n)\big) \to \frac{1}{2}$;
2. $o\big(\mathbf{T}(\mathscr{R}_n)\big) \to \frac{3}{2}$;
3. *as $n \to \infty$ the $\mathscr{R}_n$ have either larger and larger balls or longer and longer tubes (in the sense of the result of* [18] *mentioned in Section 1).*

For a hyperbolic Riemann surface $\mathscr{R}$, $\mathbf{T}(\mathscr{R})$ depends only on the conformal type of $\mathscr{R}$, not on any marking. Thus we may consider $i\big(\mathbf{T}(\mathscr{R})\big)$ and $o\big(\mathbf{T}(\mathscr{R})\big)$ as functions on the moduli space of unmarked Riemann surfaces quasiconformally equivalent to $\mathscr{R}$. In the case where $\mathscr{R}$ has finite hyperbolic area, is of genus g, and has n punctures, we denote by $\mathscr{M}^{g,n}$ this moduli space. Mumford [14] showed that if $\{\mathscr{R}_k \in \mathscr{M}^{g,n}\}_{k=1}^{\infty}$ is a sequence of Riemann surfaces leaving every compact set in $\mathscr{M}^{g,n}$, then the $\mathscr{R}_n$ are developing short geodesics. By the collar lemma (see [9]) this implies that the $\mathscr{R}_n$ are developing long tubes. In this finite volume case, the converses also clearly hold. Thus one has

Corollary 4. *$i\big(\mathbf{T}(\mathscr{R}_n)\big) \to \frac{1}{2}$ and $o\big(\mathbf{T}(\mathscr{R}_n)\big) \to \frac{3}{2}$ if and only if the $\{\mathscr{R}_n\}_{k=1}^{\infty}$ eventually leave every compacta in $\mathscr{M}^{g,n}$.*

Finally we note that these results hold, with little modification, for hyperbolic orbifolds, *i.e.* $\mathscr{R} = \mathbf{D}/\Gamma$ where Γ may have elliptic elements.

5. Obstructions to quasidisk-hood may be quite complicated

We now examine in a somewhat more general setting the obstruction to a simply connected domain being a quasidisk. First note that the set of $\alpha \in \mathbf{C}$ such that $\alpha\, dz^2/(1-z^2)^2$ is the Schwarzian derivative of a map from $\mathbf{D}$ to a quasidisk forms the interior of the bounded set bordered by the cardioid

$$\Lambda = \left\{ \alpha = 2(1 - re^{2i\theta}) : r = 4\cos^2\theta, 0 \le \theta \le \pi \right\},$$

in $\mathbf{C}$. For α exterior to this cardioid, any map from $\mathbf{D}$ to $\hat{\mathbf{C}}$ having $\alpha\, dz^2/(1-z^2)^2$ as its Schwarzian is not even univalent (see [8]). We saw in Theorem 1 above that for univalent maps $f\colon \mathbf{D} \to \hat{\mathbf{C}}$ with $\|\mathscr{S}_f\|_\infty \le \frac{1}{2}$ the point $\alpha = 2$ on Λ embodies the entire obstruction to $f(\mathbf{D})$ being a quasidisk. (Never mind the fact that all such maps are *a priori* univalent.) It was seen in [18] that if $f\colon \mathbf{D} \to \hat{\mathbf{C}}$ is univalent with $\|\mathscr{S}_f\|_\infty = \frac{3}{2}$ then the point $\alpha = -6$ embodies the obstruction to $f(\mathbf{D})$ being a quasidisk.

Now consider the space of oriented simply connected domains Ω in $\hat{\mathbf{C}}$ which are uniformized by $\mathbf{D}$. We say two such domains are equivalent, denoted by $\Omega_1 \sim \Omega_2$, precisely when there is some $\phi \in$ Möb such that $\Omega_2 = \phi(\Omega_1)$. The space of such domains mod $\sim$ is naturally parametrized by the Schwarzian derivatives of Riemann mappings to the Ω, mod precomposition by elements of Möb($\mathbf{D}$). We have the following straightforward consequence of the Nakanishi–Yamamoto theorem

Corollary 5. *There exists a quasidisk Ω such that no sequence of uniformizing maps $f_n\colon \mathbf{D} \to \Omega$ has the property that for some $\alpha \in \mathbf{C}$, $\mathscr{S}_{f_n} \to \alpha\, dz^2/(1-z^2)^2$ uniformly on compacta in $\mathbf{D}$.*

Proof. Such a quasidisk may be taken to be one of the components of the domain of discontinuity of a quasifuchsian group uniformizing two conformally distinct, quasiconformally equivalent marked finite hyperbolic area Riemann surfaces. As the geometries of the two surfaces are finite and fixed, neither big balls nor long tubes exist. □

This allows us to show the following, which includes Theorem 5 of the introduction.

Theorem 5′. *There exists a simply connected non-quasidisk domain Ω uniformized by $\mathbf{D}$ such that*

1. *no sequence of uniformizing maps $f_n\colon \mathbf{D} \to \Omega$ has the property that for some $\alpha \in \mathbf{C}$, $\mathscr{S}_{f_n} \to \alpha\, dz^2/(1-z^2)^2$ uniformly on compacta in $\mathbf{D}$,*
2. *for any uniformizing map $f\colon \mathbf{D} \to \Omega$, there exists a sequence of quasidisks Ω_k and uniformizing maps $g_k\colon \mathbf{D} \to \Omega_k$ such that $g_k \to f$ uniformly on compacta in $\mathbf{D}$ (and, moreover, $\mathscr{S}_{g_k} \to \mathscr{S}_f$ in the topology of (1.3)).*

Thus the special collection of quadratic differentials $\alpha\, dz^2/(1-z^2)^2$ does not contain the full obstruction for a simply connected domain uniformized by $\mathbf{D}$ *to be a quasidisk.*

Proof. For an example here, take a totally degenerate group on the boundary of Bers' embedding of Teichmüller space for a finite hyperbolic area Riemann surface. Once again, the finite geometry of the surface does not allow for big balls or long tubes. In fact, one may use Corollary 4 to show that $\|\mathscr{S}_f\|_\infty$ may be taken arbitrarily close to either $\frac{1}{2}$ or $\frac{3}{2}$. □

Addendum. Since the writing of this article, Ohtake's [20] has appeared, containing related results.

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